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Estimados socios:

En este nuevo número del boletín recogemos una veintena de trabajos presentados en la Conferencia Internacional *Non-autonomous and Stochastic Dynamical Systems and Multidisciplinary Applications* (NSDS'09) celebrada en honor del profesor **Peter E. Kloeden**, en conmemoración de su sexagésimo cumpleaños, que tuvo lugar en Sevilla, entre el 22 y el 26 de Junio del pasado año 2009.

El profesor Kloeden se ha destacado por sus contribuciones en todo tipo de ecuaciones diferenciales, sistemas dinámicos y sus aplicaciones, recibiendo recientemente el premio **W. T. and Idalia Reid** de la *Society for Industrial and Applied Mathematics* (SIAM) por sus contribuciones fundamentales en teoría y análisis computacional de ecuaciones diferenciales.

La Conferencia, que se centró en avances recientes en métodos topológicos, teoría ergódica y nuevos desarrollos para sistemas dinámicos no autónomos y sistemas dinámicos estocásticos, fue organizada por la Universidad de Sevilla y contó con más de una veintena de conferenciantes invitados y casi un centenar de participantes. Queremos agradecer especialmente a dos de sus organizadores, Mª José Garrido y Pedro Marín, por su ayuda en la recopilación de los trabajos que presentamos aquí.

Recibid un cordial saludo,

Grupo Editor
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PULLBACK ATTRACTOR FOR A NON-AUTONOMOUS REACTION-DIFFUSION EQUATION IN SOME UNBOUNDED DOMAINS

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Abstract

The existence of a pullback attractor in $L^2(\Omega)$ for the following non-autonomous reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t), \text{ in } \Omega \times (\tau, +\infty), \\ u = 0, \quad \text{on} \quad \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{1}$$

is proved in this paper, when the domain Ω is not necessarily bounded but satisfying the Poincaré inequality, and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. The main concept used in the proof is the asymptotic compactness of the process generated by the problem.

Key words: *pullback attractor, asymptotic compactness, evolution process, non-autonomous reaction-diffusion equation.*
AMS subject classifications: *35B41, 35Q35, 35Q30, 35K90, 37L30.*

1 Introduction and setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be an open set, not necessarily bounded and suppose that Ω satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\int_\Omega |u(x)|^2\, dx \leq \lambda_1^{-1} \int_\Omega |\nabla u(x)|^2\, dx, \quad \forall u \in H_0^1(\Omega). \tag{2}$$

Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t), \text{ in } \Omega \times (\tau, +\infty), \\ u = 0, \quad \text{on} \quad \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{3}$$

where $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $f \in C(\mathbb{R})$ satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, $l \geq 0$, and $p > 2$ such that

$$-\alpha_1 |s|^p \leq f(s)s \leq -\alpha_2 |s|^p, \tag{4}$$

$$(f(s) - f(r))(s - r) \leq l(s - r)^2 \quad \forall r, s \in \mathbb{R}. \tag{5}$$

The aim of this paper is to show the existence of a pullback attractor in the phase space $L^2(\Omega)$ for the problem (3) in the case of open domains not necessarily bounded but satisfying the Poincaré inequality. This, and the fact that the non-autonomous h belongs to the space $L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, are the main novelties of our problem.

The lack of compactness of the injection $H^1_0(\Omega) \subset L^2(\Omega)$ (in the case of unbounded domains) implies that the standard techniques previously used, particularly the one involving the so-called flatenning property (see [6], [7], [12], [14], amongst others), which have been successfully used when Ω is bounded and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, do not work in our case.

Instead, we will use the asymptotic compactness already used in the case of non-autonomous 2D-Navier-Stokes (see [1] and [2], see also [5] for a close result), and which was previously used in [11] for the autonomous case. We would like to emphasize that this technique seems to be the only one which allows to prove the main result of this paper (namely Theorem 4) concerning the existence of pullback attractor for our problem.

It is also worth mentioning that our problem has received much attention over the last years in the case of a bounded domain or for a less general term h (see [3], [7], [12], [14]).

Finally, the reader can find similar results for several variants of our model in the references [9], [10], among others.

2 Existence and uniqueness of solution

We state in this section a result on the existence and uniqueness of solution of problem (3). By $|\cdot|$ we denote the norm in $L^2(\Omega)$, by $|\nabla\cdot|$ the norm in $H^1_0(\Omega)$ and by $\|\cdot\|_*$ the norm in $H^{-1}(\Omega)$. We will use $(\cdot,\cdot)$ to denote the scalar product in $L^2(\Omega)$ and we will use $\langle\cdot,\cdot\rangle$ to denote the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Theorem 1 *Suppose that Ω satisfies (2). Assume that $f \in C(\mathbb{R})$ satisfies (4) and (5), and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. Then, for all $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, there exists a unique solution $u(t) = u(t; \tau, u_\tau)$ of (3) such that*

$$\begin{aligned}
&u \in L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \quad \forall T > \tau, \\
&\frac{d}{dt}(u(t), v) - \langle \Delta u(t), v\rangle = \langle f(u(t)), v\rangle \\
&\quad + \langle h(t), v\rangle, \ \text{in } \mathcal{D}'(\tau, \infty), \ \forall v \in H^1_0(\Omega) \cap L^p(\Omega), \\
&u(\tau) = u_\tau.
\end{aligned}$$

Moreover,

$$u \in C([\tau,\infty); L^2(\Omega)),$$

and u satisfies the energy equation,

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + |\nabla u(t)|^2 = \langle f(u(t)), u(t)\rangle \\ +\langle h(t), u(t)\rangle \quad \text{in } \mathcal{D}'(\tau,\infty). \tag{6}$$

Proof. The proof of this Theorem can be done by the method of monotony (see [8]). □

3 Preliminaries on the theory of pullback attractors

Now, we will recall the main points from the theory of pullback attractors which will be needed to prove our objective (see [1] and [2] for more details).

Let us consider a process (also called a two-parameter semigroup) U on a metric space X, i.e., a family $\{U(t,\tau);\ -\infty < \tau \leq t < +\infty\}$ of continuous mappings $U(t,\tau) : X \to X$, such that $U(\tau,\tau)x = x$, and

$$U(t,\tau) = U(t,r)U(r,\tau) \quad \text{for all } \tau \leq r \leq t. \tag{7}$$

Suppose that $\mathcal{D}$ is a nonempty class of parameterized sets $\widehat{D} = \{D(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 1 *The process $U(\cdot,\cdot)$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact (i.e. pre-compact) in X.*

Definition 2 *It is said that $\widehat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $U(\cdot,\cdot)$ if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \leq t$ such that*

$$U(t,\tau)D(\tau) \subset B(t) \quad \text{for all } \tau \leq \tau_0(t,\widehat{D}).$$

Definition 3 *The family $\widehat{A} = \{A(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pullback $\mathcal{D}$-attractor for $U(\cdot,\cdot)$ if*

1. *$A(t)$ is compact for all $t \in \mathbb{R}$,*

2. *$\widehat{A}$ is pullback $\mathcal{D}$-attracting, i.e.,*

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0,$$

for all $\widehat{D} \in \mathcal{D}$, and all $t \in \mathbb{R}$,

3. *$\widehat{A}$ is invariant, i.e.,*

$$U(t,\tau)A(\tau) = A(t), \quad \text{for } -\infty < \tau \leq t < +\infty.$$

We have the following result (see [2] for more details).

Theorem 2 *Suppose that the process $U(\cdot,\cdot)$ is pullback $\mathcal{D}$-asymptotically compact and that $\widehat{B} \in \mathcal{D}$ is a family of pullback $\mathcal{D}$-absorbing sets for $U(\cdot,\cdot)$.*

Then, the family $\widehat{A} = \{A(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ defined by $A(t) = \Lambda(\widehat{B}, t),\quad t \in \mathbb{R}$, where for each $\widehat{D} \in \mathcal{D}$

$$\Lambda(\widehat{D}, t) = \bigcap_{s \le t} \left(\overline{\bigcup_{\tau \le s} U(t,\tau) D(\tau)} \right),$$

is a pullback $\mathcal{D}$-attractor for $U(\cdot,\cdot)$ which satisfies in addition that $A(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}$, for $t \in \mathbb{R}$. Furthemore, $\widehat{A}$ is minimal in the sense that if $\widehat{C} = \{C(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that $\lim_{\tau \to -\infty} \mathrm{dist}(U(t,\tau)B(\tau), C(t)) = 0$, then $A(t) \subset C(t)$.

4 Existence of the pullback attractor

Now, we can prove our main result in this paper. First, we need a continuity result which is established in the next subsection.

4.1 Weak Continuity

Assume that the function $f \in C(\mathbb{R})$ satisfies (4) and (5), and that $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$.

Thanks to Theorem 1, we can define a process $\{U(t,\tau),\ \tau \le t\}$ in $L^2(\Omega)$, as

$$U(t,\tau)u_\tau = u(t;\tau,u_\tau)\ \forall u_\tau \in L^2(\Omega),\ \forall \tau \le t. \tag{8}$$

From the uniqueness of solution to problem (3), it follows that (8) defines a process in $L^2(\Omega)$. In addition, it can be proved that the process defined by (8) is continuous in $L^2(\Omega)$.

Moreover, U is weakly continuous, and more exactly the following result holds true. We will denote by "$\rightharpoonup$" the weak convergence in the corresponding indicated space, while "$\rightarrow$" will denote the strong convergence, as usual.

Proposition 3 *Let $\{u_{\tau_n}\} \subset L^2(\Omega)$ be a sequence converging weakly in $L^2(\Omega)$ to an element $u_\tau \in L^2(\Omega)$. Then, for all $T > \tau$, it follows*

$$U(t,\tau)u_{\tau_n} \rightharpoonup U(t,\tau)u_\tau \text{ in } L^2(\Omega)\ \forall t \ge \tau, \tag{9}$$

$$U(\cdot,\tau)u_{\tau_n} \rightharpoonup U(\cdot,\tau)u_\tau \text{ in } L^2(\tau,T;H^1_0(\Omega)), \tag{10}$$

$$U(\cdot,\tau)u_{\tau_n} \rightharpoonup U(\cdot,\tau)u_\tau \text{ in } L^p(\tau,T;L^p(\Omega)), \tag{11}$$

$$f(U(\cdot,\tau)u_{\tau_n}) \rightharpoonup f(U(\cdot,\tau)u_\tau) \text{ in } L^{p'}(\tau,T;L^{p'}(\Omega)). \tag{12}$$

If Ω is a bounded set, then

$$U(\cdot,\tau)u_{\tau_n} \longrightarrow U(\cdot,\tau)u_\tau \text{ in } L^2(\tau,T;L^2(\Omega)). \tag{13}$$

Proof. This result may be proved in much the same way as Theorem 1, and using similar arguments to [11]. □

4.2 The existence of the global pullback attractor

Let $\mathcal{R}_{\lambda_1}$ be the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t \to -\infty} e^{\lambda_1 t} r^2(t) = 0,$$

and denote by $\mathcal{D}_{\lambda_1}$ the class of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that $D(t) \subset \overline{B}(0, r_{\widehat{D}}(t))$, for some $r_{\widehat{D}} \in \mathcal{R}_{\lambda_1}$, where $\overline{B}(0, r_{\widehat{D}}(t))$ denotes the closed ball in $L^2(\Omega)$ centered at zero with radius $r_{\widehat{D}}(t)$.

Now, we can prove the following result.

Theorem 4 *Suppose that Ω satisfies (2), and suppose that $f \in C(\mathbb{R})$ satisfies (4) and (5) with $l = 0$. Let $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ be such that*

$$\int_{-\infty}^{t} e^{\lambda_1 s} \|h(s)\|^2_{H^{-1}(\Omega)} \, ds < +\infty \quad \forall t \in \mathbb{R}.$$

Then, there exists a unique global pullback $\mathcal{D}_{\lambda_1}$-attractor for the process U, which belongs to $\mathcal{D}_{\lambda_1}$, and is defined by (8).

Proof. We only give the main ideas of the proof. Let $\tau \in \mathbb{R}$, and $u_\tau \in L^2(\Omega)$ be fixed, and denote

$$u(t) = u(t; \tau, u_\tau) = U(t, \tau)u_\tau \quad \forall t \geq \tau.$$

Let $\widehat{D} \in \mathcal{D}_{\lambda_1}$ be given. Taking into account (2), (4), the energy equality and integrating between τ and t,

$$\begin{aligned} |U(t,\tau)u_\tau|^2 \quad &\leq \quad e^{-\lambda_1 t} \int_{-\infty}^{t} e^{\lambda_1 s} \|h(s)\|^2_{H^{-1}(\Omega)} \, ds \\ &\quad + e^{\lambda_1(\tau - t)} r^2_{\widehat{D}}(\tau), \end{aligned} \tag{14}$$

for all $u_\tau \in D(\tau)$ and for all $t \geq \tau$.
Denote by $R_{\lambda_1}(t)$ the nonnegative number given for each $t \in \mathbb{R}$ by

$$R^2_{\lambda_1}(t) = e^{-\lambda_1 t} \int_{-\infty}^{t} e^{\lambda_1 s} \|h(s)\|^2_{H^{-1}(\Omega)} \, ds + 1. \tag{15}$$

Observe that $R_{\lambda_1} \in \mathcal{R}_{\lambda_1}$. Now, consider the family $\widehat{B}_{\lambda_1}$ of closed balls in $L^2(\Omega)$, $\widehat{B}_{\lambda_1} = \{B_{\lambda_1}(t) : t \in \mathbb{R}\}$, defined by $B_{\lambda_1}(t) = \left\{v \in L^2(\Omega) : |v| \leq R_{\lambda_1}(t)\right\}$. It is straightforward to check that $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$, and moreover, by (14), the family $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing for the process U.
According to Theorem 2, to finish the proof of the theorem we only have to prove that U is pullback $\mathcal{D}_{\lambda_1}$-asymptotically compact.

Let us fix $\widehat{D} \in \mathcal{D}_{\lambda_1}$, a sequence $\tau_n \to -\infty$, a sequence $u_{\tau_n} \in D(\tau_n)$, and $t \in \mathbb{R}$. We have to prove that from the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ we can extract a subsequence that converges in $L^2(\Omega)$.

As the family $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, by a diagonal procedure, it is not difficult to conclude that there exist a subsequence $\left\{\left(\tau_{n'}, u_{\tau_{n'}}\right)\right\} \subset \{(\tau_n, u_{\tau_n})\}$, and a sequence $\{w_k; k \geq 0\} \subset L^2(\Omega)$ such that for all $k \geq 0$, and $w_k \in B_{\lambda_1}(t-k)$,

$$U(t-k,\tau_{n'})u_{\tau_{n'}} \rightharpoonup w_k \ \text{ in } L^2(\Omega), \tag{16}$$

and

$$|w_0| \leq \liminf_{n'\to\infty} \left|U(t,\tau_{n'})u_{\tau_{n'}}\right|. \tag{17}$$

If we now prove that also

$$\limsup_{n'\to\infty} \left|U(t,\tau_{n'})u_{\tau_{n'}}\right| \leq |w_0|, \tag{18}$$

then we will have

$$\lim_{n'\to\infty} \left|U(t,\tau_{n'})u_{\tau_{n'}}\right| = |w_0|.$$

And this, together with the weak convergence, will imply the strong convergence in $L^2(\Omega)$ of $U(t,\tau_{n'})u_{\tau_{n'}}$ to w_0.

In order to prove (18), consider $[u] := |\nabla u|^2 - \frac{\lambda_1}{2}|u|^2 - \langle f(u), u\rangle$. Taking into account (2), (4), the energy equality and integrating between τ and t, it is immediate that for all $k \geq 0$ and all $\tau_{n'} \leq t-k$,

$$\begin{aligned}
&\left|U(t,\tau_{n'})u_{\tau_{n'}}\right|^2 \\
&= \left|U(t-k,\tau_{n'})u_{\tau_{n'}}\right|^2 e^{-\lambda_1 k} \\
&+ 2\int_{t-k}^{t} e^{\lambda_1(s-t)} \left\langle h(s), U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right\rangle ds \\
&- 2\int_{t-k}^{t} e^{\lambda_1(s-t)} \left[U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right] ds.
\end{aligned} \tag{19}$$

Now we will prove that

$$\begin{aligned}
&\int_{t-k}^{t} e^{\lambda_1(s-t)} \left[U(s,t-k)w_k\right] ds \\
&\leq \liminf_{n'\to\infty} \int_{t-k}^{t} e^{\lambda_1(s-t)} \left[U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right] ds.
\end{aligned} \tag{20}$$

Denote

$$J_k(v) = J_k^{(1)}(v) + J_k^{(2)}(v),$$

where

$$J_k^{(1)}(v) = \int_{t-k}^{t} e^{\lambda_1(s-t)} \left(|\nabla v(s)|^2 - \frac{\lambda_1}{2}|v(s)|^2\right) ds,$$

and

$$J_k^{(2)}(v) = -\int_{t-k}^{t} e^{\lambda_1(s-t)} \langle f(v), v\rangle \, ds,$$

for all $v \in L^2(t-k, t; H_0^1(\Omega)) \cap L^p(t-k, t; L^p(\Omega))$.
We also obtain from (16) and using Proposition 3

$$\begin{aligned} \liminf_{n'\to\infty} (J_k^{(1)}(U(\cdot, t-k)U(t-k, \tau_{n'})u_{\tau_{n'}}) \\ \geq J_k^{(1)}(U(\cdot, t-k)w_k). \end{aligned} \tag{21}$$

Using (5) with $l = 0$, from (16) and Proposition 3 we easily obtain

$$\begin{aligned} \lim_{n'\to\infty} \inf \left(J_k^{(2)}(U(\cdot, t-k)U(t-k, \tau_{n'})u_{\tau_{n'}}) \right) \\ \geq J_k^{(2)}(U(\cdot, t-k)w_k). \end{aligned}$$

Therefore (20) is easily obtained from the last inequality and (21).
Then, taking into account that the family $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, from (16), using Proposition 3 and thanks to (19) and (20), we obtain

$$\begin{aligned} \lim_{n'\to\infty} \sup \left| U(t, \tau_{n'})u_{\tau_{n'}} \right|^2 \\ \leq R_{\lambda_1}^2(t-k)e^{-\lambda_1 k} + |w_0|^2, \end{aligned}$$

for all $k \geq 1$. Taking into account (15), we easily obtain

$$\lim_{n'\to\infty} \sup \left| U(t, \tau_{n'})u_{\tau_{n'}} \right|^2 \leq |w_0|^2.$$

□

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ELLIPTIC PROBLEMS IN THIN DOMAINS WITH HIGHLY OSCILLATING BOUNDARIES

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Abstract

We study the Laplace operator with Neumann boundary conditions in a 2-dimensional thin domain with a higly oscillating boundary. We obtain the correct limit problem for the case where the boundary is the graph of the oscillating function $\epsilon G_\epsilon(x)$ where $G_\epsilon(x) = a(x) + b(x)g(x/\epsilon)$ with g periodic and a and b not necessarily constant.

Key words: *thin domains, oscillations, homogenization*
AMS subject classifications: *35B27, 74K10.*

1 Introduction

We are interested in analyzing the behavior of the solutions, as the parameter $\epsilon \to 0$, of the following linear elliptic problem

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f^\epsilon & \text{in } R^\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \text{on } \partial R^\epsilon \end{cases} \tag{1}$$

where the domain R_ϵ is a thin domain with a highly oscillating boundary, $f^\epsilon \in L^2(R^\epsilon)$ and $N^\epsilon = (N_1^\epsilon, N_2^\epsilon)$ is the unit outward normal to ∂R^ϵ.

We define the thin domain as

$$R^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I, \quad 0 < x_2 < \epsilon\, G_\epsilon(x_1)\} \tag{2}$$

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where $I = (0,1)$, the function $G_\epsilon(x) = a(x) + b(x)g(x/\epsilon)$ where $g : \mathbb{R} \mapsto \mathbb{R}$ is an L-periodic positive function of class $\mathcal{C}^1$ and the functions $a, b : I \mapsto \mathbb{R}$ are piecewise $\mathcal{C}^1$-functions defined on $I = (0,1)$ satisfying

$$\alpha_0 \leq a(x) \leq \alpha_1 \text{ and } \beta_0 \leq b(x) \leq \beta_1. \tag{3}$$

We also assume that there exist positive constants G_0 and G_1 such that

$$0 < G_0 \leq G_\epsilon(x) \leq G_1 \text{ on } I \tag{4}$$

uniformly in $\epsilon > 0$.

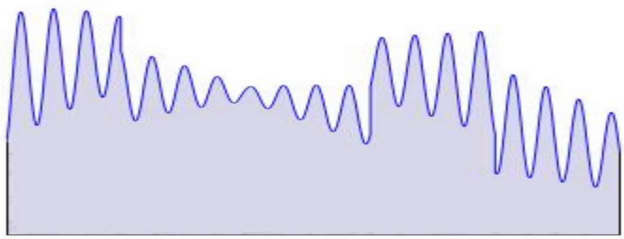

Figure 1: The thin domain R^ϵ

It is known that if the domain does not present oscillations, that is $g \equiv 0$, the 1-dimensional limiting problem is given by,

$$\begin{cases} -\dfrac{1}{a(x)}(a(x)v_x)_x + v = f, \text{ in } (0,1) \\ v_x(0) = v_x(1) = 0 \end{cases} \tag{5}$$

see for instance [6, 7, 8].

Moreover, if we consider $g(x/\epsilon^\alpha)$ for some $0 < \alpha < 1$, instead of $g(x/\epsilon)$ and if we assume that $a(x) + b(x)g(x/\epsilon^\alpha) \to h(x)$ w-$L^2(0,1)$ and $\frac{1}{a(x)+b(x)g(x/\epsilon^\alpha)} \to k(x)$ w-$L^2(0,1)$ (observe that $h(x)k(x) \geq 1$ a.e. and in general it is not true that $h(x)k(x) \equiv 1$), then the limit problem is

$$\begin{cases} -\dfrac{1}{h(x)}(\dfrac{1}{k(x)}v_x)_x + v = f, \text{ in } (0,1) \\ v_x(0) = v_x(1) = 0 \end{cases} \tag{6}$$

see [1].

In this note we are interested in addressing the case $\alpha = 1$, that is $G_\epsilon(x) = a(x) + b(x)g(x/\epsilon)$, where none of the techniques used to solve the previous two cases apply. Observe that this case is very resonant: the height of the domain, the amplitude of the oscillations at the boundary and the period of the oscillations are of the same order ϵ.

As a matter of fact, we will show that the limit equation is

$$\begin{cases} -\partial_x(q(x)\,\partial_x w_0) + p(x)w_0 = p(x)f(x), \quad x \in (0,1) \\ w_0'(0) = w_0'(1) = 0 \end{cases} \tag{7}$$

where

$$q(x) = \int_{Y_x^*} \Big\{1 - \frac{\partial X_x}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2, \qquad p(x) = |Y_x^*|$$

and the 2 dimensional domain Y_x^* depends on $x \in (0,1)$ and is given by

$$Y_x^* = \{(y_1, y_2) \in \mathbb{R}^2 \; : \; 0 < y_1 < L, \quad 0 < y_2 < a(x) + b(x)g(y_1)\} \tag{8}$$

and X_x is given by the solution of

$$\begin{cases} -\Delta X_x = 0 \text{ in } Y_x^* \\ \frac{\partial X_x}{\partial N} = 0 \text{ on } B_2^x \\ \frac{\partial X_x}{\partial N} = N_1 \text{ on } B_1^x \\ X_x(0, y_2) = X_x(L, y_2) \text{ on } B_0^x \\ \int_{Y_x^*} X_x \; dy_1 dy_2 = 0. \end{cases} \tag{9}$$

To obtain the limitting equation (7) we will divide the analysis in three cases: (1) the purely periodic case, that is, $a(\cdot)$ and $b(\cdot)$ constants; (2) the piecewise periodic case, that is, $a(\cdot)$ and $b(\cdot)$ are piecewise constant and (3) the general case, where a and b are smooth functions.

2 Basic facts and notation

To study the convergence of (1) on the thin oscillating domain R^ϵ, we consider the change of variables $(x, y) = (x_1, \epsilon x_2)$ which transform problem (1) into the equivalent linear elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon \quad \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 \quad \text{on } \partial\Omega^\epsilon \end{cases} \tag{10}$$

on the domain $\Omega^\epsilon \subset \mathbb{R}^2$ given by

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I, \quad 0 < x_2 < G_\epsilon(x_1)\}. \tag{11}$$

Observe that domain Ω_ϵ is not thin any more, although the oscillations presented at the upper boundary are very wild. Nevertheless, the $\frac{1}{\epsilon^2}$ factor in front of the diffusion in the x_2 direction compensate the very wild oscillations at the top boundary.

The variational formulation of (10) is find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} \Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi\Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi dx_1 dx_2 \; \forall \varphi \in H^1(\Omega^\epsilon). \tag{12}$$

In a natural way, we will need to consider the space $H^1(U)$ with the following norm, that we denote by $H^1_\epsilon(U)$

$$\|w\|^2_{H^1_\epsilon(U)} = \|w\|^2_{L^2(U)} + \left\|\frac{\partial w}{\partial x_1}\right\|^2_{L^2(U)} + \frac{1}{\epsilon^2}\left\|\frac{\partial w}{\partial x_2}\right\|^2_{L^2(U)}$$

given by the inner product

$$(\phi,\varphi)_{H^1_\epsilon(U)} = \int_U \left\{\frac{\partial \phi}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial \phi}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + \phi\,\varphi\right\} dx_1 dx_2$$

where U is an arbitrary open set of $\mathbb{R}^2$.

Remark that the solutions u^ϵ satisfy an uniform a priori estimate on ϵ. In fact, we can take $\varphi = u^\epsilon$ in the expression (12) and after some easy calculations, we obtain

$$\|u_\epsilon\|_{H^1_\epsilon(\Omega_\epsilon)} \le \|f^\epsilon\|_{L^2(\Omega_\epsilon)}. \tag{13}$$

We also have the following extension operator,

Lemma 1 *Let $\mathcal{O}$ and $\mathcal{O}^\epsilon$ be the domains given by*

$$\mathcal{O} = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < G_1\}$$
$$\mathcal{O}^\epsilon = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < G_\epsilon(x_1)\}$$

where $I \subset \mathbb{R}$ is an open interval, $G_\epsilon : I \mapsto \mathbb{R}$ is a $\mathcal{C}^1$-function satisfying $0 < G_0 \le G_\epsilon(x_1) \le G_1$ for all $x \in I$ and $\epsilon > 0$.

We have the following general extension operator

$$P_\epsilon \in \mathcal{L}(L^p(\mathcal{O}^\epsilon), L^p(\mathcal{O})) \cap \mathcal{L}(W^{1,p}(\mathcal{O}^\epsilon), W^{1,p}(\mathcal{O}))$$

and a constant K independent of ϵ and p such that

$$\begin{aligned} \|P_\epsilon\varphi\|_{L^p(\mathcal{O})} \le K\,\|\varphi\|_{L^p(\mathcal{O}^\epsilon)}, \quad \left\|\frac{\partial P_\epsilon\varphi}{\partial x_2}\right\|_{L^p(\mathcal{O})} \le K\left\|\frac{\partial \varphi}{\partial x_2}\right\|_{L^p(\mathcal{O}^\epsilon)} \\ \left\|\frac{\partial P_\epsilon\varphi}{\partial x_1}\right\|_{L^p(\mathcal{O})} \le K\left\{\left\|\frac{\partial \varphi}{\partial x_1}\right\|_{L^p(\mathcal{O}^\epsilon)} + \eta(\epsilon)\left\|\frac{\partial \varphi}{\partial x_2}\right\|_{L^p(\mathcal{O}^\epsilon)}\right\} \end{aligned} \tag{14}$$

for all $\varphi \in W^{1,p}(\mathcal{O}^\epsilon)$ where $1 \le p \le \infty$ and $\eta(\epsilon) = \sup_{x\in I}\{|G'_\epsilon(x)|\}$.

Proof. We extend the functions in the vertical direction by reflection across the oscillating boundary, see [3] for details. □

3 The purely periodic case

With the aid of the multiple scale method, we can obtain our candidate to limit problem. Once the limit problem is obtained, we may use the oscillatory test function method of Tartar to show the convergence. The ideas to obtain this problem follow very closely the arguments of [5, 4].

We define the basic cell

$$Y^* = \{(y,z) \in \mathbb{R}^2 \ : \ 0 < y < L \text{ and } 0 < z < g(y)\} \tag{15}$$

and we call B_0 the lateral boundary, B_1 the upper boundary and B_2 the lower boundary of ∂Y^*. So that, $\partial Y^* = B_0 \cup B_1 \cup B_2$.

In this cell we solve the problem,

$$\begin{cases} -\Delta_{y,z} X(y,z) = 0 \quad \text{ in } Y^* \\ \frac{\partial X}{\partial N}(y, g(y)) = -\frac{g'(y)}{\sqrt{1+(g'(y))^2}} \quad \text{ on } B_1, \\ \frac{\partial X}{\partial N}(y,0) = 0 \quad \text{ on } B_2 \\ X(0,z) = X(L,z) \quad z \in B_0, \\ \int_{Y^*} X \, dy_1 dy_2 = 0 \end{cases}$$

and consider

$$q = \int_{Y^*} \Big\{1 - \frac{\partial X}{\partial y}(y,z)\Big\} dydz, \qquad p = |Y^*|.$$

Then, the limitting problem is,

$$\begin{cases} -q\dfrac{d^2 w_0}{dx^2}(x) + p w_0(x) = p f(x), \quad x \in (0,1) \\ u_0'(0) = u_0'(1) = 0. \end{cases} \tag{16}$$

4 The piecewise periodic case

We consider in this section the case where the functions a and b are *locally constant functions* defined on $I = (0,1)$. That is, we suppose there exists a set of points

$$0 = x_0 < x_1 < ... < x_N = 1 \tag{17}$$

such that the functions a and b are constants, say a_i and b_i, on each interval $I_i = (x_{i-1}, x_i)$ with $1 \leq i \leq N$.

Considering the weak formulation (12) of problem (10) and using the extension operator from Lemma 1 in each of the intervals (x_i, x_{i+1}), we can obtain that if we define the family of basic cells

$$Y_i^* = \{(y_1,y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L, \quad 0 < y_2 < a_i + b_i g(y_1)\} \tag{18}$$

and if we solve the family of problems in each of the basic cells for $i = 1, \dots, n$,

$$\begin{cases} -\Delta X_i = 0 \text{ in } Y_i^* \\ \frac{\partial X_i}{\partial N} = 0 \text{ on } B_2^i \\ \frac{\partial X_i}{\partial N} = N_1 \text{ on } B_1^i \\ X_i(0,y_2) = X_i(L,y_2) \text{ on } B_0^i \\ \int_{Y_i^*} X_i \, dy_1 dy_2 = 0 \end{cases} \tag{19}$$

where B_0^i is the lateral boundary, B_1^i is the upper boundary and B_2^i is the lower boundary of ∂Y_i^* for all $i = 1, ..., N$ and define

$$q_i = \int_{Y_i^*} \Big\{1 - \frac{\partial X_i}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2, \qquad p_i = |Y_i^*|$$

then the variational formulation of the limit problem is

$$\int_0^1 q(x) \frac{\partial u_0}{\partial x} \frac{\partial \varphi}{\partial x} + p(x) u_0 \varphi = \int_0^1 p(x) f \varphi, \quad \forall \varphi \in H^1(0,1) \tag{20}$$

where $q(x)$ and $p(x)$ are the piecewise constant function $q(x) = q_i$, $p(x) = p_i$ for $x \in (x_i, x_{i+1})$, $i = 1, \dots, N$.

As a matter of fact, the following result can be proved,

Proposition 2 *For each sequence $\epsilon \to 0$, there exists a subsequence, that we also denote by ϵ, and a function $f_0 \in L^2(0,1)$ such that if $u_0 \in H^1(0,1)$ is the weak solution of (20) then*

$$P_\epsilon u^\epsilon \rightharpoonup u_0 \quad w - H^1((x_i, x_{i+1}) \times (0, G_1)), \quad i = 1, \dots, N$$

where P_ϵ is the extension operator given by Lemma 1 and where we assume that u_0 has been extended constantly in the x_2 direction.

5 The general case

We consider in this section the case where a and b are smooth functions not necessarily piecewise constant. We will obtain the limit equation and the convergence of the solution of (10) to the solution of the limit problem by approximating the functions $a(\cdot)$ and $b(\cdot)$ in $L^\infty(0,1)$ by piecewise constant functions $a_\delta(\cdot)$, $b_\delta(\cdot)$, using the results from the previous section and passing to the limit when $\delta \to 0$. Observe that if $\|a - a_\delta\|_{L^\infty(0,1)} \to 0$ and $\|b - b_\delta\|_{L^\infty(0,1)} \to 0$ as $\delta \to 0$, then $\|G_\epsilon - G_\epsilon^\delta\| \to 0$ as $\delta \to 0$, uniformly in ϵ. This new parameter δ introduces new difficulties in the problem since now we will need to consider problem (10) with two parameters, ϵ and δ and the solution $u = u_\delta^\epsilon$. In order to be able to pass to the limit appropriately, we will prove a result on the continuous dependence of the solutions of (10) with respect to the functions a and b uniformly in ϵ. This uniformity will allow us to interchange the limit and obtain the correct limit problem.

More precisely, assume a, $\hat{a}$, b and $\hat{b}$ are real functions uniformly bonded on I satisfying (3) and consider the associated oscillating domains Ω^ϵ and $\hat{\Omega}^\epsilon$ given by

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I, \quad 0 < x_2 < G_\epsilon(x_1)\},$$
$$\hat{\Omega}^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I, \quad 0 < x_2 < \hat{G}_\epsilon(x_1)\}$$

with

$$G_\epsilon(x) = a(x) + b(x) g(x/\epsilon) \qquad \hat{G}_\epsilon(x) = \hat{a}(x) + \hat{b}(x) g(x/\epsilon)$$

satisfying (4).

Let u^ϵ and $\hat{u}^\epsilon$ be the solutions of the problem (10) in the oscillating domains Ω^ϵ and $\hat{\Omega}^\epsilon$ respectively with $f^\epsilon \in L^2(\mathbb{R}^2)$. Then we have the following result:

Proposition 3 *There exists a positive real function $\rho : [0,\infty) \mapsto [0,\infty)$ such that*

$$\|u^\epsilon - \hat{u}^\epsilon\|^2_{H^1_\epsilon(\Omega^\epsilon\cap\hat{\Omega}^\epsilon)} + \|u^\epsilon\|^2_{H^1_\epsilon(\Omega^\epsilon\setminus\hat{\Omega}^\epsilon)} + \|\hat{u}^\epsilon\|^2_{H^1_\epsilon(\hat{\Omega}^\epsilon\setminus\Omega^\epsilon)} \le \rho(\delta) \tag{21}$$

with $\rho(\delta) \to 0$ as $\delta \to 0$ uniformly for all

- $\epsilon > 0$;
- *piecewise C^1 functions $a, b, \hat{a}, \hat{b}$ with $\|a-\hat{a}\|_{L^\infty(0,1)} \le \delta$, $\|b-\hat{b}\|_{L^\infty(0,1)} \le \delta$, and $\alpha_0 \le a(x), \hat{a}(x) \le \alpha_1$, $\beta_0 \le b(x), \hat{b}(x) \le \beta_1$,*
- $f_\epsilon \in L^2(\mathbb{R}^2)$, $\|f_\epsilon\|_{L^2(\mathbb{R}^2)} \le 1$.

Idea of the proof. We use that u^ϵ and $\hat{u}^\epsilon$ are the minima in $H^1(\Omega_\epsilon)$ and $H^1(\hat{\Omega}_\epsilon)$, respectively of the functionals

$$\begin{aligned} V_\epsilon(\varphi) &= \frac{1}{2}\int_{\Omega^\epsilon} \Big\{ \frac{\partial\varphi}{\partial x_1}^2 + \frac{1}{\epsilon^2}\frac{\partial\varphi}{\partial x_2}^2 + \varphi^2 \Big\} dx_1 dx_2 - \int_{\Omega^\epsilon} f^\epsilon \varphi dx_1 dx_2 \\ \hat{V}_\epsilon(\hat{\varphi}) &= \frac{1}{2}\int_{\hat{\Omega}^\epsilon} \Big\{ \frac{\partial\hat{\varphi}}{\partial x_1}^2 + \frac{1}{\epsilon^2}\frac{\partial\hat{\varphi}}{\partial x_2}^2 + \hat{\varphi}^2 \Big\} dx_1 dx_2 - \int_{\hat{\Omega}^\epsilon} f^\epsilon \hat{\varphi} dx_1 dx_2. \end{aligned} \tag{22}$$

That is,

$$V_\epsilon(u^\epsilon) = \min_{\varphi\in H^1(\Omega^\epsilon)} V_\epsilon(\varphi), \qquad \hat{V}_\epsilon(\hat{u}^\epsilon) = \min_{\hat{\varphi}\in H^1(\hat{\Omega}^\epsilon)} \hat{V}_\epsilon(\hat{\varphi}).$$

To be able to obtain an estimate like (21) we will need to be able to somehow plug the function u_ϵ in the minimization problem for $\hat{u}_\epsilon$ and also plug $\hat{u}_\epsilon$ in the minimization for u_ϵ and operate wisely to obtain the estimates. Since the domains Ω_ϵ and $\hat{\Omega}_\epsilon$ are different and they do not necessarily have any inclusion relation, we will need to "extend" the function u_ϵ to $\hat{\Omega}_\epsilon$ and viceversa, "extend" the function $\hat{u}_\epsilon$ to Ω_ϵ. But if we use the extension operator defined in Lemma 1 then the constants involved will depend on the derivative of the functions a, b and $\hat{a}, \hat{b}$. But this is a serious drawback since the functions a, b $\hat{a}, \hat{b}$ are not smooth enough and they do not need to be close in the C^1 topology. Therefore, we cannot use the extension operator from Lemma 1.

Instead of this extension operator we define an operator which consists in "stretching" a function defined in Ω_ϵ only in the x_2-direction by a factor $(1+\eta)$ and restrict the "stretched" function to the domain $\hat{\Omega}_\epsilon$. That is, in general, let us define the operator

$$\begin{aligned} P_{1+\eta} &: H^1(U) \mapsto H^1(U(1+\eta)) \\ (P_{1+\eta}\varphi)(x_1,x_2) &= \varphi\left(x_1, \frac{x_2}{1+\eta}\right) \quad (x_1,x_2)\in U \end{aligned} \tag{23}$$

where

$$U(1+\eta) = \{(x_1, (1+\eta)x_2) \in \mathbb{R}^2 \,|\, (x_1, x_2) \in U\}$$

and $U \subset \mathbb{R}^2$ is an arbitrary open set.

With this operator, we can show that if u^ϵ is a solution of problem (10) then

$$\|u^\epsilon\|^2_{H^1_\epsilon(\Omega^\epsilon \setminus \Omega^\epsilon(\frac{1}{1+\eta}))} + \|P_{1+\eta}u^\epsilon\|^2_{H^1_\epsilon(\Omega^\epsilon(1+\eta)\setminus\Omega^\epsilon)} + \|u^\epsilon - P_{1+\eta}u^\epsilon\|^2_{H^1_\epsilon(\Omega^\epsilon)} \le C\sqrt{\eta}$$

where $C = C(G_1, \|f^\epsilon\|_{L^2})$ independent of $\epsilon \in (0,1)$.

This last estimate allows us to consider the function $P_{1+\eta}u_\epsilon|_{\hat{\Omega}_\epsilon}$ as a test function in the problem in $\hat{\Omega}_\epsilon$ and $P_{1+\eta}\hat{u}_\epsilon|_{\Omega_\epsilon}$ as a test function in the problem in Ω_ϵ. The fact that $\|a-\hat{a}\|_{L^\infty(0,1)}, \|b-\hat{b}\|_{L^\infty(0,1)} \le \delta$ permits us to take η small uniformly in ϵ.

With Proposition 3 and using that estimate (21) is uniformly in ϵ, will allow us to show that the limit problem in the general case is given by (7).

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MORE ON FINITE-TIME HYPERBOLICITY

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Abstract

A solution of a nonautonomous ordinary differential equation is finite-time hyperbolic, i.e. hyperbolic on a compact interval of time, if the linearisation along that solution exhibits a strong exponential dichotomy. In analogy to classical asymptotic facts, it is shown that finite-time hyperbolicity is robust, that is, it persists under small perturbations. Eigenvalues and -vectors may be misleading with regards to hyperbolicity. This is demonstrated by means of simple examples.

Key words: *Hyperbolicity, exponential dichotomy, finite-time dynamics.*
AMS subject classifications: *34A30, 37B55, 37D05.*

Hyperbolicity is widely recognised as a fundamental notion of dynamical systems theory. While extensions and refinements of the classical, that is, asymptotic concept continue to play a vital role in modern dynamics, much attention has recently been drawn to the systematic study of suitable finite-time analogues. This note contributes to *finite-time dynamics* a brief discussion of two practical aspects of the hyperbolicity concept developed and utilised e.g. in [1, 3, 4, 6, 8].

1 Hyperbolicity is robust

Consider the nonautonomous ordinary differential equation

$$\dot{x} = f(t, x)\,, \tag{1}$$

where $f : I \times U \to \mathbb{R}^d$ is C^1, $I = [t_-, t_+]$ with $-\infty < t_- < t_+ < +\infty$, and $U \subset \mathbb{R}^d$ is a non-empty open set. The linearisation of (1) along any solution $\mu : I \to U$ is

$$\dot{y} = D_x f\big(t, \mu(t)\big) y\,. \tag{2}$$

To quantify growth and decay of solutions of (2), arbitrary inner product norms $\|\cdot\|_\Gamma = \sqrt{\langle \cdot, \Gamma \cdot \rangle}$ are considered, where $\Gamma \in \mathbb{R}^{d\times d}$ is any symmetric positive definite matrix, i.e. $\Gamma^\top = \Gamma > 0$, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^d$;

the symbol $\|\cdot\|_\Gamma$ also denotes the induced norm on $\mathbb{R}^{d\times d}$. Quantities depending on Γ have their dependence made explicit by a subscript which is suppressed only if Γ equals $\mathrm{id}_{d\times d}$, the $d\times d$ identity matrix.

To define finite-time hyperbolicity, instead of (2) consider more generally any nonautonomous *linear* equation

$$\dot{y} = A(t)y\,, \tag{3}$$

where $A : I \to \mathbb{R}^{d\times d}$ is continuous. Let $\Phi : I \times I \to \mathbb{R}^{d\times d}$ denote the associated evolution operator, i.e., $y : t \mapsto \Phi(t,s)\eta$ is, for any $\eta \in \mathbb{R}^d$, the unique solution of (3) satisfying $y(s) = \eta$. A projection-valued function $P : I \to \mathbb{R}^{d\times d}$ is an *invariant projector* for (3) if $P(t)\Phi(t,s) = \Phi(t,s)P(s)$ for all $t,s \in I$. Note that $t \mapsto P(t)$ is continuous, and $\mathrm{rk}P(t)$ is constant, for any invariant projector.

Definition 1 *Let* $\Gamma^\top = \Gamma > 0$*. Equation* (3) *is* hyperbolic (*on* I *w.r.t.* $\|\cdot\|_\Gamma$) *if there exists an invariant projector* P *for* (3)*, together with constants* $\alpha, \beta > 0$*, such that for every* $y \in \mathbb{R}^d$*,*

$$\big\|\Phi(t,s)P(s)y\big\|_\Gamma \le e^{-\alpha(t-s)}\big\|P(s)y\big\|_\Gamma\,, \quad \forall t \ge s\,, \tag{4}$$

$$\big\|\Phi(t,s)\big(\mathrm{id}_{d\times d} - P(s)\big)y\big\|_\Gamma \le e^{\beta(t-s)}\big\|\big(\mathrm{id}_{d\times d} - P(s)\big)y\big\|_\Gamma\,, \quad \forall t \le s\,. \tag{5}$$

A solution μ *of* (1) *is hyperbolic if the associated linearisation* (2) *is hyperbolic.*

The estimates in Definition 1 incorporate a finite-time variant of the classical notion of an *exponential dichotomy* that is more restrictive than the latter because an arbitrary multiplicative constant on the right-hand side of (4) or (5) would render the concept meaningless. Consequently, to establish the robustness of finite-time hyperbolicity, classical arguments using Gronwall-type estimates (see e.g. [10]) do not apply directly. Instead, the alternative argument presented in Lemma 2 below makes use of [3, Lem.9], restated here as

Proposition 1 *Equation* (3) *is hyperbolic on* I *w.r.t.* $\|\cdot\|_\Gamma$*, with invariant projector* P *and constants* $\alpha, \beta > 0$*, if and only if, for all* $t \in I$ *and* $y \in \mathbb{R}^d$*,*

$$\frac{d}{dt}\|\Phi(t,t_-)P(t_-)y\|_\Gamma \le -\alpha\|\Phi(t,t_-)P(t_-)y\|_\Gamma\,, \tag{6}$$

as well as

$$\frac{d}{dt}\big\|\Phi(t,t_-)\big(\mathrm{id}_{d\times d} - P(t_-)\big)y\big\|_\Gamma \ge \beta\big\|\Phi(t,t_-)\big(\mathrm{id}_{d\times d} - P(t_-)\big)y\big\|_\Gamma\,. \tag{7}$$

Lemma 2 *Let* $A, \widetilde{A} : I \to \mathbb{R}^{d\times d}$ *be continuous, and assume* (3) *is hyperbolic, with constants* $\alpha, \beta > 0$*. Then, given* $0 < \widetilde{\alpha} < \alpha$, $0 < \widetilde{\beta} < \beta$*, there exists* $\delta > 0$ *such that*

$$\dot{y} = \widetilde{A}(t)y \tag{8}$$

is hyperbolic as well, with constants $\widetilde{\alpha}, \widetilde{\beta}$*, whenever* $\max_{t\in I}\|\widetilde{A}(t) - A(t)\|_\Gamma < \delta$.

Proof. For every continuous $B : I \to \mathbb{R}^{d\times d}$, let $\|B\|_\infty := \max_{t\in I} \|B(t)\|_\Gamma$, and denote by Φ and $\widetilde{\Phi}$ the evolution operators associated with (3) and (8), respectively. Also, recall the trivial estimate

$$e^{-|t-s|\|A\|_\infty}\|y\|_\Gamma \le \|\Phi(t,s)y\|_\Gamma \le e^{|t-s|\|A\|_\infty}\|y\|_\Gamma\,, \quad \forall t,s\in I\,, \tag{9}$$

and note that $\widetilde{P} : t \mapsto \widetilde{\Phi}(t,t_-)P(t_-)\widetilde{\Phi}(t,t_-)^{-1}$ is an invariant projector for (8). For the latter equation, the variation of constants formula yields

$$\widetilde{\Phi}(t,t_-) - \Phi(t,t_-) = \int_{t_-}^{t} \Phi(t,\tau)\big(\widetilde{A}(\tau) - A(\tau)\big)\widetilde{\Phi}(\tau,t_-)\,\mathrm{d}\tau\,,$$

which together with (9) implies that, for all $t \in I$,

$$\begin{aligned}
\big\|\widetilde{\Phi}(t,t_-) - \Phi(t,t_-)\big\|_\Gamma &\le \int_{t_-}^{t} e^{(t-\tau)\|A\|_\infty}\|\widetilde{A}-A\|_\infty e^{(\tau-t_-)\|\widetilde{A}\|_\infty}\mathrm{d}\tau \\
&\le e^{(t-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty \int_{t_-}^{t} e^{(\tau-t_-)\|\widetilde{A}-A\|_\infty}\mathrm{d}\tau \\
&\le e^{(t-t_-)\|A\|_\infty}\left(e^{(t-t_-)\|\widetilde{A}-A\|_\infty}-1\right) \\
&\le 2(t-t_-)e^{(t-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty\,,
\end{aligned}$$

provided that $\|\widetilde{A}-A\|_\infty < \delta_1 := (t_+ - t_-)^{-1}$. Given $y \in \mathbb{R}^d$, define the two C^1-functions $\phi, \widetilde{\phi} : I \to \mathbb{R}$ as

$$\phi : t \mapsto \tfrac{1}{2}\|\Phi(t,t_-)P(t_-)y\|_\Gamma^2\,, \quad \widetilde{\phi} : t \mapsto \tfrac{1}{2}\|\widetilde{\Phi}(t,t_-)P(t_-)y\|_\Gamma^2\,.$$

For notational convenience, let $\eta = P(t_-)y$. It follows from the estimate

$$\begin{aligned}
\big|\dot{\widetilde{\phi}} - \dot{\phi}\big| &= \big|\langle\Gamma\widetilde{A}\widetilde{\Phi}\eta, \widetilde{\Phi}\eta\rangle - \langle\Gamma A\Phi\eta, \Phi\eta\rangle\big| \\
&\le \big|\langle\Gamma(\widetilde{A}-A)\widetilde{\phi}\eta, \widetilde{\phi}\eta\rangle\big| + \big|\langle\Gamma A\widetilde{\Phi}\eta, \widetilde{\Phi}\eta\rangle - \langle\Gamma A\Phi\eta, \Phi\eta\rangle\big| \\
&\le 2\|\widetilde{A}-A\|_\infty\widetilde{\phi} + \|A\|_\infty\|(\widetilde{\Phi}-\Phi)\eta\|_\Gamma(\|\widetilde{\Phi}\eta\|_\Gamma + \|\Phi\eta\|_\Gamma) \\
&\le 2\|\widetilde{A}-A\|_\infty\widetilde{\phi} + \|A\|_\infty(t_+-t_-)e^{(t_+-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty\|\eta\|_\Gamma(\sqrt{8\widetilde{\phi}}+\sqrt{8\phi}) \\
&\le 2\|\widetilde{A}-A\|_\infty\left(\widetilde{\phi} + 2\|A\|_\infty(t_+-t_-)e^{(t_+-t_-)(2\|A\|_\infty+\|\widetilde{A}-A\|_\infty)}(\widetilde{\phi}+\phi)\right)\,,
\end{aligned}$$

which is valid whenever $\|\widetilde{A}-A\|_\infty < \delta_1$, that

$$\big|\dot{\widetilde{\phi}}(t) - \dot{\phi}(t)\big| \le C\|\widetilde{A}-A\|_\infty\big(\widetilde{\phi}(t)+\phi(t)\big)\,, \quad \forall t\in I\,,$$

where C depends only on $t_+ - t_- + \|A\|_\infty$. With Proposition 1, therefore,

$$\begin{aligned}
\dot{\widetilde{\phi}} &\le \dot{\phi} + C\|\widetilde{A}-A\|_\infty(\widetilde{\phi}+\phi) \;\le\; -2\alpha\phi + C\|\widetilde{A}-A\|_\infty(\widetilde{\phi}+\phi) \\
&\le -2(\alpha - C\|\widetilde{A}-A\|_\infty)\widetilde{\phi} + (2\alpha + C\|\widetilde{A}-A\|_\infty)\big|\widetilde{\phi}-\phi\big|\,,
\end{aligned} \tag{10}$$

whenever $\|\widetilde{A} - A\|_\infty < \delta_1$. Under the latter condition, observe that also

$$\begin{aligned}
\big|\widetilde{\phi} - \phi\big| &\le \tfrac{1}{2}\|(\widetilde{\phi} - \phi)\eta\|_\Gamma(\|\widetilde{\phi}\eta\|_\Gamma + \|\phi\eta\|_\Gamma) \\
&\le (t_+ - t_-)e^{(t_+ - t_-)\|A\|_\infty}\|\widetilde{A} - A\|_\infty\|\eta\|_\Gamma\left(\sqrt{2\widetilde{\phi}} + \sqrt{2\phi}\right) \\
&\le 2(t_+ - t_-)e^{(t_+ - t_-)\|A\|_\infty}\|\widetilde{A} - A\|_\infty\left(e^{(t_+ - t_-)\|\widetilde{A}\|_\infty}\widetilde{\phi} + e^{(t_+ - t_-)\|A\|_\infty}\phi\right) \\
&\le 2(t_+ - t_-)e^{1 + 2(t_+ - t_-)\|A\|_\infty}\|\widetilde{A} - A\|_\infty(\widetilde{\phi} + \phi) \\
&\le 2C\|\widetilde{A} - A\|_\infty\widetilde{\phi} + C\|\widetilde{A} - A\|_\infty\big|\widetilde{\phi} - \phi\big| \, ,
\end{aligned}$$

which in turn implies that

$$\big|\widetilde{\phi}(t) - \phi(t)\big| \le 4C\|\widetilde{A} - A\|_\infty\widetilde{\phi}(t) \, , \quad \forall t \in I \, , \tag{11}$$

provided that $\|\widetilde{A} - A\|_\infty < \delta_2 := (2C)^{-1} < \delta_1$. Combining (10) and (11) yields

$$\dot{\widetilde{\phi}}(t) \le -2\big(\alpha - 2C(1 + 2\alpha)\|\widetilde{A} - A\|_\infty\big)\widetilde{\phi}(t) \, , \quad \forall t \in I \, ,$$

whenever $\|\widetilde{A} - A\|_\infty < \delta_2$. With $\delta := \dfrac{\min(1, \alpha - \widetilde{\alpha})}{2C(1 + 2\alpha)} > 0$ therefore $\|\widetilde{A} - A\|_\infty < \delta$ implies that $\dot{\widetilde{\phi}}(t) \le -2\widetilde{\alpha}\widetilde{\phi}(t)$ for all $t \in I$. This establishes $(\widetilde{6})$. A completely analogous argument proves $(\widetilde{7})$. Overall, $\|\widetilde{A} - A\|_\infty < \delta$ ensures that $(\widetilde{8})$ is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$, with invariant projector $\widetilde{P}$ and constants $\widetilde{\alpha}, \widetilde{\beta}$. □

Remark 1 (i) Note that δ in Lemma 2 depends only on $\alpha - \widetilde{\alpha}$, $\beta - \widetilde{\beta}$, and $t_+ - t_- + \|A\|_\infty$. Usually, it is not possible to choose $\widetilde{\alpha} = \alpha$ or $\widetilde{\beta} = \beta$, not even if (3) and (8) are autonomous.

(ii) It was shown in [3, Exp.24] that, perhaps somewhat surprisingly,

$$\dot{y} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y$$

is hyperbolic for every I and Γ. Thus, by Lemma 2,

$$\dot{y} = \begin{bmatrix} a_1 & 1 \\ a_2 & a_3 \end{bmatrix} y \tag{12}$$

is hyperbolic as well, provided that $\max_{t\in I}\sum_{i=1}^3 |a_i(t)|$ is sufficiently small for the continuous functions $a_1, a_2, a_3 : I \to \mathbb{R}$. If so, even though the (possibly time-dependent) eigenvalues of (12) may be both positive or negative, the rank of any invariant projector according to Definition 1 equals *one.*

The desired robustness result is an immediate consequence of Lemma 2. It asserts that hyperbolicity according to Definition 1 is robust under variations of the initial data and C^1-small perturbations of the right-hand side in (1).

Theorem 3 *Assume the solution μ of* (1) *is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$. Then there exists $\delta > 0$ such that for every C^1-function $\widetilde{f} : I \times U \to \mathbb{R}^d$ with*

$$\sup_{t\in I}\Big(\big\|\widetilde{f}\big(t,\mu(t)\big) - f\big(t,\mu(t)\big)\big\|_\Gamma + \big\|D_x\widetilde{f}\big(t,\mu(t)\big) - D_x f\big(t,\mu(t)\big)\big\|_\Gamma\Big) < \delta\,, \quad (13)$$

every solution $\widetilde{\mu} : I \to U$ of

$$\dot{x} = \widetilde{f}(t,x) \quad (14)$$

is hyperbolic as well, provided that $\|\widetilde{\mu}(t_0) - \mu(t_0)\|_\Gamma < \delta$ for some $t_0 \in I$.

Proof. Given $\varepsilon > 0$, choose $\delta_1 > 0$ so small that

$$T_{\delta_1} := \big\{(t,x) : t \in I, \|x - \mu(t)\|_\Gamma \le \delta_1\big\} \subset I \times U$$

and $\|D_x\widetilde{f}(t,x) - D_x\widetilde{f}(t,y)\|_\Gamma < \frac{1}{2}\varepsilon$ whenever $x, y \in T_{\delta_1}$ and $\|x-y\|_\Gamma < \delta_1$. Also, pick $\delta_2 > 0$ small enough to ensure that $\max_{t\in I}\|\widetilde{f}\big(t,\mu(t)\big) - f\big(t,\mu(t)\big)\|_\Gamma < \delta_2$ and $\|x_0 - \mu(t_0)\| < \delta_2$ for some $t_0 \in I$ imply that the solution of (14) with $x(t_0) = x_0$ exists for all $t \in I$ and satisfies $\max_{t\in I}\|x(t) - \mu(t)\|_\Gamma < \delta_1$. With $\delta := \min(\frac{1}{2}\varepsilon, \delta_1, \delta_2)$, it follows from (13) that

$$\begin{aligned}
&\big\|D_x\widetilde{f}\big(t,\widetilde{\mu}(t)\big) - D_x f\big(t,\mu(t)\big)\big\|_\Gamma \\
&\quad\le \big\|D_x\widetilde{f}\big(t,\widetilde{\mu}(t)\big) - D_x\widetilde{f}\big(t,\mu(t)\big)\big\|_\Gamma + \big\|D_x\widetilde{f}\big(t,\mu(t)\big) - D_x f\big(t,\mu(t)\big)\big\|_\Gamma \\
&\quad\le \tfrac{1}{2}\varepsilon + \delta < \varepsilon\,,
\end{aligned}$$

if only $\|\widetilde{\mu}(t_0) - \mu(t_0)\|_\Gamma < \delta$ for some $t_0 \in I$. Since $\varepsilon > 0$ was arbitrary, Lemma 2 applies with $A(t) = D_x f\big(t,\mu(t)\big)$ and $\widetilde{A}(t) = D_x\widetilde{f}\big(t,\widetilde{\mu}(t)\big)$. □

2 How (not) to detect hyperbolicity

If the system (3) is *autonomous*, then it has a (classical) exponential dichotomy if and only if no eigenvalue of A lies on the imaginary axis. It thus seems natural to use eigenvalues as a tool to detect hyperbolicity: If the eigenvalues and -vectors vary sufficiently little over time then, hopefully, some insight concerning finite-time behaviour can be gained from them. In this spirit and for $d = 2$ and $\Gamma = \mathrm{id}_{2\times 2}$, [6, Thm.1] and [9, Thm.1] present conditions on the spectral data of A that ensure finite-time hyperbolicity.

Relying on spectral data in a finite-time context does have its pitfalls, though. This fact, already hinted at by Remark 1(ii), is elucidated further through the following simple example which is phrased in the terminology of [7] so as to make it directly accessible to readers of that paper. Specifically, a family $\mathcal{L} = \{\mathcal{L}_t : t \in I\}$ of C^1-curves $\mathcal{L}_t : \mathbb{R} \to \mathbb{R}^d$ is referred to as a *material line* of (1) if it is invariant in the sense that, for any $s, t \in I$,

$$x_0 \in \mathcal{L}_s(\mathbb{R}) \quad \text{if and only if} \quad x(t; s, x_0) \in \mathcal{L}_t(\mathbb{R})\,;$$

here $x(\cdot; s, x_0)$ denotes the unique solution of (1) with $x(s) = x_0$. The obvious fluid dynamical interpretation is that, at each time t, the set $\mathcal{L}_t(\mathbb{R})$ represents a smooth curve of fluid particles advected by the velocity field f. A material line $\mathcal{L}$ is *attracting* if for every solution μ of (1) with $\mu(t) \in \mathcal{L}_t(\mathbb{R})$ for some (and hence every) $t \in I$, there exists $\alpha > 0$ and a smooth family X of $(d-1)$-dimensional subspaces, invariant under the linearisation (2) along μ, i.e. $\Phi(t,s)X(s) = X(t)$ for all $s, t \in I$, such that $X(t)$ is, for every $t \in I$, transversal to $T_{\mu(t)}\mathcal{L}_t(\mathbb{R})$, and

$$\|\Phi(t,s)x\| \le e^{-\alpha(t-s)}\|x\|, \quad \forall t \ge s, x \in X(s). \tag{15}$$

For any $\kappa > 0$, consider now the autonomous linear equation

$$\dot{x} = \begin{bmatrix} -1 & 6 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & \kappa \end{bmatrix} x. \tag{16}$$

Since the (x_1, x_2)-plane and the x_3-axis are both invariant under the flow generated by (16), corresponding respectively to two negative and one positive eigenvalue, it seems plausible that e.g. the x_3-axis is an attracting material line. In fact, Case 1 of [7, Thm.1], asserts that *every* solution of (16) is contained in an attracting material line, and hence (16) allows for *many* attracting material lines. Plausible though this may be, it is actually not true:

Claim 4 *No material line of* (16) *is attracting.*

To verify this claim, suppose $\mathcal{L}$ was an attracting material line of (16) and μ a solution in $\mathcal{L}$. Denote by $G_{2,3}$ the set of all two-dimensional subspaces of $\mathbb{R}^3$. It follows from (15) that $\frac{d}{dt}\frac{1}{2}\|\Phi(t,s)x\|^2\big|_{t=s} \le -\alpha\|x\|^2$ for all $x \in X(s)$, where $X(s) \in G_{2,3}$ is transversal to $T_{\mu(s)}\mathcal{L}_s(\mathbb{R})$. Note that $\frac{d}{dt}\frac{1}{2}\|\Phi(t,s)x\|^2\big|_{t=s} = \langle Cx, x\rangle$ with the symmetric matrix

$$C = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -7 & 0 \\ 0 & 0 & \kappa \end{bmatrix}.$$

Thus Claim 4 will follow immediately once it is demonstrated that

$$\max_{x \in X, \|x\|=1} \langle Cx, x\rangle \ge 0, \quad \forall X \in G_{2,3}. \tag{17}$$

To prove (17), first recall the following elementary fact from linear algebra.

Proposition 5 *Let $X \neq \{0\}$ be a subspace of $\mathbb{R}^d$, and $C, D \in \mathbb{R}^{d\times d}$ symmetric matrices with $D > 0$. Then $\{\langle Cx, x\rangle : x \in X, \langle Dx, x\rangle = 1\} = [\rho_-, \rho_+]$, where ρ_+ and ρ_- denote, respectively, the largest and smallest eigenvalue of $[\langle Cb_i, b_j\rangle][\langle Db_i, b_j\rangle]^{-1} \in \mathbb{R}^{l\times l}$, and $\{b_1, \ldots, b_l\}$ is any basis of X.*

Denote by $X_{\vartheta,\varphi} \subset \mathbb{R}^3$ the two-dimensional space

$$X_{\vartheta,\varphi} = \begin{bmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ \sin\vartheta \end{bmatrix}^{\perp}, \quad 0 \le \vartheta \le \tfrac{1}{2}\pi,\ 0 \le \varphi \le 2\pi;$$

every $X \in G_{2,3}$ equals $X_{\vartheta,\varphi}$ for the appropriate ϑ, φ. To apply Proposition 5 with $D = \mathrm{id}_{3\times 3}$ and $X = X_{\vartheta,\varphi}$, deduce from a straightforward computation that $[\langle Cb_i, b_j\rangle][\langle Db_i, b_j\rangle]^{-1}$ is similar to $\kappa\,\mathrm{id}_{2\times 2} + E_1E_2$, where

$$E_1 = -\kappa\,\mathrm{id}_{2\times 2} + \begin{bmatrix} -1 & 3 \\ 3 & -7 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1-\cos^2\vartheta\cos^2\varphi & -\cos^2\vartheta\cos\varphi\sin\varphi \\ -\cos^2\vartheta\cos\varphi\sin\varphi & 1-\cos^2\vartheta\sin^2\varphi \end{bmatrix}.$$

It follows that the maximum of $\{\langle Cx, x\rangle : x \in X_{\vartheta,\varphi}, \|x\| = 1\}$ is $\kappa + \tau$, with τ denoting the largest zero of the quadratic function

$$p_{\vartheta,\varphi} : t \mapsto t^2 + \Big(2(\kappa+4) - \cos^2\vartheta\big(\kappa+4-3\sqrt{2}\sin(2\varphi+\tfrac{1}{4}\pi)\big)\Big)t + (\kappa^2+8\kappa-2)\sin^2\vartheta\,.$$

If $0 < \kappa \le 3\sqrt{2}-4$ then $p_{\vartheta,\varphi}(0) \le 0$ and hence $\tau \ge 0$. On the other hand,

$$p_{\vartheta,\varphi}(3\sqrt{2}-4-\kappa) = 3\sqrt{2}(3\sqrt{2}-4-\kappa)\cos^2\vartheta\big(1+\sin(2\varphi+\tfrac{1}{4}\pi)\big) \le 0$$

whenever $\kappa > 3\sqrt{2}-4$, so that $\kappa + \tau \ge 3\sqrt{2}-4$ in this case. Overall therefore

$$\max_{x\in X, \|x\|=1}\langle Cx, x\rangle \ge \min(\kappa, 3\sqrt{2}-4) > 0\,, \quad \forall X \in G_{2,3}\,.$$

Clearly, this strengthened form of (17) proves Claim 4.

Remark 2 (i) A straightforward computation confirms that (16) is hyperbolic w.r.t. $\|\cdot\|$ if and only if $t_+ - t_- < \frac{1}{6}\log\frac{9+4\sqrt{2}}{7} \approx 0.1232$. In this case, the rank of any invariant projector for (16) according to Definition 1 equals *one*, and not *two* as might be expected.

(ii) If A is constant and has no eigenvalue on the imaginary axis, then there always exist uncountably many $\Gamma = \Gamma^\top > 0$ such that (3) is hyperbolic w.r.t. $\|\cdot\|_\Gamma$ on *every* compact interval I, see [1, Rem.2] and [2, Thm.2.9]. For example, (16) is hyperbolic on every I w.r.t. $\|\cdot\|_\Gamma$, where

$$\Gamma = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, if the definition of attractivity is adapted in that $\|\cdot\|$ in (15) is replaced by $\|\cdot\|_\Gamma$, then every trajectory of (16) is indeed contained in an attracting material line. Not restricting oneself to the Euclidean norm may thus be beneficial even in the most elementary of circumstances.

(ii) The reader may wonder exactly which part of the alleged proof of [7, Thm.1] is problematic. The answer is simple: As the above example shows, linear changes of coordinates do generally not preserve finite-time hyperbolicity, not even if they are time-*independent*. Concretely, $x = My$ with the appropriate non-singular matrix M transforms (16) into $\dot{y} = \mathrm{diag}\,[-1, -7, \kappa]$, for which e.g. every trajectory not contained in the (y_1, y_2)-plane, and hence in particular the y_3-axis is an attracting material line.

(iv) The usage of time-dependent spectral data to detect finite-time hyperbolicity can be avoided altogether. Based on a *dynamic partition* of the extended phase space, [1, Cor.9] presents a neat condition guaranteeing that a solution μ of (1) is hyperbolic. The dynamic partition does not involve eigenvalues or -vectors but rather utilises a classification of the points in $I \times U$ according to their qualitative instantaneous behaviour. The interested reader may want to consult [1, 2, 5, 6, 8] where aspects of this useful concept are developed in detail.

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RECENT RESULTS ON NON–AUTONOMOUS DISCRETE SYSTEMS

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Abstract

The aim of this paper is to present some recent results concerning non–autonomous discrete systems (non–autonomous order one difference equations) and state some problems that arise in this field.

Key words: *Non–autonomous discrete systems, non–autonomous difference equations, dynamical systems, Li–Yorke chaos.*
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1 Introduction

Let X be a compact metric space and consider a sequence of continuous maps $f_n : X \to X$, $n \in \mathbb{N}$, denoted by $f_{1,\infty} = (f_n)_{n=1}^{\infty}$. This sequence defines a non–autonomous discrete system $(X, f_{1,\infty})$. The orbit of any $x \in X$ is given by the sequence $(f_1^n(x)) = \text{Orb}(x, f_{1,\infty})$, where $f_1^n = f_n \circ ... \circ f_1$ for $n \geq 1$, and f_1^0 is the identity map. If $f_n = f$ for any $n \in \mathbb{N}$, then the pair (X, f) is an autonomous discrete dynamical system. We point out that all definitions that we will introduce in this paper for the non–autonomous case have an autonomous well–known equivalent definition in the setting of discrete dynamical systems.

Non–autonomous discrete systems where introduced in [20], although they also have appeared connected to some non–autonomous difference equations (see [14] or [15]). Note that the orbit of $x \in X$ is given by the solution of the non–autonomous difference equation

$$\begin{cases} x_{n+1} = f_n(x_n), \\ x_1 = x. \end{cases}$$

It is obvious that, if we do not add any condition on the sequence $f_{1,\infty}$, in general we cannot characterize the behavior of the orbits of the system. So, we are going to present some particular cases of non–autonomous discrete systems for which something can be said on this behavior.

The set of limit points of an orbit $\mathrm{Orb}(x, f_{1,\infty})$ is the ω–limit set of x, which is denoted by $\omega(x, f_{1,\infty})$. A point $x \in X$ is said to be recurrent if $x \in \omega(x, f_{1,\infty})$. We denote by $\Lambda(f_{1,\infty}) = \cup_{x\in X}\omega(x, f_{1,\infty})$. Finally, $x \in X$ is a non–wandering point if for any open neighborhood U of x, there is a positive integer n such that $f_1^n(U) \cap U \neq \emptyset$. Note that

$$\mathrm{R}(f_{1,\infty}) \subseteq \omega(f_{1,\infty}) \subseteq \Omega(f_{1,\infty}),$$

where $\mathrm{R}(f_{1,\infty})$ and $\Omega(f_{1,\infty})$ denote the sets of recurrent and non–wandering points, respectively.

The paper is organized as follows. In the next section we analyze some results concerning periodic sequences of maps. Later on, we study dynamic properties of sequences which converge uniformly to continuous maps. Finally, in the last section we show some results concerning the dynamical complexity of the last class of sequences of maps.

2 Periodic sequences

Let us assume the existence of a minimal positive integer k such that $f_{n+k} = f_n$ for all $n \geq 1$, and therefore, the sequence $f_{1,\infty}$ is periodic. The interest for such sequences comes from biological and economical sciences. Let us emphasize that some scientists working on population dynamics use such kind of systems to model the population growth of species under some periodic changes in the environment (see e.g. [9] and [15]). On the other hand, periodic sequences of period $k = 2$ are deeply connected to duopoly models. A duopoly is a market in which two firms produce the same or equivalent goods. Hence, the future production are given by the so–called reaction functions f_i, $i = 1, 2$. In some cases, such reaction functions have an one dimensional domain and the productions are given by the systems $(f_1, f_2, f_1, f_2, ...)$ and $(f_2, f_1, f_2, f_1, ...)$, respectively (see e.g. [21], [22] and [24]).

It is just simple to prove that for any $x \in X$,

$$\begin{aligned}\omega(x, f_{1,\infty}) &= \omega(x, f_k \circ ... \circ f_1) \cup \omega(f_1(x), f_1 \circ f_k \circ ... \circ f_2) \\ &\quad \cup ... \cup \omega(f_1^{k-1}(x), f_{k-1} \circ ... \circ f_1 \circ f_k),\end{aligned}$$

and hence one can wonder whether the behavior of $f_{1,\infty}$ can be deduced from the behavior of the compositions $f_k \circ ... \circ f_1$, $f_1 \circ f_k \circ ... \circ f_2$, ... and $f_{k-1} \circ ... f_1 \circ f_k$. This idea produces others positive results. For instance, in [20] it is proved that

$$h(f_{1,\infty}) = \frac{1}{k} h(f_k \circ ... \circ f_1) = ... = \frac{1}{k} h(f_{k-1} \circ ... \circ f_1 \circ f_k),$$

where $h(f_{1,\infty})$ is the topological entropy of $f_{1,\infty}$, which is a measure of the dynamical complexity of a system. Additionally, in [10], the characterization of metric attractors (roughly speaking sets which are ω–limit sets of almost all

$x \in [0,1]$) of periodic of order two where the maps have negative Schwarzian derivative[1] have been investigated. The same authors investigate in [11] the Pitchfork bifurcation in these kind of systems. Finally, in [1] a characterization of periodic solutions of one dimensional non–autonomous difference equations has been found in terms of the Sharkovsky's result for one dimensional maps (see e.g. [26] and for a simple proof see [12]).

However, we must point out that the above compositions may need not have the same dynamic properties (see e.g. [4] or [8]). Additionally, some dynamic properties of a sequence $f_{1,\infty}$ cannot be obtained from the above compositions; for instance, the existence of periodic orbits of odd period of a non–autonomous system $f_{1,\infty} = (f_1, f_2, f_1, f_2, ...)$ defined on the unit interval $[0,1]$ cannot be deduced from the periodic orbits of $f_1 \circ f_2$ and $f_2 \circ f_1$ (see [7]).

Even when the compositions of maps $f_1, ..., f_k$ can describe a dynamical property of $f_{1,\infty}$, sometimes it is not good in practice. For instance, it is well–known that the logistic family $f_\mu(x) = \mu x(1-x)$, $x \in [0,1]$ and $\mu \in [1,4]$, has a fixed point which is an attractor for all orbits in $(0,1)$ in the case that $1 \leq \mu \leq 3$. It is an open question (see [14]) to check the conditions of the parameters μ_i, $i = 1, ..., k$, such that the periodic sequence $(f_{\mu_1}, ..., f_{\mu_k}, ...)$ has a periodic orbit of period k which also is an attractor for all trajectories in $(0,1)$. Of course, this periodic orbit $(x_1, x_2, ..., x_k, ...)$ has to satisfy

$$|f'_{\mu_1}(x_1) f'_{\mu_2}(x_2) ... f'_{\mu_k}(x_k)| < 1,$$

but the family of parameters which makes possible the above equality is very difficult to characterize in practice.

A similar problem can be found in [23], where an economic model is presented. This model is given by a periodic sequence $(f_1, ..., f_k, ...)$ and all maps have the same fixed point, which is usually called the Cournot equilibrium. The local stability of this point is very important in the microeconomic theory. The local equilibrium $\mathbf{x}_0 \in \mathbb{R}^n$ will be stable provided the Jacobian matrix

$$\mathbf{J}(f_k \circ ... \circ f_1)(\mathbf{x}_0) = \mathbf{J}(f_k)(\mathbf{x}_0) \cdot \mathbf{J}(f_{k-1})(\mathbf{x}_0) \cdot ... \cdot \mathbf{J}(f_1)(\mathbf{x}_0)$$

has spectral radius smaller than one. But in practice this is impossible to verify from $\mathbf{J}(f_i)(\mathbf{x}_0)$, $i = 1, ..., k$, because these matrices have spectral radius at least one. So, checking the stability of the Cournot equilibrium is a difficult technical problem.

3 Convergent sequences

Now, assume that the sequence $f_{1,\infty}$ converges uniformly to a continuous map f. In general, it is not true that an ω–limit set $\omega(x, f_{1,\infty})$ is also an ω–limit set of f; for example, from [16] can be constructed a sequence $f_{1,\infty}$, which converges to the identity on $[0,1]$, and such that there are $x \in [0,1]$ with the property that $\omega(x, f_{1,\infty}) = [0,1]$, while any ω–limit set of the limit map is a single point.

[1] The Schwarzian derivative of a good enough map f is $Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$.

However, we can introduce some results which characterize these ω–limit sets. The next one can be seen in [19].

Theorem 1 *Let $f_n : X \to X$ be continuous such that f_n converges uniformly to f. Then:*

(a) For any $x \in X$, the set $\omega(x, f_{1,\infty})$ is compact and strongly invariant by f [$f(\omega(x, f_{1,\infty})) = \omega(x, f_{1,\infty})$].

(b) Let $X = [0,1]$ and assume that every periodic orbit of f is a fixed point. Then for any $x \in I$, $\omega(x, f_{1,\infty}) = [a,b] \subset \mathrm{F}(f)$, $0 \le a \le b \le 1$, , where $\mathrm{F}(f)$ denotes the set of fixed points of f.

If we consider additional properties for the limit map f, we can improve the last result. Let $\delta > 0$. A sequence x_n is a δ–pseudo orbit of f if $d(x_{n+1}, f(x_n)) < \delta$ for $n \ge 1$. Given $\varepsilon > 0$, we say that $\mathrm{Orb}(x, f)$ ε–shadows x_n if $d(x_n, f^n(x)) < \varepsilon$ for $n \ge 1$. The map f has the shadowing property if for any $\varepsilon > 0$ there is $\delta > 0$ such that any δ–pseudo orbit is ε–shadowed by an orbit of f (see [2] or [17]). The map f has the limit shadowing property (see [13]) if $\lim_{n\to\infty} d(x_{n+1}, f(x_n)) = 0$, which implies that there is $x \in X$ such that $\lim_{n\to\infty} d(x_n, f^n(x)) = 0$. With this notation, the following results from [5] can be understood.

Theorem 2 *Assume $f_{1,\infty} = (f_n)$, $f_n : X \to X$ continuous for all n, converges uniformly to f which has the shadowing property. Then any limit point of any trajectory of $f_{1,\infty}$ is in $\Omega(f)$. In addition, if f has the limit shadowing property, then for any $x \in X$, there is $z \in X$ such that $\omega(x, f_{1,\infty}) = \omega(z, f)$.*

In the interval case, we can state a stronger result.

Theorem 3 *Assume $f_{1,\infty} = (f_n)$ is a sequence on continuous interval maps converging uniformly to f, which has the shadowing property. Then for any $x \in [0,1]$ there is $z \in [0,1]$ such that $\omega(x, f_{1,\infty}) = \omega(z, f)$.*

It is unclear whether Theorem 3 holds when the phase space is not the compact interval. We conjecture that it remains true for continuous tree and graph maps, but it is false in general for maps defined on two–dimensional spaces. Finally, it is also interesting to investigate what conditions are necessary to get recurrence, that is, when is non–empty $\mathrm{R}(f_{1,\infty})$?

4 Chaos and related notions

In the seminal paper [20] the following result concerning the topological entropy $h(f_{1,\infty})$ can be found.

Theorem 4 *Let $f_n : X \to X$ be continuous such that f_n converges uniformly to f. Then:*

$$h(f_{1,\infty}) \le h(f).$$

Since the topological entropy of a map is a measure of the dynamical complexity of such map (see e.g. [3]), the above result suggests the general idea that if f is simple then $f_{1,\infty}$ is also simple, and the complexity of $f_{1,\infty}$ will imply the complexity of f. As we will see, this idea is false for Li–Yorke chaos.

One of the most well–known definition of chaos in discrete dynamical systems is due to Li and Yorke (see [25]). A non–autonomous discrete system $f_{1,\infty}$ is said to be chaotic in the sense of Li–Yorke if there is an uncountable subset $S \subset I$ such that for any $x, y \in S$, $x \neq y$, it is held that

$$\liminf_{n\to\infty} |f_1^n(x) - f_1^n(y)| = 0$$

and

$$\limsup_{n\to\infty} |f_1^n(x) - f_1^n(y)| > 0.$$

The set S is called a scrambled set of $f_{1,\infty}$. Note that when $f_n = f$, this definition agrees with the classical definition of Li–Yorke in the case of discrete dynamical systems.

Let us start with the negative results. It can be deduced from [16] the existence of a sequence $f_{1,\infty}$ chaotic in the sense of Li and Yorke, which converges uniformly to the identity. That is, the limit map is simple while the non–autonomous system $f_{1,\infty}$ is complicated.

Continuous interval maps which are not chaotic in the sense of Li–Yorke are in fact dynamically simple. Recall that $x \in I$ is periodic if there is $k \in \mathbb{N}$ such that $f^k(x) = x$. We say that an orbit $\mathrm{Orb}(x, f)$ is approximated by periodic orbits if for any $\epsilon > 0$ there are $n_0 \in \mathbb{N}$ and a periodic point x_0 such that $|f^n(x) - f^n(x_0)| < \epsilon$ for all $n \geq n_0$. Then it is proved in [18] and [27] that an interval map is either Li–Yorke chaotic or any orbit is approximated by periodic orbits. We can refer the complexity of $f_{1,\infty}$ to the complexity of f as follows (see [6]).

Theorem 5 *Let $f_{1,\infty}$ be a sequence of surjective continuous interval maps converging to a map f.*

(a) If the map f has positive topological entropy, then $f_{1,\infty}$ is Li–Yorke chaotic.

(b) If the map f has the shadowing property, then $f_{1,\infty}$ is Li–Yorke chaotic if and only if f is Li–Yorke chaotic.

Note that Li and Yorke chaotic maps with zero topological entropy have not the shadowing property (see [17]). So, does such kind of maps satisfy Theorem 5? More precisely, if $f_{1,\infty}$ converges to a chaotic map f with zero topological entropy, is $f_{1,\infty}$ also chaotic in the Li–Yorke sense?

With the shadowing property hypothesis, we can state the following, which is a dichotomy between simplicity and complexity for this kind of sequences.

Theorem 6 *Assume that the sequence of surjective continuous interval maps $f_{1,\infty}$ converges uniformly to a map f which has the shadowing property. Then $f_{1,\infty}$ is either Li–Yorke chaotic or for any $x \in [0,1]$ and $\epsilon > 0$ there is a periodic point y of f, and $n_0 \in \mathbb{N}$ such that $|f_1^n(x) - f_1^n(y)| < \epsilon$ for any $n \geq n_0$.*

Surjectivity condition in Theorems 5 and 6 is an easy way to avoid the existence of the next critical example. Consider the sequence $f_{1,\infty} = (g, f, f, f, ...)$ where $g(x) = 0$ and $f(x) = 4x(1-x)$ for $x \in [0,1]$. Then $f_{1,\infty}$ converges uniformly to f, which is a chaotic map, but all the orbits of $f_{1,\infty}$ are eventually constant to 0. It is an open question to find another non–drastic conditions that guarantee the validity of such results.

The number of non-equivalent definitions of chaos and simplicity for discrete dynamical systems is huge (see for example several of them in [3]). So, it is a natural question to wonder when similar results to Theorems 5 and 6 are true for these chaos definitions. Additionally, it is also natural to wonder about these question when the sequence $f_{1,\infty}$ does not converge uniformly to any continuous limit map f.

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STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH ADDITIVE NOISE ON TIME-VARYING DOMAINS

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Abstract

Stochastic partial differential equations (SPDE) with additive noise of the reaction-diffusion type are formulated on time-varying domains, where the domains are obtained by a regular temporally dependent spatially diffeomorphic transformation of a reference domain, which is bounded and has a smooth boundary. The main issue considered here is the interpretation of the notions of noise and solution on time-varying domains.

Key words: *Stochastic partial differential equations, time-varying domains, additive noise*

AMS subject classifications: *35K20, 35R60, 60H15*

1 Introduction

Deterministic partial differential equations on cylindrical or time-varying domains have attracted considerable attention, with the focus mainly on existence and regularity issues, see [1, 7, 8, 12]. Only very recently have other issues such as controllability and the existence of attractors been investigated, see [4, 5, 6].

The theory of stochastic partial differential equations is now well established [2, 11], but to our knowledge, always uses a fixed spatial domain. Here we consider a class of such equations, with additive noise, on time-varying domains which are obtained by a regular temporally dependent spatially diffeomorphic transformation of a reference domain, which is bounded and has a smooth boundary. This requires an appropriate interpretation of the notions of noise and solution, which are given here.

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2 Parabolic PDE on time-varying domains

Let $\mathcal{O}$ be a nonempty bounded open subset of $\mathbb{R}^N$ with C^2 boundary $\partial\mathcal{O}$, and $r = r(y,t)$ a vector function

$$r \in C^1(\overline{\mathcal{O}} \times [0,\infty); \mathbb{R}^N), \tag{1}$$

such that

$$r(\cdot,t) : \mathcal{O} \to \mathcal{O}_t := r(\mathcal{O},t) \quad \text{is a } C^2\text{-diffeomorphism for all } t \in [0,\infty). \tag{2}$$

We define

$$Q := \bigcup_{t\in(0,+\infty)} \mathcal{O}_t \times \{t\},$$

$$\Sigma := \bigcup_{t\in(0,+\infty)} \partial\mathcal{O}_t \times \{t\}.$$

The set Q is an open subset of $\mathbb{R}^{N+1}$, with boundary

$$\partial Q = \Sigma \cup (\mathcal{O}_0 \times \{0\}).$$

We will also assume that the function $\bar{r} = \bar{r}(x,t)$, where $\bar{r}(\cdot,t) = r^{-1}(\cdot,t)$ denotes the inverse of $r(\cdot,t)$, satisfies

$$\bar{r} \in C^{2,1}(\bar{Q}; \mathbb{R}^N), \tag{3}$$

i.e., $\bar{r}$, $\dfrac{\partial \bar{r}}{\partial t}$, $\dfrac{\partial \bar{r}}{\partial x_i}$ and $\dfrac{\partial^2 \bar{r}}{\partial x_i \partial x_j}$ belong to $C(\bar{Q};\mathbb{R}^N)$, for all $1 \leqslant i,j \leqslant N$.

We consider the following initial boundary value problem for a nonlinear parabolic partial differential equation of reaction-diffusion type with homogeneous Dirichlet boundary condition,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(u) = 0 \quad \text{in } Q, \\ u = 0 \ \text{ on } \Sigma, \\ u(x,0) = u_0(x), \quad x \in \mathcal{O}_0, \end{cases} \tag{4}$$

where $u_0 : \mathcal{O}_0 \to \mathbb{R}$ and $f \in C^1(\mathbb{R})$ are given. We will assume that f satisfies that there exist nonnegative constants α_1, α_2, β and l, and $p \geqslant 2$, such that

$$-\beta + \alpha_1 |s|^p \leqslant f(s)s \leqslant \beta + \alpha_2 |s|^p \quad \forall\, s \in \mathbb{R} \tag{5}$$

and

$$f'(s) \geqslant -l \quad \forall\, s \in \mathbb{R}. \tag{6}$$

.

Following [6], we set

$$v(y,t) = u(r(y,t),t) \quad \text{for } y \in \mathcal{O}, t \geq 0,$$

or, equivalently,

$$u(x,t) = v(\bar{r}(x,t),t) \quad \text{for } x \in \mathcal{O}_t, t \geq 0.$$

Then, the PDE (4) can be transformed to (see also [5])

$$\begin{cases} \dfrac{\partial v}{\partial t}(y,t) - \sum\limits_{k,j=1}^{N} \dfrac{\partial v}{\partial y_j}(a_{jk}(y,t)\dfrac{\partial v}{\partial y_k}(y,t)) + b(y,t)\cdot\nabla_y v(y,t) + f(v(y,t)) = 0 \\ \text{in } \mathcal{O}\times(0,\infty), \\ v = 0 \ \text{ on } \partial\mathcal{O}\times(0,\infty), \\ v(y,0) = u_0(r(y,0)), \quad y \in \mathcal{O}, \end{cases} \tag{7}$$

where $\cdot$ denotes the inner product of $\mathbb{R}^N$,

$$a_{jk}(y,t) = \sum_{i=1}^{N} \frac{\partial \bar{r}_k}{\partial x_i}(r(y,t),t)\frac{\partial \bar{r}_j}{\partial x_i}(r(y,t),t), \quad j,k = 1,\cdots,N;$$

and $b(y,t) = (b_1(y,t),\cdots,b_N(y,t)) \in \mathbb{R}^N$ is defined by

$$b_k(y,t) = \frac{\partial \bar{r}_k}{\partial t}(r(y,t),t) - \Delta_x \bar{r}_k(r(y,t),t) + \sum_{j=1}^{N} \frac{\partial a_{jk}}{\partial y_j}(y,t), \quad k = 1,2,\cdots,N.$$

The proof of the following result can be seen in [5].

Lemma 1 *For any* $0 < T < \infty$, $a_{jk} \in C^1(\overline{\mathcal{O}}\times[0,T])$, $b_k \in C^0(\overline{\mathcal{O}}\times[0,T])$. *In particular,* $a_{jk}, \dfrac{\partial a_{jk}}{\partial y_j}, b_k \in L^\infty(\mathcal{O}\times(0,T)), \quad j,k = 1,2,\cdots,N.$

Moreover, there exists a $\delta = \delta(r,T) > 0$ *such that for any* $(y,t) \in \mathcal{O}\times[0,T]$,

$$\sum_{j,k=1}^{N} a_{jk}(y,t)\xi_j\xi_k \geqslant \delta|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^N.$$

The existence and uniqueness solutions of the above PDE were established in [5], see also [4].

3 A stochastic process on variable domains

Consider fixed a probability space $(\Omega,\mathcal{F},P)$ and a sequence $\{\beta_j(t) : t \geq 0\}_{j\geq 1}$ of mutually independent normalized real Wiener processes defined on it.

Let $\{\phi_j\}_{j\geq 1} \subset L_2(\mathcal{O})$ be a sequence of functions such that

$$\sum_{j=1}^{\infty} \|\phi_j\|^2_{L^2(\mathcal{O})} < \infty, \tag{8}$$

and define

$$\psi_j(x,t) := \phi_j(\bar{r}(x,t)) \quad x \in \mathcal{O}_t, \quad t \in [0,\infty), \quad j = 1,2,.... \tag{9}$$

Observe that

$$\begin{aligned} \|\psi_j(t)\|^2_{L^2(\mathcal{O}_t)} &= \int_{\mathcal{O}_t} (\phi_j(\bar{r}(x,t)))^2\, dx \\ &= \int_{\mathcal{O}} (\phi_j(y))^2 \mathrm{Jac}(r,y,t)\, dy \\ &\leq C_{r,t} \|\phi_j\|^2_{L^2(\mathcal{O})} \end{aligned} \tag{10}$$

for any $t \in [0,\infty)$, where we have denoted $\mathrm{Jac}(r,y,t)$ the absolute value of the determinant of the Jacobi matrix $\left(\frac{\partial r_i}{\partial y_j}(y,t)\right)_{N\times N}$, and

$$C_{r,t} := \max_{y\in\overline{\mathcal{O}}} \mathrm{Jac}(r,y,t).$$

We consider the process

$$M(t) := \sum_{j=1}^{\infty} \beta_j(t)\psi_j(t) \quad t \geq 0. \tag{11}$$

Let us denote by $\mathbb{E}$ the expectation with respect the probability P. Observe that, thanks to the pairwise independence of the β_j, for any $t \geq 0$ and integers $m > n \geq 1$, we have

$$\mathbb{E}\left\|\sum_{j=n}^{m} \beta_j(t)\psi_j(t)\right\|^2_{L^2(\mathcal{O}_t)} = t\sum_{j=n}^{m} \|\psi_j(t)\|^2_{L^2(\mathcal{O}_t)},$$

and therefore, by (8) and (10), the equality (11) defines for any $t \geq 0$ an element $M(t) \in L^2(\mathcal{O}_t \times \Omega)$ which is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t$ is the sub-σ-algebra of $\mathcal{F}$ generated by the random variables $\{\beta_j(s) :\ s \in [0,t],\ j \geq 1\}$.

Thus $\{M(t) :\ t \geq 0\}$ can be viewed as an $\mathcal{F}_t$-adapted process with values in $L^2(\mathcal{O}_t)$. Observe that

$$\mathbb{E}M(t) = 0$$

and

$$
\begin{aligned}
\mathbb{E}\|M(t)\|^2_{L^2(\mathcal{O}_t)} &= t\sum_{j=1}^{\infty}\|\psi_j(t)\|^2_{L^2(\mathcal{O}_t)} \\
&\leq tC_{r,t}\sum_{j=1}^{\infty}\|\phi_j\|^2_{L^2(\mathcal{O})}
\end{aligned}
$$

for all $t \geq 0$.

4 A Stochastic PDE on time-varying domains

We now consider the additive noise version of (4), i.e., the stochastic parabolic PDE with additive noise and homogeneous Dirichlet boundary condition,

$$
\begin{cases}
dU(t) = [\Delta U(t) - f(U(t))]\ dt + dM(t) \quad \text{in} \quad Q \\
U = 0 \ \text{ on } \Sigma, \\
U(x,0) = u_0(x), \quad x \in \mathcal{O}_0.
\end{cases}
\tag{12}
$$

Here we interpret $dM(t)$ as follows. Assuming enough regularity for the ϕ_j, formally we obtain from (11)

$$
dM(t) = \sum_{j=1}^{\infty}\psi_j(t)d\beta_j(t) + \sum_{j=1}^{\infty}\beta_j(t)\frac{\partial\psi_j}{\partial t}(t)dt,
$$

where

$$
\begin{aligned}
\frac{\partial\psi_j}{\partial t}(x,t) &= \frac{\partial}{\partial t}(\phi_j(\bar{r}(x,t))) \\
&= \nabla_y\phi_j(\bar{r}(x,t))\cdot\frac{\partial\bar{r}}{\partial t}(x,t).
\end{aligned}
$$

Thus,

$$
dM(x,t) = \sum_{j=1}^{\infty}\phi_j(\bar{r}(x,t))d\beta_j(t) + \sum_{j=1}^{\infty}\beta_j(t)\nabla_y\phi_j(\bar{r}(x,t))\cdot\frac{\partial\bar{r}}{\partial t}(x,t)dt. \tag{13}
$$

Now, making the change

$$
V(y,t) = U(r(y,t),t) \quad \text{for } y \in \mathcal{O}, t \geq 0,
$$

or, equivalently,

$$
U(x,t) = V(\bar{r}(x,t),t) \quad \text{for } x \in \mathcal{O}_t, t \geq 0, \tag{14}
$$

and using (13), the problem (12) is transformed in the following problem on Q:

$$
\begin{cases}
dV(y,t) = \Big[\sum_{k,j=1}^{N} \dfrac{\partial}{\partial y_j}(a_{jk}(y,t)\dfrac{\partial V}{\partial y_k}(y,t) - b(y,t)\cdot\nabla_y V(y,t) \\
\qquad\qquad + f(V(y,t)) + R(y,t)\Big]\, dt + dW(y,t) \\
\text{in } \mathcal{O}\times(0,\infty), \\
V = 0 \ \text{ on } \partial\mathcal{O}\times(0,\infty), \\
V(y,0) = u_0(r(y,0)), \quad y\in\mathcal{O},
\end{cases} \tag{15}
$$

where

$$W(y,t) := \sum_{j=1}^{\infty} \phi_j(y)\beta_j(t),$$

and

$$R(y,t) := \sum_{j=1}^{\infty} \beta_j(t)\nabla_y\phi_j(y)\cdot\frac{\partial\bar{r}}{\partial t}(r(y,t),t).$$

Observe first that by (8) and the independence of the β_j, the process $W(t) := W(\cdot,t)$ is a Wiener process with values in $L^2(\mathcal{O})$. For the convergence of $R(t) := R(\cdot,t)$ we need some additional assumptions on the ϕ_j. More exactly, we will assume that

$$\{\phi_j\}_{j\ge 1}\subset H^1(\mathcal{O}) \quad\text{and}\quad \sum_{j=1}^{\infty}\|\phi_j\|^2_{H^1(\mathcal{O})} < \infty. \tag{16}$$

Under this assumption, $R(t)$ is a well defined process with values in $L^2(\mathcal{O})$, and more exactly

$$\mathbb{E}\|R(t)\|^2_{L^2(\mathcal{O})} \le t\max_{y\in\mathcal{O}}|\frac{\partial\bar{r}}{\partial t}(r(y,t),t)|^2_{\mathbb{R}^N}\sum_{j=1}^{\infty}\|\phi_j\|^2_{H^1(\mathcal{O})} \quad \forall\, t\ge 0.$$

Thus, $R(t)$ is an $\mathcal{F}_t$-adapted process belonging to $L^\infty(0,T;L^2(\Omega\times\mathcal{O})$ for all $T>0$.

Thus, taking into account Lemma 1, from the results for nonlinear monotone SPDE obtained in [9] (see also [10]) we get existence and uniqueness of variational solution for problem (15). More exactly, we have:

Theorem 2 *Under the assumptions* (1), (2), (3), (5), (6) *and* (16), *for any* $u_0\in L^2(\mathcal{O})$ *there exists a unique* $\mathcal{F}_t$*-adapted process*

$$V\in L^2(\Omega\times(0,T);H^1_0(\mathcal{O}))\cap L^p(\Omega\times(0,T);L^p(\mathcal{O}))\cap L^2(\Omega;C([0,T];L^2(\mathcal{O})))$$

for all $T > 0$, variational solution of (15)*, i.e., such that*

$$\begin{aligned} V(\omega,y,t) &= u_0(r(y,0)) + W(\omega,y,t) \\ &\quad + \int_0^t \left[\sum_{k,j=1}^N \frac{\partial}{\partial y_j}(a_{jk}(y,s)\frac{\partial V}{\partial y_k}(\omega,y,s) - b(y,s)\cdot\nabla_y V(\omega,y,s) \right. \\ &\quad \left. + f(V(\omega,y,s)) + R(\omega,y,s) \right] ds \end{aligned}$$

for all $t \geq 0$, P-a.s. in Ω, where the equality must be understood in the sense of $H^{-1}(\mathcal{O}) + L^{p'}(\mathcal{O})$.

Then, the process U given by (14) *can be interpreted as the unique solution of* (12)*.*

Existence of random attractors (as in [3]) will be considered elsewhere.

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GENERALIZATION OF THE FUCIK-KUFNER RESULT WITH APPLICATIONS TO OBSTACLE PROBLEMS

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Abstract

In this paper we present an extension of the Fucik-Kufner result [3] to the case of n-variational inequalities in a Hilbert space. Then we adapt that extension to simplify derivation of useful inequalities concerning solutions of various types of elliptic obstacle problems.

Key words: *Variational inequality, obstacle problems.*
AMS subject classifications: *49J40*

1 Introduction

In the 1970's there was considerable interest in the analysis of obstacle problems. This was connected with the development of research on variational inequalities and has been studied by many authors (see [7], [10] and references therein). The majority of results concentrated on, natural from a mathematical point of view, problems of existence and uniqueness of the solutions. However, in case of variational inequalities corresponding to obstacle problems additional questions regarding e.g. the regularity of the solutions ([5], [6]) or convergence of the solutions ([4], [9]) or comparing of the solutions can be posed. These problems seem to be interesting due to possible applications.

Comparison theorems for the solutions of the global obstacle problems were introduced by H. Brezis [1], G. Duvant, J. Lions [2] and U. Mosco [8]. However those results allowed the comparison of the different solutions of obstacle problems of the same type.

Fucik-Kufner theorem [7] describes the constructive approach for the comparison of the two solutions of different variational inequalities.

The generalization presented here seems to represent a very simple, unified and straightforward method for comparing solutions of various types of obstacle problems simultaneously.

2 Comparison theorem

Firstly, we articulate and prove the following result representing the generalization of the Fucik-Kufner theorem [7] for the case of n variational inequalities.

Theorem 1 *Let $\{K_i\}_{i=1}^n$ be nonempty, closed, convex subset of a Hilbert space H, f be a functional in the dual space H^*, $a(\cdot,\cdot)$ a coercive, bilinear form defined on $H \times H$ and $u_i \in K_i$ be the solution of the variational problem: Find u such that*

$$a(u, v-u) \geq \langle f, v-u\rangle \quad \text{for any } v \in K_i, \tag{1}$$

Let $w_i \in K_i$ for $i = 1, 2, \ldots, n$ such that

$$w_1 + w_2 + \ldots + w_n = u_1 + u_2 + \ldots + u_n$$

and

$$\sum_{i=1, i\neq k}^{n} a(u_k - w_i, u_i - w_i) = 0,$$

for some k, then $w_i = u_i$ for $i = 1, 2, \ldots, n$.

Proof. Put $v = w_i$ in (1)

$$\begin{cases} a(u_1, w_1 - u_1) \geq \langle f, w_1 - u_1\rangle \\ a(u_2, w_2 - u_2) \geq \langle f, w_2 - u_2\rangle \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ a(u_n, w_n - u_n) \geq \langle f, w_n - u_n\rangle \end{cases}$$

Let us sum

$$\sum_{i=1}^{n} a(u_i, w_i - u_i) \geq \langle f, w_1 + \ldots + w_n - u_1 - \ldots - u_n\rangle = \langle f, 0\rangle = 0,$$

since f is a linear functional. We calculate as follows

$$\begin{aligned} 0 \ &\leq \ \sum_{i=1}^{n} a(u_i, w_i - u_i) = \sum_{i=1, i\neq k}^{n} a(u_i, w_i - u_i) + a(u_k, w_k - u_k) \\ &= \sum_{i=1, i\neq k}^{n} a(u_i, w_i - u_i) + a(u_k, \sum_{i=1, i\neq k}^{n} (u_i - w_i)) \\ &= \sum_{i=1, i\neq k}^{n} a(u_i, w_i - u_i) + \sum_{i=1, i\neq k}^{n} a(u_k, u_i - w_i) \\ &= \sum_{i=1, i\neq k}^{n} a(u_k - u_i, u_i - w_i) = \sum_{i=1, i\neq k}^{n} a(u_k - w_i + w_i - u_i, u_i - w_i) \end{aligned}$$

$$= \sum_{i=1, i\neq k}^{n} a(u_k - w_i, u_i - w_i) + \sum_{i=1, i\neq k}^{n} a(w_i - u_i, u_i - w_i)$$
$$= -\sum_{i=1, i\neq k}^{n} a(w_i - u_i, w_i - u_i) \leq -\sum_{i=1, i\neq k}^{n} \mu \parallel u_i - w_i \parallel^2,$$

where $\mu > 0$ is the constant of coerciveness. This means that every norm must be zero, hence

$$w_1 = u_1, w_2 = u_2, \ldots, w_n = u_n.$$

□

In the next chapter we present an application of the above result, where we compare solutions of three different types of obstacle problems. It is worth pointing out that due to this result it will be possible to obtain such comparisons simultaneously.

3 Application

We start with introducing some basic notations connected with obstacle problems.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with the boundary $\partial\Omega$ of $C^{1,1}$ class,

L is an elliptic operator

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_i})$$

with coefficients $a_{ij} : \overline{\Omega} \to \mathbb{R},\ a_{ij} \in C^1(\overline{\Omega})$ for $1 \leq i, j \leq n$, which satisfy the ellipticity condition i.e. there exists a positive constant μ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

The operator L for $u, v \in H_0^1(\Omega)$ determines (see [7]) the bilinear, continuous and coercive form on $H_0^1(\Omega)$, in the following way

$$\langle Lu, v\rangle = a(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)u_{x_i}(x)v_{x_j}(x)dx.$$

One version of an obstacle problem is to find the solution to the variational inequality

$$a(u, v - u) \geq \langle f, v - u\rangle, \forall v \in K,$$

where $f \in H^{-1}(\Omega)$ and K is a so-called admissible set whose definition depends on the type of the obstacle problem considered.

Let $\psi, \varphi \in H^{1,p}(\Omega)$, $\psi \le \varphi$ on Ω, $\psi \le 0$ on $\partial\Omega$ and $\varphi \ge 0$ on $\partial\Omega$. We define the sets

$$\begin{aligned} K_1 &= \{v \in H_0^1(\Omega) : \psi \le v \text{ in } \Omega\}, \\ K_2 &= \{v \in H_0^1(\Omega) : \psi \le v \le \varphi \text{ in } \Omega\}, \\ K_3 &= \{v \in H_0^1(\Omega) : v \le \varphi \text{ in } \Omega\}. \end{aligned}$$

It is well known (see [10]) that there exist the unique solutions of obstacle problems with the admissible set K_1, K_2 or K_3, respectively.

Now we show some relations between solutions of the above obstacle problems. Let us define

$$\begin{aligned} w_1 &= \max(u_1, u_2), \\ w_2 &= u_1 + u_2 + u_3 - w_1 - w_3, \\ w_3 &= \min(u_2, u_3). \end{aligned}$$

We see that $u_1 + u_2 + u_3 = w_1 + w_2 + w_3$ and

$$\begin{aligned} &a(u_2 - w_1, u_1 - w_1) + a(u_2 - w_3, u_3 - w_3) \\ =&a(u_2 - \max(u_1, u_2), u_1 - \max(u_1, u_2)) + a(u_2 - \min(u_2, u_3), u_3 - \min(u_2, u_3)) \\ =&a(\min(u_2 - u_1, 0), \min(0, u_1 - u_2)) + a(\max(0, u_2 - u_3), \max(u_3 - u_2, 0)) \\ =&a(\max(u_1 - u_2, 0), -\min(u_1 - u_2, 0) + a(\max(u_2 - u_3, 0), -\min(u_2 - u_3, 0)) \\ =&a\big((u_1 - u_2)^+, (u_1 - u_2)^-\big) + a\big((u_2 - u_3)^+, (u_2 - u_3)^-\big) \\ =&\int_\Omega \sum_{i,j=1}^n a_{ij}(x)(u_1 - u_2)^+_{x_i}(u_1 - u_2)^-_{x_j} dx \\ +&\int_\Omega \sum_{i,j=1}^n a_{ij}(x)(u_2 - u_3)^+_{x_i}(u_2 - u_3)^-_{x_j} dx = 0. \end{aligned}$$

Finally we must show that $w_i \in K_i$ for $i = 1, 2, 3$. One can notice that

$$w_1 = \begin{cases} u_1 \ge \psi, \text{ if } u_1 \ge u_2 \\ u_2 \ge \psi, \text{ if } u_1 < u_2, \end{cases}$$

thus, $w_1 \in K_1$.

$$w_2 = \begin{cases} u_1 & \text{, if } u_2 > u_1 \text{ and } u_2 \ge u_3 \Rightarrow \psi \le u_1 < u_2 \le \varphi \\ u_1 + u_3 - u_2 & \text{, if } u_3 > u_2 > u_1 \Rightarrow \psi \le u_1 < u_1 + u_3 - u_2 < u_3 \le \varphi \\ u_2 & \text{, if } u_1 \ge u_2 \ge u_3 \Rightarrow \psi \le u_2 \le \varphi \\ u_3 & \text{, if } u_1 \ge u_2 \text{ and } u_3 > u_2 \Rightarrow \psi \le u_2 < u_3 \le \varphi, \end{cases}$$

thus, $w_2 \in K_2$.

$$w_3 = \begin{cases} u_2 \le \varphi, \text{ if } u_2 < u_3 \\ u_3 \le \varphi, \text{ if } u_2 \ge u_3, \end{cases}$$

thus, $w_3 \in K_3$.

Theorem 1 indicates that $w_1 = \max(u_1, u_2) = u_1$ and $w_3 = \min(u_2, u_3) = u_3$, so we can deduce the following relations

$$u_1 \geq u_2 \geq u_3.$$

One can observe that the procedure connected with comparing the solutions of variational inequalities is not complicated and the result is strictly following the intuitive approach.

The advantage of the method relies on comparing n different problems simultaneously instead of performing $n-1$ repetitions of the similar techniques connected with finding the relations between different pairs of solutions.

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PLANAR BIMODAL PIECEWISE LINEAR SYSTEMS. BIFURCATION DIAGRAMS

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Abstract

The set of planar bimodal linear control systems is partitioned into a finite number of differentiable strata, each of them consisting of those systems having canonical forms (for the equivalence relation which corresponds to admissible changes of basis) differing only in the values of the continuous invariants. Bifurcation diagrams with regard to this stratification are derived.

Key words: *Canonical form, stratification, bifurcation diagram.*
AMS subject classifications: *93B10, 93B27, 93C10*

1 Introduction

Piecewise linear systems have attracted the interest of researchers because of their interesting dynamical properties and the wide range of applications. The most common piecewise linear systems found in practice are in two or three dimensions. See for example [3], [4], [5].

In this paper, we tackle bifurcation diagrams for planar bimodal piecewise control systems. We consider $2D$ control linear systems acting on complementary half-planes and the equivalence relation defined by basis changes, preserving continuity along a given line ("admissible basis changes"). As the set of equivalence classes is not locally finite, we consider the union of equivalence classes differing only in the continuous invariants in the canonical form under this equivalence relation found in [6]. There are a finite number of sets in this partition, each of them is proved to be a differentiable manifold, therefore constitutes a finite stratification of the space of systems. This is the starting point to obtain bifurcation diagrams, with regard to this classification. Moreover, canonical forms can be applied to study controllability and other dynamical properties in each stratum.

In section 2, we state the definitions of bimodal piecewise linear systems and admissible basis changes. In section 3, we recall the canonical forms for order two bimodal systems. In section 4, we stratify the set of triples of matrices defining order two bimodal systems. Finally, in section 5, we show a bifurcation diagram.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers, $M_{n\times m}(\mathbb{R})$ the set of matrices with m rows and n columns (in the particular case where $m = n$ we will denote the set simply by $M_n(\mathbb{R})$), $Gl_n(\mathbb{R})$ the set of all invertible matrices in $M_n(\mathbb{R})$ and by$(e_1, \ldots, e_n)$ the natural basis of the Euclidian space $\mathbb{R}^n$.

2 Bimodal Piecewise Linear Systems

Bimodal piecewise linear systems consist of two linear dynamics acting on each side of a given hyperplane. Most of elementary non-linear circuits found in practice may be modeled with two linear regions separated by parallel boundaries hyperplanes, with two or three state variables. See [3], [4], [7], [8], where different topics about these systems are studied.

Bimodal (piecewise) linear systems can be defined by two control linear systems:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + B_1, \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t), \end{cases} \quad \text{if } \boldsymbol{y}(t) \leq 0, \quad \begin{cases} \dot{\boldsymbol{x}}(t) = A_2\boldsymbol{x}(t) + B_2, \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t), \end{cases} \quad \text{if } \mathbf{y}(t) \geq 0$$

where $A_1, A_2 \in M_n(\mathbb{R})$; $B_1, B_2 \in M_{n\times 1}(\mathbb{R})$; $C \in M_{1\times n}(\mathbb{R})$, being the dynamics continuous along a separating hyperplane $C\boldsymbol{x} = 0$ for some matrix $C \in M_{1\times n}(\mathbb{R})$. For simplicity, we will consider $C = (1\ 0 \ldots 0) \in M_{1\times n}(\mathbb{R})$ and that the dynamics is continuous along the hyperplane $H = \{\boldsymbol{x} \in \mathbb{R}^n : C\boldsymbol{x} = 0\}$, and hence: $H = \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = 0\}$.

Then continuity along H is equivalent to:

$$B_2 = B_1, \qquad A_2e_i = A_1e_i, \quad 2 \leq i \leq n.$$

We will simply write $B = B_1 = B_2$. Thus any bimodal piecewise linear system can be defined by a triple of matrices (A_1, A_2, B), where A_1, A_2 differ only in the first column.

Notation *Throughout the paper, $\mathcal{X}$ will denote the set of triples of matrices defining bimodal piecewise linear systems,*

$$\mathcal{X} = \{(A_1, A_2, B) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_{n\times 1}(\mathbb{R}) \mid A_2e_i = A_1e_i,\ 2 \leq i \leq n\}$$

which is obviously a differentiable manifold (of dimension $n^2 + 2n$).

As in [6], we consider basis changes preserving the hiperplanes $x_1(t) = k$ in order to allow the results below to be also applied in the cases where a separating hyperplane $x_1(t) = \delta$, $\delta \neq 0$, are considered (see, for example, [3]).

Definition 1 *Basis changes in the state variables space preserving the hyperplanes $x_1(t) = k$ will be called* admissible basis changes. *Thus, they are basis changes given by a matrix $S \in Gl_n(\mathbb{R})$,*

$$S = \begin{pmatrix} 1 & 0 \\ U & T \end{pmatrix}, \quad T \in Gl_{n-1}(\mathbb{R}).$$

Let us denote by $\mathcal{S}$ the Lie subgroup of $Gl_n(\mathbb{R})$

$$\mathcal{S} := \left\{ S \in Gl_n(\mathbb{R}) \,\middle|\, S = \begin{pmatrix} 1 & 0 \\ U & T \end{pmatrix},\ T \in Gl_{n-1}(\mathbb{R}) \right\}$$

We consider the equivalence relation in the set of matrices $\mathcal{X}$ which corresponds to admissible basis changes.

Definition 2 *Two triples of matrices* $(A_1, A_2, B), (A'_1, A'_2, B') \in \mathcal{X}$ *are said to be* equivalent *if there exists a matrix* $S \in \mathcal{S}$ *(representing an admissible basis change) such that* $(A'_1, A'_2, B') = (S^{-1}A_1S, S^{-1}A_2S, S^{-1}B)$.

This equivalence relation partitions $\mathcal{X}$ into finer equivalence classes than the similarity equivalence relation.

3 Canonical forms for $n = 2$

A canonical form is a representative in each equivalence class which is easier to deal with, and therefore calculations become simpler using it. In [3], canonical forms were obtained, assuming observability. In [6] canonical forms in the non-observable case are obtained, in the case where the observability matrix of the system rank equal to $n-1$. In particular, in the case $n = 2$ these canonical forms and the matrices S which correspond to admissible basis changes are listed below. We will use (CFN), N $= 1, 2, \ldots$ to label them.

Let us consider a triple of matrices defining an order two bimodal system

$$\left(\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}, \begin{pmatrix} \gamma_1 & a_3 \\ \gamma_2 & a_4 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right).$$

Let us assume that the system is observable ($a_3 \neq 0$). Then (see [3]), the corresponding canonical forms A_1^c, A_2^c, B^c for the matrices A_1, A_2 and B respectively, are:

- **<u>Case 0:</u>** $a_3 \neq 0$,

$$A_1^c = \begin{pmatrix} a_1 + a_4 & 1 \\ a_2a_3 - a_1a_4 & 0 \end{pmatrix} = \begin{pmatrix} \operatorname{tr} A_1 & 1 \\ \det A_1 & 0 \end{pmatrix},$$

$$A_2^c = \begin{pmatrix} \gamma_1 + a_4 & 1 \\ a_3\gamma_2 - a_4\gamma_1 & 0 \end{pmatrix} = \begin{pmatrix} \operatorname{tr} A_2 & 1 \\ \det A_2 & 0 \end{pmatrix},$$

$$B^c = \begin{pmatrix} b_1 \\ a_3b_2 - a_4b_1 \end{pmatrix}; S = \begin{pmatrix} 1 & 0 \\ \frac{a_4}{a_3} & \frac{1}{a_3} \end{pmatrix}. \qquad \text{(CF1)}$$

From now on, we will assume that the system is unobservable: $a_3 = 0$. We distinguish several cases.

- **<u>Case 1:</u>** $a_3 = 0$, $a_1 \neq a_4, \gamma_1 \neq a_4$.

– If $\gamma_2 = a_2 \frac{a_4-\gamma_1}{a_4-a_1}$, $b_2 + \dfrac{a_2 b_1}{a_4 - a_1} = 0$,

$$A_1^c = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} \gamma_1 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ 0 \end{pmatrix};$$
$$S = \begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_4-a_1} & t \end{pmatrix},\ \text{for any } t \neq 0. \tag{CF2}$$

– If $\gamma_2 = a_2 \frac{a_4-\gamma_1}{a_4-a_1}$, $b_2 + \frac{a_2 b_1}{a_4-a_1} \neq 0$,

$$A_1^c = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} \gamma_1 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ 1 \end{pmatrix};$$
$$S = \begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_4-a_1} & b_2 + b_1 \frac{a_2}{a_4-a_1} \end{pmatrix}. \tag{CF3}$$

– If $\gamma_2 \neq a_2 \frac{a_4-\gamma_1}{a_4-a_1}$,

$$A_1^c = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} \gamma_1 & 0 \\ 1 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ \frac{b_2 + b_1 \frac{a_2}{a_4-a_1}}{\gamma_2 - a_2 \frac{a_4-\gamma_1}{a_4-a_1}} \end{pmatrix};$$
$$S = \begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_4-a_1} & \gamma_2 - a_2 \frac{a_4-\gamma_1}{a_4-a_1} \end{pmatrix}. \tag{CF4}$$

- **Case 2:** $a_3 = 0$, $a_1 = a_4, \gamma_1 \neq a_4$.

– If $a_2 = 0, b_2 + \frac{\gamma_2 b_1}{a_4-\gamma_1} = 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} \gamma_1 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ 0 \end{pmatrix};$$
$$S = \begin{pmatrix} 1 & 0 \\ -\frac{\gamma_2}{a_4-\gamma_1} & t \end{pmatrix}\ \text{for any t } \neq 0. \tag{CF5}$$

– If $a_2 = 0, b_2 + \frac{\gamma_2 b_1}{a_4-\gamma_1} \neq 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} \gamma_1 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ 1 \end{pmatrix};$$
$$S = \begin{pmatrix} 1 & 0 \\ -\frac{\gamma_2}{a_4-\gamma_1} & b_2 + b_1 \frac{\gamma_2}{a_4-\gamma_1} \end{pmatrix}. \tag{CF6}$$

– If $a_2 \neq 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 1 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} \gamma_1 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ \frac{1}{a_2}\left[b_2 + b_1 \frac{\gamma_2}{a_4-\gamma_1}\right] \end{pmatrix};$$
$$S = \begin{pmatrix} 1 & 0 \\ -\frac{\gamma_2}{a_4-\gamma_1} & a_2 \end{pmatrix}. \tag{CF7}$$

- **Case 3:** $a_3 = 0$, $a_1 \neq a_4, \gamma_1 = a_4$.
  - If $\gamma_2 = 0, b_2 = 0$,

$$A_1^c = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ 0 \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_4 - a_1} & t \end{pmatrix},\ \text{for any } t \neq 0. \tag{CF8}$$

  - If $a_2 = 0, b_2 \neq 0$,

$$A_1^c = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ 1 \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_4 - a_1} & b_2 \end{pmatrix}. \tag{CF9}$$

  - If $a_2 \neq 0$,

$$A_1^c = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} a_4 & 0 \\ 1 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ \frac{1}{\gamma_2}\left[b_2 + b_1 \frac{a_2}{a_4 - a_1}\right] \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_4 - a_1} & \gamma_2 \end{pmatrix}. \tag{CF10}$$

- **Case 4:** $a_3 = 0$, $a_1 = a_4 = \gamma_1$.
  - If $a_2 = 0, \gamma_2 = 0, b_1 \neq 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} b_1 \\ 0 \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ \frac{b_2}{b_1} & t \end{pmatrix}\ \text{for any } t \neq 0. \tag{CF11}$$

  - If $a_2 = 0, \gamma_2 = 0, b_1 = 0,\ b_2 = 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ u & t \end{pmatrix},\ \text{for any } t \neq 0, u. \tag{CF12}$$

  - If $a_2 = 0, \gamma_2 = 0, b_1 = 0, b_2 \neq 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ A_2^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix},\ B^c = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ u & b_2 \end{pmatrix}\ \text{for any } u. \tag{CF13}$$

– If $a_2 = 0, \gamma_2 \neq 0, b_1 \neq 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix}, \; A_2^c = \begin{pmatrix} a_4 & 0 \\ 1 & a_4 \end{pmatrix}, \; B^c = \begin{pmatrix} b_1 \\ 0 \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ \frac{b_2}{b_1} & \gamma_2 \end{pmatrix}. \tag{CF14}$$

– If $a_2 = 0, \gamma_2 \neq 0, b_1 = 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 0 & a_4 \end{pmatrix}, \; A_2^c = \begin{pmatrix} a_4 & 0 \\ 1 & a_4 \end{pmatrix}, \; B^c = \begin{pmatrix} 0 \\ \frac{b_2}{\gamma_2} \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ u & \gamma_2 \end{pmatrix} \text{ for any } u. \tag{CF15}$$

– If $a_2 \neq 0, b_1 \neq 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 1 & a_4 \end{pmatrix}, \; A_2^c = \begin{pmatrix} a_4 & 0 \\ \frac{\gamma_2}{a_2} & a_4 \end{pmatrix}, \; B^c = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 0 \\ \frac{b_2}{b_1} & a_2 \end{pmatrix}. \tag{CF16}$$

– If $a_2 \neq 0, b_1 = 0$,

$$A_1^c = \begin{pmatrix} a_4 & 0 \\ 1 & a_4 \end{pmatrix}, \; A_2^c = \begin{pmatrix} a_4 & 0 \\ \frac{\gamma_2}{a_2} & a_4 \end{pmatrix}, \; B^c = \begin{pmatrix} 0 \\ \frac{b_2}{a_2} \end{pmatrix};$$

$$S = \begin{pmatrix} 1 & 0 \\ u & a_2 \end{pmatrix}, \quad \text{for any } u. \tag{CF17}$$

4 Stratification

A finite partition of the differentiable manifold $\mathcal{X}$ may be deduced from that in equivalence classes: consider the sets consisting of all equivalence classes with canonical forms of the "same type", but with different values for the parameters. The sets thus obtained are disjoint sets and, as we will show, differentiable manifolds. Therefore, they constitute a stratification of $\mathcal{X}$.

In order to use Arnold's techniques (see [1]), the starting point is that equivalence classes are the orbits of the Lie group action of $\mathcal{S}$ on $\mathcal{X}$ defined by $\alpha(S, (A_1, A_2, B)) = (S^{-1}A_1S, S^{-1}A_2S, S^{-1}B)$.

Given $(A_1, A_2, B) \in \mathcal{X}$, we will denote by $\mathcal{O}(A_1, A_2, B)$ its orbit and consider the partition of $\mathcal{X}$ into sets, each of them corresponding to the union of orbits or equivalence classes having associated a canonical form of the same type; namely, E_1 is the set of all triples of matrices having canonical form of type CF1, E_2 the set of all those having canonical form of type CF2, and so on. Note that these orbits are differentiable manifolds (see [9]).

Theorem 1 *The sets E_i, $i = 1, \ldots, 17$ are differentiable manifolds.*

Proof. E_i, $i \neq 2,5,8$ are open sets of linear varieties. E_2, E_5 and E_8 are defined by quadratic equations, giving rise to implicit manifolds with no singular points. Thus they all are differentiable manifolds. □

Corollary 2 $\mathcal{X} = \left(\bigcup_{i=1}^{17} E_i\right)$ *is a finite stratification of* $\mathcal{X}$.

Proof. Clearly, these sets are disjoint sets and constitute a partition of $\mathcal{X}$. From Theorem 1 they are differentiable manifolds, thus a stratification of $\mathcal{X}$. □

Next Table shows the dimensions of the strata above.

Stratum	Dimension	Stratum	Dimension	Stratum	Dimension
E_1	8	E_2	5	E_3	6
E_4	7	E_5	5	E_6	5
E_7	6	E_8	5	E_9	4
E_{10}	6	E_{11}	3	E_{12}	1
E_{13}	3	E_{14}	4	E_{15}	3
E_{16}	5	E_{17}	4		

5 Bifurcation diagrams

A bifurcation diagram of a family of bimodal systems,

$$\Lambda : \mathbb{R}^d \longrightarrow M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_{n\times 1}(\mathbb{R})$$

is a partition of the parameter space $\mathbb{R}^d$ according to the canonical form of the triple of matrices, and induced by the stratification which was given in Section 4. In particular, this stratification provides the information about which canonical forms are near each other in the sense of local perturbations.

Let us show as an example about how a bifurcation diagram may be obtained.

Example 1 *Consider the triple of matrices* $\left(\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ *and the effect of a perturbation on it:*

$$\left(\begin{pmatrix} 2 & \varepsilon_1 \\ 1+\varepsilon_2 & 3+\varepsilon_3 \end{pmatrix}, \begin{pmatrix} 1 & \varepsilon_1 \\ -2 & 3+\varepsilon_3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \textit{for small } \varepsilon_1, \varepsilon_2, \varepsilon_3.$$

If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$, *we obtain the initial triple, which belongs to* E_2. *If* $\varepsilon_1 = 0$, $\varepsilon_3 \neq 0$, *we obtain a triple in* E_4. *If* $\varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 \neq 0$, *we obtain a triple in* E_3. *Finally, in the case where* $\varepsilon_1 \neq 0$, *we obtain a triple in* E_1.

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DIFFERENCE COMBINATION PARAMETRIC RESONANCE; APPLICATION TO THE GARDEN HOSE PROBLEM.

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Abstract

This paper discusses combination resonance phenomena in parametric systems of two or more degrees of freedom starting from a theoretical result by Mailybayev & Seyranian. We present, to the best of our knowledge, the first example of difference combination resonance in a mechanical system. That is, where the system may exhibit significant response when forced with an external frequency that is the difference between its two internal resonant frequencies. The model system studied is a double pendulum with a follower load, a non-conservative force that would be described for example by an oscillating jet of fluid, like an idealized garden hose. For this example, after the inclussion of gravity, the difference combination frequency may be *lower* than the two individual resonant frequencies, a surprising effect that should be taken into account when analyzing the stability of other high-dimensional systems.

Key words: *combination resonance, follower pendulum, parametrically excited dynamical systems*
AMS subject classifications: *37B55, 70K28*

1 Introduction

Dynamics systems subject to a parametric excitation are such that the forcing terms appear as (usually periodic) time-varying coefficients of the state variables. The canonical example is the Mathieu equation, which can be written in dimensionless form

$$\ddot{x} + (a + b\cos(t))x = 0\,, \tag{1}$$

in which the period of excitation is scaled to 2π. Here the parameter b represents the strength of the applied parametric forcing (scaled by the square of the frequency) and a is the square of the ratio of natural frequency to forcing frequency. The shape and properties of the instability tongues are well known, as are their properties if one adds a small amount of damping, see e.g. [3, 6, 1].

In particular, for small forcing and damping harmonic instabilities (which correspond to pitchfork bifurcations of the trivial solution) occur in thin tongues that originate from every value of a that is equal to the square of an integer; and sub-harmonic (period-doubling) instability tongues originate from point every $a = (2n - 1/2)^2$, $n = 1, 2, \ldots$.

For example, the Mathieu equation arises if one looks for instabilities of the trivial solution to a simple pendulum whose support is subject to periodic acceleration equal to $\Delta \cos(\Omega t)$. The equations of motion of such a system in the presence of small linear damping can be written in the form

$$\ddot{\theta} + c\dot{\theta} + [\kappa + \delta \cos(\Omega t)] \sin(\theta) = 0\,, \tag{2}$$

where c is the damping coefficient, $\kappa = g/l$, g is acceleration due to gravity, l is the length of the pendulum and $\delta = \Delta/l$.

This paper concerns a different phenomenon for multi-degree-of-freedom systems, namely that of combination resonance, where an instability occurs for the trivial solution when the parametric excitation frequency Ω is close to the sum or difference of two of the natural frequencies of the system $\omega_1 \pm \omega_2$. There are many examples of such combination resonances in the literature. For example, the book by Nayfeh [7] considers many such cases, especially of mechanical systems where the two modes are derived from a Galerkin reduction of a continuum system such as a plate or beam. In all these examples, though, it is a *sum* combination resonance that is excited $\Omega \approx \omega_1 + \omega_2$. However, there do not seem to be any concrete examples of *difference* resonances in the literature. In fact, such a mechanical device might be somewhat strange. If we had $\omega_1 \approx \omega_2$, then $\Omega = \omega_1 - \omega_2$ would be small, perhaps several orders of magnitude smaller. So a difference combination resonance would give a response at a high frequency from low frequency excitation. This would be like making a drum vibrate by sending it up and down in an elevator!

This paper presents an example of a system that has just such a property. Based on some theoretical results by Mailybayev and Seyranian [4] (summarized in the next section) we show in Section 3, that a difference combination resonance may occur in theory in a simplified model of a hose with time varying flow, namely a double pendulum with a combination of purely follower and constant-directional loads. Section 4 then carries out a preliminary numerical parameter sweep to verify that this effect is indeed seen in practice. Finally section 5 draws conclusions and points to future work.

2 A Parametric resonance theorem

Consider a linear m-degree-of-freedom linear system ($m \geq 2$) with periodic coefficients that can be written in matrix form as

$$\mathbf{M}\ddot{\mathbf{y}} + \gamma \mathbf{D}\dot{\mathbf{y}} + (\mathbf{C} + \delta \mathbf{B}(\Omega t))\mathbf{y} = 0\,. \tag{3}$$

Here $\mathbf{M}, \mathbf{D}$ and $\mathbf{C}$ are symmetric, positive definite matrices, $\mathbf{B}(\tau)$ is a piecewise-continuous 2π-periodic matrix function of Ωt that contains the parametric excitation terms, $\mathbf{y}$ is an m-dimensional vector of generalized coordinates and γ and δ are small parameters.

Let ω_i and ω_j be two normal modes frequencies (that is natural frequencies of the problem when $\delta = \gamma = 0$). Then we define:

Fundamental resonances to occur when $\Omega = 2\omega_j/k$ with $j = 1, \dots, m$ and $k = 1, 2, \dots$. If k is even then these are equivalent to the harmonic (pitchfork) bifurcations of the Mathieu equation. If k is odd these are the sub-harmonic (period-doubling) bifurcations.

Combination resonances: $\Omega = (\omega_i \pm \omega_j)/k$ with $\omega_i > \omega_j$ and $k = 1, 2, \dots$. The sign '+' corresponds to sum combination resonances, and '−' to difference combination resonances.

Theorem 1 ([4]) *If $\mathbf{B}(\tau)$ is symmetric, then the system may be subjected only to fundamental and sum combination resonances.*

If $\mathbf{B}(\tau) = \phi(\tau)\mathbf{B_0}$ one obtains fundamental resonances and, combination resonances for $\Omega = \omega_1 sign(c_{ij})\omega_2$, where

$$c_{ij} = \mathbf{u}_i^T \mathbf{B_0} \mathbf{u}_j \mathbf{u}_j^T \mathbf{B_0} \mathbf{u}_i \,,$$

where $\mathbf{u}_j$ are the eigenvectors of the conservative system

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\mathbf{y} = 0$$

To explain this result, it is sufficient to consider (3) in the case $m = 2$, $\gamma = 0$ and where $B(t)$ is a constant matrix times a sinusoidal function, $B = B_0 \cos(\Omega t)$. Suppose further that we change coordinates so that the system with $\delta = 0$ is written in diagonal form, and finally that time has been rescaled so that $\Omega = 1$. Hence we obtain a system of the form

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \left[\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} + \delta \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \cos(t) \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \qquad (4)$$

where $\alpha_1 = \omega_1^2/\Omega^2$, $\alpha_2 = \omega_2^2/\Omega^2$ and the matrix $\hat{B} = \{b_{ij}\}$ is the original constant matrix B_0 written in the transformed coordinates. Recalling how one computes stability curves for the Mathieu equation using Floquet theory, see e.g [3], we can look for solutions to (4) in the form

$$x_1 = \sum_{n=-\infty}^{\infty} c_n e^{int/\sqrt{2}}, \qquad x_2 = \pm \sum_{n=-\infty}^{\infty} c_n e^{inst/\sqrt{2}},$$

where $s = \pm 1$ and c_n are the Fourier coefficients. We find an infinite system of algebraic equations. It is straightforward to see that in the case $\delta = 0$, there is a non-trivial solution with $c_n = 0$ for all $n \neq k$ and $c_k \neq 0$, whenever $\alpha_1 + s\alpha_2 = k^2$. This would suggest that both sum and difference combination resonances are possible. However, when looking at the conditions for the

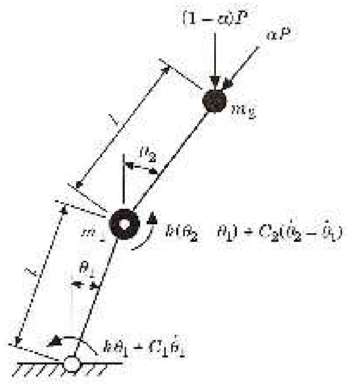

Figure 1: Schematic diagram of a planar double pendulum with stiff joints, and a (non-conservative) follower force and a (conservative) parametric force acting at the free end of the device. Here $\alpha \in [0, 1]$ represents the relative contribution of these two end-forces. The dimensionless equations of motion are given in (6). If $g > 0$ then gravity is assumed to be acting in the vertically upwards direction.

bifurcation equations to have a real solution for nonzero δ, one finds the following condition

$$c_{12} = \text{sign}(b_{12}b_{21}) = s. \tag{5}$$

That is, to excite a sum resonance, the off-diagonal entries of B_0 must be of the same sign, and to excite a difference resonance these diagonal entries must be of opposite sign. In particular, if $\hat{B}$ is symmetric matrix (as is the case in many mechanical applications) then only sum combinations can be excited. In fact, it can be argued (see references in [4]) that pure Hamiltonian systems can never excite difference combination resonance; in other words, if difference parametric resonance is possible at all, then the matrix $B(\Omega t)$ must contain non-conservative terms.

3 The follower pendulum

Canonical examples of mechanical systems that contain non-conservative forces are those that involve fluid-structure interaction [5]. The simplest form of such systems arise in models for hose pipes or structures with attached jets, where the fluid inside the mechanism is only modelled via a so-called "follower force" that is aligned with the end of the mechanism. A particular simple example is that of a pendulum with a follower force, which is represented in Fig. 1.

Following [2] (see also [8]) the equations of motion of such a device can be written in dimensionless form as

$$\begin{aligned}&(1+m)\ddot{\theta}_1+\cos(\theta_1-\theta_2)\ddot{\theta}_2+2c\dot{\theta}_1-c\dot{\theta}_2+2\theta_1-\theta_2+\dot{\theta}_2^2\sin(\theta_1-\theta_2)\\&=p(\Omega t)(1-\alpha)\sin\theta_1+\alpha\sin(\theta_1-\theta_2)-g(1+m)\sin(\theta_1)\\&\cos(\theta_1-\theta_2)\ddot{\theta}_1+\ddot{\theta}_2+c(\dot{\theta}_1-\dot{\theta}_2)-\theta_1+\theta_2+\dot{\theta}_1^2\sin(\theta_1-\theta_2)\\&=p(\Omega t)(1-\alpha)\sin\theta_2-g\sin(\theta_2)\,.\end{aligned}\tag{6}$$

Here it is assumed that the two joints have equal stiffness and damping, $m=m_2/m_1$ represents the ratio between the moments of inertia of the two pendulums, $c \ll 1$ is a dimensionless damping coefficient and we assume that the forcing term $p(\Omega t)$ is assumed to be a 2π-periodic function of its argument (specifically for the numerical computations in the next section we take $p(t)=\delta\cos(\Omega t)$). The original motivation for including the parameter α was to introduce a homotopy that enables one to pass from a purely conservative system (if $c=0$ also) when $\alpha=0$ to a non-conservative system when $\alpha=1$. The new ingredient here is to additionally include the effects of gravity via the terms proportional to g which represents the ratio of gravitational to stiffness forces.

Taking the case $g=0$, after linearization about the trivial equilibrium position $\theta_1=\theta_2=0$, a straightforward calculation for (6) reveals that for $\alpha=0$

$$c_{12}=-\frac{1}{4}(1+m)^2>0,$$

whereas for $\alpha=1$,

$$c_{12}=-1.$$

Thus, since c_{12} is a continuous function of α, we conclude that for sufficiently large follow forces, the system is indeed of the right form to excite difference combination resonances.

A detailed two-timescale perturbation expansion was carried out in [9] for $g=0$, $\alpha=1$, in which it was found that for small c and p the difference combination resonance does indeed lead to nontrivial responses of the system for (6). Rather than reproduce this lengthy, but standard, analysis, we turn instead to numerical results to illustrate the occurrence of the difference combination resonance.

4 Numerical computations

The response of the systems has been computed as the time averaged norm of the position and velocities after 200 time units staring from small amplitude random initial conditions. The resonances will be detected as an increase of this response function as we sweep in frequency. To illustrate the validity and difficulties of this detection method we plot in Figure 2 the behavior of the response function for the Mathieu equation (2). We would expect some structure for $\Omega\sim 1$ and $\Omega\sim 2$ (corresponding to crossing of the branching

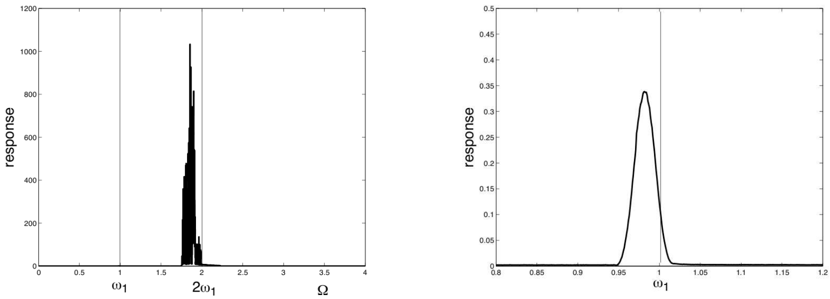

Figure 2: (Left) Response of the Mathieu system as a function of the frequency Ω for $\delta = 0.5$, $c = 10^{-5}$ and a small amplitude random initial position. (Right) A zoom around the frequency 1 displays a much weaker resonance.

point (BP) and period doubling (PD) curves in the typical Mathieu tongues diagram). A strong resonance structure around the period doubling frequency is clearly visible inidicating that the system is in the highly nonlinear region. The BP resonance is only visible after a zoom process and is shown in the right panel of Figure 2.

Performing a similar numerical computation for the follower pendulum system (6), we obtain the response function shown in Figure 3 and 4 for the cases of zero gravity (double pendulum in an horizontal table) and non vanishing gravity (hanging double pendulum).

In agreement with the theoretical prediction the sum combination resonance is present for the purely conservative case ($\alpha = 0$) whereas the difference combination resonances is only possible for the purely follower situation ($\alpha = 1$). The transition from one case to the other and the interaction with the other fundamental resonances visible in the numerical experiments will be subject of future study.

It is worth noting that the presence of a difference combination resonance in the latter case (with gravity) occurs for frequencies much lower than any of the internal frequencies of the system (normal modes). This unexpected results may have relevant implications while evaluating the stability of analogous structures or, in the positive side, to take advantage of the increase of response at or close to the resonance in a "energy harvesting" device.

5 Conclusion

This paper has produced as far as we are aware the first physically realizable example of a system that can excite difference combination resonances. The

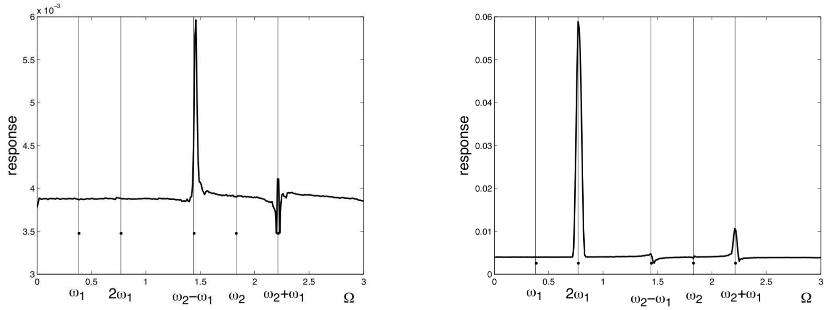

Figure 3: Response of the follower pendulum without gravity for $\alpha = 0$ (left) and $\alpha = 1$. The vertical lines are a guide for the eye to locate the position of the predicted fundamental and combination resonances

reason why such systems have previously not been observed appears to be because of the requirement (5) with $s = -1$, namely that the appropriate portion of the parametric forcing matrix must have skew-symmetric terms. In rotating systems it is well known that such skew symmetric terms can arise in stiffness and interia matrices $\mathbf{M}$ and $\mathbf{A}$ due to Coriolis forces or gyroscopic effects, and indeed can give rise to Hopf bifurcations. However, it seems hard to imagine a mechanical system for which the unforced system does not have these rotational effects but the parametric excitation terms do. Instead, we have found an example of mechanical systems with follower forces where we have been able to show the required condition (5) is satisfied with $s = -1$.

This paper presents just a preliminary study of the difference parametric resonance phenomenon. Future work will present detailed numerical continuation results that map out in the parameter space of (6) regions in which difference combination resonance can arise, and in particular to search for regions in which $\omega_1 \approx \omega_2$ so that the difference resonance can exist for frequencies way below the two fundamental resonances. We will also consider the implications for the nonlinear dynamics of the system.

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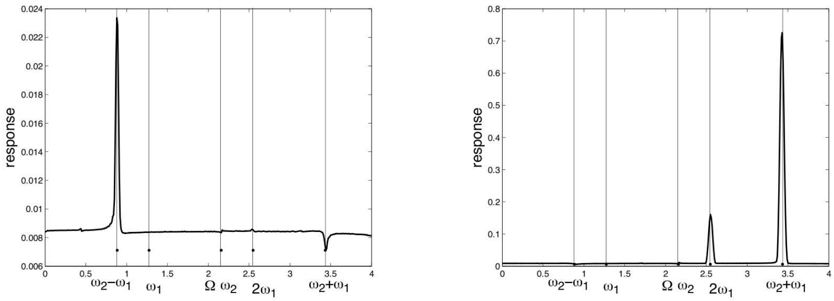

Figure 4: Response function with gravity for $\alpha = 0$ (left) and $\alpha = 1$.

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ON FRACTIONAL BROWNIAN MOTIONS AND RANDOM DYNAMICAL SYSTEMS

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Abstract

In this paper we consider a class of nonlinear stochastic partial differential equations (SPDEs) driven by a fractional Brownian motion with the Hurst parameter bigger than 1/2. We show that these SPDEs generate random dynamical systems.

Key words: *Fractional Brownian motions, Random dynamical systems, Stochastic differential equations.*
AMS subject classifications: *60H15, 37H10, 60H05.*

1 Introduction

A central mathematical object in Stochastics and Stochastic Processes is the Ito integral. It plays an important role in many areas of pure and applied mathematics including mathematical finance, population dynamics, fluid dynamics, statistics, signal processing, control, particle systems, to name a few. The integrator of such an integral is often chosen to be the Brownian motion (the Wiener process) or its semimartingale generalizations. These random functions are of unbounded total variation, so that their Stieltjes integrals do not exist. Special properties of the integrators and the integrands are necessary to generalize the definition of the Stieltjes integral to the Ito integral, and enable the definition of solutions of differential equations driven by Brownian motion.

A property of paramount importance to this effect for Brownian motion is the independence of its increments. To move beyond integrals and processes constructed using this property is one of the most important tasks in the theory of Stochastics. We are most interested in using the fractional Brownian motion (fBm) process B^H where $H \in (0,1)$ is fixed. It is a type of stochastic process which deviates significantly from Brownian motion and semimartingales.

As a centered Gaussian process, it is characterized by the stationarity of its increments and a medium- or long-memory property which is in sharp contrast with martingales and Markov processes. It also exhibits power scaling and path regularity properties with Holder parameter H, which are very distinct from Brownian motion (note that the Brownian motion is included in this family of models when considering $H = 1/2$). Fractional Brownian motion has become a popular choice of late for applications where classical processes cannot model these non-trivial properties; for instance long memory, which is also known as persistence, and corresponds to the case $H \in (1/2, 1)$, is of fundamental importance for financial data and in internet traffic, see [12], [16] . Fractional Brownian motion is also a good candidate to model random long time influences in climate systems, see [15].

Ever since the pioneering works of Zähle [17], Decreusefond and Üstünel [5], and Lyons [11], the main thrust has been to understand how to perform stochastic integration with respect to fBm in a way which is consistent with some properties of the classical Ito theory for Brownian motion. In the case of higher regularity ($H > 1/2$), simple trajectorial methods, labelled as pathwise, can be used which make it easy to translate one integration theory into another, as fractional derivatives allow a pathwise estimate of the integrals in terms of integrand and integrator using special norms. Pathwise integrals historically gave the first cases where adequate solutions to stochastic differential equations (SDEs) were established, e.g. Nualart and Rascanu [14]; infinite-dimensional equations have been treated with the same success as finite-dimensional ones, e.g. Nualart and Maslowski [13], Garrido-Atienza *et al.* [6].

In this paper, we aim to investigate the equations' asymptotics. There are two theories dealing with the asymptotic qualitative behavior for general SDEs: the theory of random dynamical systems (RDS) and the theory of existence and uniqueness of invariant measures for the associated Markov semigroup. However, similarly to fBm itself, equations driven by fBm do not generate a Markov process; this precludes the study of invariant measures using classical tools for fBm-driven systems. This motivates our plan to concentrate on the study of fBm-driven SDEs as RDS.

The theory of RDS, developed by L. Arnold and coworkers, see [1], can be used to describe the asymptotical and qualitative behavior of systems of random and stochastic differential/difference equation in terms of stability, Lyapunov exponents, invariant manifolds, and attractors.

As we have said, considering fBm instead of Brownian motion has some advantages because of the nice properties that the fBm enjoys and the Brownian motion does not. Another crucial advantage is the following: for many Brownian-driven SPDEs with non–trivial diffusion coefficients, it is not known if these equations generate a RDS. The reason is that usually stochastic differential equations are only defined almost surely where the exceptional set may depend on ω since this exceptional set is related to the definition of an Ito integral which is defined as a limit of random variables in probability. And such a family of exceptional sets does not allow to use the theory of RDS. But we can overcome such exceptional sets dealing with SPDEs driven by a fBm with $H > 1/2$, provided the stochastic integrals are interpreted in the pathwise sense.

2 Preliminaries on random dynamical systems

In this section we review some basic concepts and results on random dynamical systems that will be used later.

In the next definition, we introduce a system that models the evolution of a noise.

Definition 1 *A metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{T}})$ *with two-sided time* $\mathbb{T}$ *(which is* $\mathbb{R}$ *in the continuous case and* $\mathbb{Z}$ *in the discrete one) consists of a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a family of transformations* $\{\theta_t\}_{t\in\mathbb{T}}$ *such that:*
1. It is a one-parameter group, i.e.

$$\theta_0 = \mathrm{id}_\Omega, \quad \theta_{t+s} = \theta_t\theta_s, \forall t, s \in \mathbb{T},$$

2. $(t,\omega) \in \mathbb{T}\times\Omega \to \theta_t\omega$ *is measurable,*
3. $\mathbb{P}$ *is invariant with respect to* θ*, i.e.,* $\theta_t\mathbb{P} = \mathbb{P}$*, for all* $t \in \mathbb{T}$*, which means that* $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$*, for all* $A \in \mathcal{F}$ *and all* $t \in \mathbb{T}$*.*
4. $\mathbb{P}$ *is ergodic with respect to* θ*, i.e, for any* $\{\theta_t\}_{t\in\mathbb{T}}$*-invariant set* $B \in \mathcal{F}$*, which means that* $\theta_t B = B$ *for all* $t \in \mathbb{T}$*, we have either* $\mathbb{P}(B) = 0$ *or* $\mathbb{P}(B) = 1$*.*

We now introduce a couple of examples of metric dynamical systems. Let $V = (V, \|\cdot\|, (\cdot,\cdot))$ be a separable Hilbert space.

Consider first the Brownian motion. We choose for Ω the set of continuous functions $C_0^V = C_0(\mathbb{R}, V)$ on $\mathbb{R}$ with values in V which are zero at zero. On this set we introduce the compact open topology given by the uniform convergence on compact intervals in $\mathbb{R}$. The Borel–σ–algebra over this space is denoted by $\mathcal{B}(C_0^V)$. $\mathbb{P}_{\frac{1}{2}}$ is the Wiener measure. The existence of such a canonical process $(C_0^V, \mathcal{B}(C_0^V), \mathbb{P}_{\frac{1}{2}})$ follows by Kolmogorov's theorem about the existence of a continuous modification of a process, see Bauer [2]. The flow θ is given by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad \omega \in \Omega \tag{1}$$

which is called the Wiener shift. The Wiener shift is measurable, see Arnold [1] Page 544, because C_0^V is separable and $(t,\omega) \mapsto \theta_t\omega$ is continuous. We emphasize that this metric dynamical system is ergodic, see Boxler [3].

Now let us introduce the fractional Brownian motion. Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H(t)$, $t \in \mathbb{R}$, with the covariance function

$$\mathbb{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t,\, s \in \mathbb{R}$$

is called a *two–sided one-dimensional fractional Brownian motion* (fBm), and H is the *Hurst parameter.*

Assume that Q is a bounded and symmetric linear operator on V which is of trace class, i.e., there exist a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in V and a sequence of nonnegative numbers $\{\lambda_i\}_{i\in\mathbb{N}}$ such that $trQ = \sum_{i=1}^{\infty}\lambda_i < \infty$ and $Qe_i = \lambda_i e_i$, $i \in \mathbb{N}$. A continuous *V-valued fractional Brownian motion* B^H with incremental covariance operator Q and Hurst parameter H is defined by

$$B^H(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i \beta_i^H(t), \qquad t \in \mathbb{R}$$

where $\{\beta_i^H(t)\}_{i\in\mathbb{N}}$ is a sequence of stochastically independent one-dimensional fBm. Notice that the above series is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ since $\sum_{i=1}^{\infty}\lambda_i < \infty$ and $\mathbb{E}(\beta_i^H(t))^2 = |t|^{2H}$ for $t \in \mathbb{R}$.

Remark 1 *$B^{1/2}$ is the Brownian motion.*

Using the definition of B^H, Kolmogorov's theorem ensures that B^H has a continuous version. Thus we can consider the canonical interpretation of an fBm: let $\Omega = C_0(\mathbb{R}, V)$, equipped again with the compact open topology. Let $\mathcal{F}$ be the associated Borel-σ-algebra and $\mathbb{P}_H$ the distribution of the fBm B^H, and $\{\theta_t\}_{t\in\mathbb{R}}$ be the flow of Wiener shifts defined by (1). Then the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a metric dynamical system which is ergodic, see [9]. Furthermore,

$$B^H(\cdot, \omega) = \omega(\cdot), \quad B^H(\cdot, \theta_r\omega) = B^H(\cdot + r, \omega) - B^H(r, \omega) = \omega(\cdot + r) - \omega(r). \tag{2}$$

We now introduce the concept of random dynamical systems that is used to describe the dynamics of systems under the influence of a noise.

Definition 2 *A random dynamical system (RDS) with one-sided time $\mathbb{T}^+$ and phase space V is a pair consisting of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and a mapping $\varphi : \mathbb{T}^+ \times \Omega \times V \to V$ which is $(\mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(V), \mathcal{B}(V))$–measurable and satisfies the cocycle property*

$$\varphi(t, \theta_\tau\omega, \cdot) \circ \varphi(\tau, \omega, \cdot) = \varphi(t + \tau, \omega, \cdot), \quad \textit{for } t, \tau \in \mathbb{T}^+, \omega \in \Omega,$$
$$\varphi(0, \omega, \cdot) = \mathrm{id}_V.$$

A typical example of cocycle mapping is the solution operator of finite or infinite dimensional differential equations with random coefficients satisfying particular regularity assumptions. Another example is the solution operator of finite dimensional Ito-equations. As we announced in the Introduction, for infinite dimensional Ito-equations with non-trivial diffusion coefficients this problem is rather unsolved.

Notice that the cocycle property is the generalization of the semigroup property; in fact, if we deleted all ω-dependence in the cocycle property we would just get the semigroup property.

We want to stress that we have required the MDS to be defined on two-sided time $\mathbb{T}$, while the RDS is only required to be defined on one-sided time $\mathbb{T}^+$. The reason is that we cannot expect the mapping φ to be defined on $\mathbb{T}$, since it is given, for instance, by the solution operator of a SPDE, which is not invertible in general. However, we can consider expressions of the following type: $\varphi(t, \theta_{-t}\omega, x)$, for $x \in V$, $\omega \in \Omega$, $t \in \mathbb{T}^+$, expressions that play a crucial role when analyzing the existence of random fixed points or random attractors associated to the RDS φ, see [8].

As we have mentioned, the purpose of this paper is to show that an infinite dimensional stochastic differential equation driven by an fBm with general diffusion coefficients generates a random dynamical system.

3 Main results

In this section we first introduce some basic concepts and results on fractional calculus and stochastic integrals with respect to the fBm β^H and B^H.

For $T > 0$, let $W^{\alpha,1}(0,T;V)$ be the space of measurable functions $f : [0,T] \to V$ such that

$$|f|_\alpha = \int_0^T \left(\frac{\|f(s)\|}{s^\alpha} + \int_0^s \frac{\|f(s) - f(\zeta)\|}{(s-\zeta)^{\alpha+1}} d\zeta \right) ds < \infty,$$

where $1 - H < \alpha < \frac{1}{2}$ is fixed, so we need to consider from now on $H \in (1/2, 1)$.

Following Zähle [17], for $f \in W^{\alpha,1}(0,T;V)$ we define the stochastic integral as the generalized Stieltjes integral

$$\int_0^T f d\beta^H = (-1)^\alpha \int_0^T D^\alpha_{0+} f(s) D^{1-\alpha}_{T-} \beta^H_{T-}(s) ds, \tag{3}$$

$$\int_s^t f d\beta^H = \int_0^T f \mathbf{1}_{(s,t)} d\beta^H, \qquad \text{for } 0 \le s < t \le T,$$

where, in general, for $0 \le a < b \le T$, $\beta^H_{b-}(s) := \beta^H(s) - \beta^H(b)$, and for $a < t < b$ the Weyl derivatives are given by

$$D^\alpha_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(\zeta)}{(t-\zeta)^{\alpha+1}} d\zeta \right),$$

$$D^{1-\alpha}_{b-} \beta^H_{b-}(t) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\beta^H(t) - \beta^H(b)}{(b-t)^{1-\alpha}} + (1-\alpha) \int_t^b \frac{\beta^H(t) - \beta^H(\zeta)}{(\zeta - t)^{2-\alpha}} d\zeta \right),$$

where Γ denotes the Gamma function. It can be proved (see, for instance, Nualart and Răşcanu [14], Decreusefond and Üstünel [5], Zähle [17]) that the stochastic integral (3) exists.

Now we define the stochastic integral with respect to the infinite dimensional fBm B^H. Let $L(V)$ denote the space of linear bounded operators on V and let $G : \Omega \times [0,T] \to L(V)$ be an operator such that $G(\omega, \cdot) e_i \in W^{\alpha,1}(0,T;V)$ for each $i \in \mathbb{N}$ and $\omega \in \Omega$. We define

$$\int_0^T G d\omega = \sum_{i=1}^\infty \int_0^T G(s) Q^{1/2} e_i d\beta^H_i(s) = \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^T G(s) e_i d\beta^H_i(s), \tag{4}$$

where the convergence of the sums in (4) is understood in V.

The following result establish that when making a change of variable in the stochastic integral, we not only have to shift the integration interval and the variable but also the path of the fBm (for the proof, see [6]).

Lemma 1 *For $a, b, r \in \mathbb{R}$, assuming that both integrals are well-defined,*

$$\int_a^b G(s)d\omega(s) = \int_{a-r}^{b-r} G(s+r)d\theta_r\omega(s).$$

Consider now the following stochastic evolution equation in V

$$\begin{cases} du(t) = (Au(t) + F(u(t)))dt + G(u(t))d\omega(t), \\ u(0) = u_0 \in V \end{cases} \tag{5}$$

where ω denotes the infinite dimensional fBm B^H (see (2)).

Assume that A is the infinitesimal generator of an analytic semigroup $S(\cdot)$, and that $F : V \to V$ is Lipschitz continuous with Lipschitz constant L_F, and $G : V \to L(V)$ and $G' : V \to L(V, L(V))$ are Lipschitz continuous in the following senses:

$$\sup_{i\in\mathbb{N}} \|G(v_1)e_i - G(v_2)e_i\| \le L_G\|v_1 - v_2\|, \tag{6}$$

$$\sup_{i\in\mathbb{N}} \|G'(v_1)e_i - G'(v_2)e_i\|_{L(V)} \le L'_G\|v_1 - v_2\|, \tag{7}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is the complete orthonormal basis in V introduced in Section 2.

The solution of (5) on $[0, T]$ is a V-valued process u whose paths are for every $\omega \in \Omega$ elements of $W^{\alpha,1}(0, T; V)$, for an $\alpha \in (1 - H, \frac{1}{2})$, and

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)G(u(s))d\omega, \quad t \in [0, T], \tag{8}$$

where the stochastic integral has to be understood according to (4).

For such an $\alpha \in (1 - H, \frac{1}{2})$, denote by $W^{\alpha,\infty}_{\xi,\sigma}(0, T; V)$ the Banach space of measurable functions $x : [0, T] \to V$ such that

$$\|x\|_{\alpha,\xi,\sigma} = \sup_{t\in[0,T]} e^{-\sigma t}\left(\|x(t)\| + t^\xi \int_0^t \frac{\|x(t) - x(r)\|}{(t-r)^{1+\alpha}}dr\right) < \infty$$

for $\sigma \ge 1$, and $\xi \in [\alpha, 1 - \alpha)$. The role of the factor t^ξ is crucial when proving the following existence theorem, which proof can be found in [6].

Theorem 2 *Let $\alpha \in (1 - H, \frac{1}{2})$, $\sigma \ge 1$ and $\xi \in [\alpha, 1 - \alpha)$. Assume F is Lipschitz continuous, and that G and G' satisfy (6) and (7). Then, for each initial point $u_0 \in V$ there exists a unique solution to equation (8) with its paths in $W^{\alpha,\infty}_{\xi,\sigma}(0, T; V)$. In addition, the mapping $\Phi : V \to W^{\alpha,\infty}_{\xi,\sigma}(0, T; V)$ given by $\Phi : u_0 \mapsto u$ is continuous for $\omega \in \Omega$.*

Theorem 3 *The solution u of (8) defines a random dynamical system $\varphi : \mathbb{R}^+ \times \Omega \times V \to V$, given by*

$$\varphi(t,\omega,u_0) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)G(u(s))d\omega.$$

Proof. The measurability follows by [4] Lemma III.14.

Trivially $\varphi(0,\omega,x) = u_0$. Let us check then the cocycle property: for t, $\tau \in \mathbb{R}^+$, $\omega \in \Omega$ and $u_0 \in V$, we have

$$\begin{aligned}\varphi(t+\tau,\omega,u_0) &= S(t+\tau)u_0 + \int_0^{t+\tau} S(t+\tau-s)F(u(s))ds \\ &+ \int_0^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s) \\ &= S(t)\left(S(\tau)u_0 + \int_0^{\tau} S(\tau-s)F(u(s))ds + \int_0^{\tau} S(\tau-s)G(u(s))d\omega(s)\right) \\ &+ \int_{\tau}^{t+\tau} S(t+\tau-s)F(u(s))ds + \int_{\tau}^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s).\end{aligned}$$

Making the change of variable $s-\tau = r$, applying Lemma 1,

$$\int_{\tau}^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s) = \int_0^t S(t-r)G(u(r+\tau))d\theta_\tau\omega(r),$$

and then, setting $y(s) = u(s+\tau)$, for $s \in [0,t]$,

$$\begin{aligned}\varphi(t+\tau,\omega,u_0) &= S(t)y(0) + \int_0^t S(t-r)F(y(r))dr + \int_0^t S(t-r)G(y(r))d\theta_\tau\omega(r) \\ &= \varphi(t,\theta_\tau\omega,\cdot)\circ\varphi(\tau,\omega,u_0).\end{aligned}$$

□

Proving that our stochastic equation (8) generates a RDS is the starting point to analyze its asymptotic behavior. One possibility, which is a key concept describing the dynamics of RDS generated by fBm-driven SDEs, is the so-called global attractor, which is an invariant compact random set attracting other bounded random sets. The essential dynamics take place in a neighborhood of the attractor (see [8]). Another option to discuss the stability of fBm-driven SDEs is to study the existence of stable and unstable manifolds and Lyapunov exponents, see [10] and [7]. Such smooth manifolds are invariant under the dynamics of the systems, and on them, the states are attracted or repelled by a steady state.

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LYAPUNOV EXPONENTS AND STABILITY IN INTERVAL MAPS

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Abstract

Determination of stability or instability of a given orbit of a scalar interval map is investigated in terms of the sign of the Lyapunov exponent of the orbit. It is proved that an orbit of such a C^2 map with a negative Lyapunov exponent is stable. To prove instability, the classical notion of Lyapunov exponent is strengthened by introducing a new quantity called strong Lyapunov exponent. Then, it is proved that an orbit of a C^1 interval map with a positive strong Lyapunov exponent is unstable, or equivalently, exhibits sensitive dependence on initial conditions. It is also shown that positive Lyapunov exponent suffices if an additional assumption is made about the critical points of the interval map.

Key words: *Interval maps, Lyapunov exponent, stability, strong Lyapunov exponent, sensitive dependence on initial conditions.*
AMS subject classifications: *37C75, 37D45, 37E05.*

1 Introduction

Let us consider a mapping $f : [0, 1] \to [0, 1]$ of the unit interval into itself and a positive orbit $\{x_n\}_{n=0}^{\infty}$ through an initial value $x_0 \in [0, 1]$, where $x_{n+1} = f(x_n)$. A question of paramount interest is the determination of stability or instability of a given such orbit. For the sake of concreteness, we proceed with the definitions of these classical notions.

Definition 1 *(Lyapunov stability) Let $f : [0, 1] \to [0, 1]$ be a mapping of the interval. The positive orbit $\{x_n\}_{n=0}^{\infty}$ through an initial value $x_0 \in [0, 1]$ is said to be Lyapunov stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|y - x_0| < \delta$ then $|f^n(y) - f^n(x_0)| < \varepsilon$ for all $n \geq 0$.*

In recent times, Lyapunov instability, equivalent to sensitive dependence on initial conditions, has played a prominent role in “chaotic” dynamics [1, 2, 5].

Definition 2 *(Sensitive dependence) Let* $f : [0, 1] \to [0, 1]$ *be a mapping of the interval. The positive orbit* $\{x_n\}_{n=0}^{\infty}$ *through an initial value* $x_0 \in [0, 1]$ *exhibits sensitive dependence on initial conditions, if there exists* $\varepsilon > 0$ *such that given any* $\delta > 0$ *there exists* y *with* $|y - x_0| < \delta$ *and* $N > 0$ *such that* $|f^N(y) - f^N(x_0)| \geq \varepsilon$.

The most significant scalar quantity attached to an orbit $\{x_n\}_{n=0}^{\infty}$, that does not include a critical point of the map and for which $f'(x_n)$ exists for $n \geq 0$, is its Lyapunov exponent.

Definition 3 *(Lyapunov exponent) The Lyapunov exponent* $\lambda(x_0)$ *of a positive orbit* $\{x_n\}_{n=0}^{\infty}$ *of an interval map* $f : [0, 1] \to [0, 1]$ *is defined as the number*

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \ln |f'(x_k)| \ ,$$

if the limit exists.

It is a popular practice, especially in experimental dynamics, to associate a positive Lyapunov exponent with instability and a negative Lyapunov exponent with stability of an orbit. However, this practice is without a firm mathematical foundation unless certain restrictions are imposed on a map. Indeed, recently Demir and Koçak [3] have constructed a piecewise linear continuous map of the interval with an orbit which has a positive Lyapunov exponent but the orbit does not exhibit sensitive dependence on initial conditions. They also produced another piecewise linear continuous map of the interval with an orbit which has a negative Lyapunov exponent but the orbit does exhibit sensitive dependence on initial conditions. In this paper we announce three theorems regarding the determination of stability or instability of an orbit of a scalar map from the Lyapunov exponent of the orbit.

2 Summary of Results

The first theorem establishes the stability of an orbit of a C^2-scalar interval map with a negative Lyapunov exponents. This simple differentiability assumption proves sufficient to overcome the difficulty demonstrated by the example of Demir and Koçak [3] referred to above.

Theorem 1 *Suppose* $f : [0,1] \mapsto [0,1]$ *is* C^2*. If an orbit* $\{x_n\}_{n=0}^{\infty}$ *has negative Lyapunov exponent* $\lambda(x_0) < 0$*, then the orbit is Lyapunov stable (in fact it is exponentially stable).*

The proof of this theorem is similar to the proof of the analogous theorem for differential equations which goes back to Lyapunov. For the details, see [4].

Dealing with the pathology exhibited by the second example of Demir and Koçak [3] proved to be more challenging. A simple differentiability assumption does not, in general, appear to be sufficient for a positive Lypaunov exponent to imply sensitive dependence. To obtain a reasonably general result, we were forced to strengthen the notion of Lyapunov exponent.

Definition 4 *The strong Lyapunov exponent of an orbit* $\{x_n\}_{n=0}^{\infty}$ *is defined as the number*

$$\Lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=i}^{i+n-1} \ln|f'(x_k)|,$$

if the limit exists uniformly with respect to i.

Now, with this new notion of strong Lyapunov exponent, we can prove the following result:

Theorem 2 *Suppose* $f : [0,1] \to [0,1]$ *is* C^1. *If an orbit* $\{x_n\}_{n=0}^{\infty}$ *of* f *has a positive strong Lyapunov exponent* $\Lambda(x_0) > 0$, *then the orbit has sensitive dependence on initial conditions.*

The proof of this theorem follows from a more general statement where the assumption that the Lyapunov exponent is uniform is replaced by the weaker assumption that the orbit stays away from a critical point. This stronger theorem is provided by showing that the assumption that a neighbouring orbit always stays nearby leads to a contradiction. For the details, see [4].

The preceding theorem could not, in general, be applied to a chaotic map as such maps usually have critical points and most orbits would be dense and hence come arbitrarily close to critical points and such orbits cannot have strong Lyapunov exponents. In the theorem below we exhibit a class of maps with critical points for which a positive Lyapunov exponent does imply sensitive dependence even for orbits which come arbitrarily close to critical points. This class includes the map $f(x) = 4x(1-x)$.

Theorem 3 *Let* $f : [0,1] \mapsto [0,1]$ *be a* C^2 *map such that* $f'(c) = 0$ *for a unique* c *and such that* $f''(c) \neq 0$ *and there exists* $m > 0$ *such that* $f^m(c) = q$ *is fixed and* $|f'(q)| > 1$. *Then if* $\{x_n\}_{n=0}^{\infty}$ *is a nonconstant orbit of* f *with a positive Lyapunov exponent* $\lambda(x_0) > 0$, *the orbit exhibits sensitive dependence on initial conditions.*

The proof of this theorem is rather more delicate. Problems arise when the orbit goes near a critical point. However then some time later it passes near an expanding fixed point and this forces a nearby orbit to separate. The details of the proof are given in a forthcoming paper [4].

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DISCRETE APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

It is shown how stochastic Itô-Taylor schemes for stochastic ordinary differential equations can be embedded into standard concepts of consistency, stability and convergence. An appropriate choice of function spaces and norms, in particular a stochastic generalization of Spijker's norm (1968), leads to two-sided estimates for the strong error of convergence under the usual assumptions.

Key words: *SODE, stochastic differential equations, Itô-Taylor schemes, discrete approximation, bistability, two-sided error estimates, stochastic Spijker norm*
AMS subject classifications: *65C20, 65C30, 65J15, 65L20, 65L70.*

1 Introduction

The invention of Itô-Taylor schemes was a major breakthrough in numerical analysis of stochastic ordinary differential equations (SODEs). We refer to the pioneering book [7] and the influential monographs [9] and [10].

In this paper we show how the strong convergence theory of these schemes can be embedded into the standard framework of consistency, stability and convergence as it is formulated in abstract terms in the theory of discrete approximations (see [14]). Moreover, by a special choice of norms, namely a stochastic version of the deterministic Spijker norm (see [12],[13],[6, Ch.III.8]), we are able to derive two-sided estimates for the strong convergence error.

While our notion of consistency and (numerical) stability goes back to the work of F. Stummel [14] there already exist other concepts in the literature. One can find notions of consistency and local truncation errors in the books [7, 9, 10]. We refer to [3] for a discussion. Other authors, who have considered the question of stability, are for instance [2, 4].

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To be more precise, we deal with the numerical approximation of $\mathbb{R}^d$-valued stochastic processes X, which satisfy an ordinary Itô stochastic differential equation of the form

$$\begin{aligned} dX(t) &= b^0(t,X(t))dt + \sum_{k=1}^{m} b^k(t,X(t))dW^k(t),\ t \in [0,T], \\ X(0) &= X_0. \end{aligned} \tag{1}$$

We assume that the initial value X_0 has finite second moment. By W^k, $k = 1,\dots,m$, we denote real and pairwise independent standard Brownian motions and we also assume that the drift and diffusion coefficient functions $b^k : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ fulfill the usual global Lipschitz and linear growth conditions such that (1) has a unique solution [1].

Note that the corresponding integral form of the SODE (1) has the representation

$$X(t) = X_0 + \int_0^t b^0(s,X(s))ds + \sum_{k=1}^{m} \int_0^t b^k(s,X(s))dW^k(s), \quad t \in [0,T]. \tag{2}$$

Itô-Taylor schemes are based on an iterated application of Itô's formula on the integrands of (2), provided that all appearing integrals and derivatives exist. Again, we refer to the books [7, 9, 10] for a rigorous derivation.

Let $\mathcal{M}$ be the set of all multi-indices $\alpha = (j_1,\dots,j_l)$, $l \in \mathbb{N}$, $j_i \in \{0,\dots,m\}$, $i = 1,\dots,l$. By $\ell(\alpha) \in \mathbb{N}$ and $n(\alpha) \in \mathbb{N}$ we denote the length of $\alpha \in \mathcal{M}$ and the number of zeros in $\alpha \in \mathcal{M}$ respectively. For $\gamma \in \{\frac{n}{2} :\ n \in \mathbb{N}\}$ consider the finite set of multi-indices (c.f. [7])

$$\mathcal{A}_\gamma = \left\{ \alpha \in \mathcal{M} : 1 \le \ell(\alpha) + n(\alpha) \le 2\gamma \text{ or } \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2} \right\}.$$

For a time grid $0 = t_0 < t_1 < \ldots < t_N = T$ with (for simplicity) equidistant step size $h = \frac{T}{N}$, $N \in \mathbb{N}$, the Itô-Taylor scheme of order γ is given by

$$\begin{aligned} &X_h(t_0) = X_0, \\ &X_h(t_k) = X_h(t_{k-1}) + \textstyle\sum_{\alpha \in \mathcal{A}_\gamma} f_\alpha(t_{k-1}, X_h(t_{k-1})) I_{\alpha,k},\ k \ge 1, \end{aligned} \tag{3}$$

with the iterated (stochastic) integrals

$$I_{\alpha,k} := \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s_1} \cdots \int_{t_{k-1}}^{s_{l-1}} dW^{j_1}(s_l) \ldots dW^{j_l}(s_1), \tag{4}$$

where $\alpha = (j_1,\dots,j_l)$ and $dW^0(s) = ds$. For the same α the coefficient function $f_\alpha : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$f_\alpha(t,x) = (L^{j_1} \cdots L^{j_j} f)(t,x), \tag{5}$$

where $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the projection with respect to the second coordinate, i.e. $f(t,x) = x$, and the L^k are differential operators of the form

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{d} b^{0,i} \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} b^{k,i} b^{k,j} \frac{\partial^2}{\partial x_i \partial x_j},$$

$$L^k = \sum_{j=1}^{d} b^{k,j} \frac{\partial}{\partial x_j}, \quad k = 1, \dots, m.$$

Example 1 *If we choose $\gamma = \frac{1}{2}$ then the set $\mathcal{A}_{\frac{1}{2}}$ just consists of all multi-indices of length* 1*, i.e. $\mathcal{A}_{\frac{1}{2}} = \{(0),(1),\dots(m)\}$, and the coefficient functions f_α simplify to the drift and diffusion coefficient functions of the SODE* (1)*, i.e. $f_{(k)} = b^k$ for $k = 0,\dots,m$. Since $I_{(0),k} = h$ and $I_{(j),k} = W^j(t_k) - W^j(t_{k-1})$, the Itô-Taylor scheme of order $\gamma = \frac{1}{2}$ is the well-known Euler-Maruyama scheme. One also easily checks that the choice $\gamma = 1$ leads to the Milstein method.*

It is well-known (see for example [7, 9, 10]) that the Itô-Taylor scheme of order γ converges at least with order γ in the strong sense, i.e. there exists a constant $C > 0$, independent of the step size h, such that

$$\max_{0 \le i \le N} \left(\mathbb{E} \left(|X(t_i) - X_h(t_i)|^2 \right) \right)^{\frac{1}{2}} \le C h^\gamma, \tag{6}$$

where X is the analytic solution to (1) and X_h denotes the numerical solution. Note that [7, 9, 10] use an even stronger norm, where max occurs inside the expectation. It is an open problem whether our approach can handle this norm as well.

In order to embed the Itô-Taylor scheme into the discrete approximation framework, we will write the equations (3) as $A_h(X_h) = R_h$ with a suitable operator A_h and right-hand side R_h. We use the norm

$$\|Y_h\|_{0,h} = \max_{0 \le i \le N} \|Y_h(t_i)\|_{L^2(\Omega)}, \tag{7}$$

and the following generalization of Spijker's norm

$$\|Y_h\|_{-1,h} = \max_{0 \le i \le N} \|\textstyle\sum_{j=0}^{i} Y_h(t_j)\|_{L^2(\Omega)}. \tag{8}$$

Here $\| \cdot \|_{L^2(\Omega)}$ denotes the L^2-norm of random variables.

The key to our two-sided error estimate is the following bistability inequality

$$C_1 \|A_h(Y_h) - A_h(Z_h)\|_{-1,h} \le \|Y_h - Z_h\|_{0,h} \le C_2 \|A_h(Y_h) - A_h(Z_h)\|_{-1,h}. \tag{9}$$

In the following section we show how the Itô-Taylor scheme fits into the discrete approximation theory. In Section 3 we give a precise formulation of our main result together with all assumptions.

2 Writing Itô-Taylor schemes as discrete approximations

In the discrete approximation theory the concepts of consistency, (numerical) stability and convergence are defined in a very general way. Our notions of bistability and of the local truncation error are directly related to the abstract framework invented by F. Stummel [14]. We present the basic ideas behind Stummel's theory in this section. Simultaneously we embed the Itô-Taylor scheme into the framework.

The starting point of the discrete approximation theory is an equation of the form $A(X) = Y$. Here, the operator $A : E \to F$ is a mapping between two sets E and F. For a given $Y \in F$ our aim is to find a discrete approximation of the solution X. To this end we assume the existence of two sequences of metric spaces $(E_h)_{h\in\mathcal{I}}$ and $(F_h)_{h\in\mathcal{I}}$ and operators $A_h : E_h \to F_h$, $h \in \mathcal{I}$, for some index set $\mathcal{I}$. With the help of two sequences of restriction operators $r_h^E : E \to E_h$ and $r_h^F : F \to F_h$, for $h \in \mathcal{I}$, the discrete spaces E_h and F_h are connected to the original spaces E and F respectively. Figure 1 visualizes the setting.

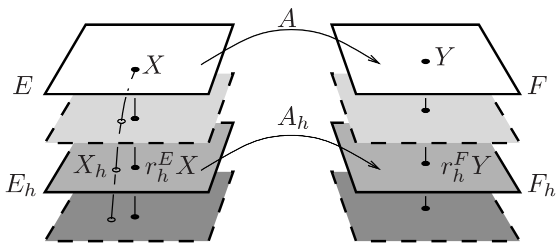

Figure 1: Visualisation of the discrete approximation theory

By solving equations of the form $A_h(X_h) = r_h^F Y$ we obtain a sequence of discrete approximations $(X_h)_{h\in\mathcal{I}}$. Now, the theory of F. Stummel answers the questions, in which sense and under which conditions the sequence $(X_h)_{h\in\mathcal{I}}$ converges to the solution X. Let us first show how the SODE (1) and the Itô-Taylor scheme (3) can be embedded into figure 1.

Since the existence of a unique solution X to (1) is guaranteed by our assumptions we consider the trivial operator

$$A : \begin{array}{l} E \to F \\ X \mapsto A(X) \end{array} \tag{10}$$

where $E := \{X\}$ and $F := \{Y = (X_0, 0)\}$ are singletons (with the second component of Y being the stochastic process which is P-a.s. equal to $0 \in \mathbb{R}^d$) and the operator A is given by

$$A(X) = \Bigg(X(0),$$
$$\Big(X(t) - X(0) - \textstyle\int_0^t b^0(s, X(s))ds - \sum_{k=1}^m \int_0^t b^k(s, X(s))dW^k(s) \Big)_{0\le t\le T} \Bigg).$$

In order to define the discrete metric spaces we denote the time grid by $\tau_h := \{t_i = ih \,|\, i = 0, \ldots, N\}$. As our underlying discrete space we consider the set $\mathcal{G}_h := \mathcal{G}(\tau_h, L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d))$ of all adapted and $L^2(\Omega)$-valued grid functions, that is, for $Z_h \in \mathcal{G}_h$, the random variables $Z_h(t_i)$ are square-integrable and $\mathcal{F}_{t_i}$-measurable random variables for all $t_i \in \tau_h$. Here $(\mathcal{F}_t)_{t\in[0,T]}$ denotes the filtration which is generated by the Wiener processes W^k, $k = 1, \ldots, m$. Now, we choose the metric spaces E_h and F_h to be the vector space $\mathcal{G}_h$ endowed with the metric induced by the norm

$$\|Z_h\|_{0,h} = \max_{0\le i\le N} \|Z_h(t_i)\|_{L^2(\Omega)} \tag{11}$$

and the stochastic version of Spijker's norm

$$\|Z_h\|_{-1,h} = \max_{0\le i\le N} \|\textstyle\sum_{j=0}^{i} Z_h(t_j)\|_{L^2(\Omega)}, \tag{12}$$

respectively. Note that E_h and F_h are Banach spaces.

Next, define the two sequences of restriction operators

$$r_h^E : \begin{array}{l} E \to E_h \\ X \mapsto r_h^E X, \quad [r_h^E X](t_i) = X(t_i) \quad \text{for} \quad t_i \in \tau_h, \end{array} \tag{13}$$

$$r_h^F : \begin{array}{l} F \to F_h \\ Y \mapsto r_h^F Y \end{array} \qquad [r_h^F Y](t_i) = \begin{cases} X_0 & i = 0, \\ 0 & i = 1, \ldots N. \end{cases} \tag{14}$$

Finally, for $h > 0$, we introduce the operator

$$A_h : \begin{array}{l} E_h \to F_h \\ X_h \mapsto A_h(X_h) \end{array}$$

by the relationship

$$\begin{aligned} &[A_h(X_h)](t_0) = X_h(t_0), \\ &[A_h(X_h)](t_i) = X_h(t_i) - X_h(t_{i-1}) - \textstyle\sum_{\alpha\in\mathcal{A}_\gamma} f_\alpha(t_{i-1}, X_h(t_{i-1})) I_{\alpha,i}, \end{aligned} \tag{15}$$

for $1 \le i \le N$. Under the assumption that all Itô-Taylor coefficient functions f_α satisfy a linear growth condition, $[A_h(X_h)](t_i)$ is an adapted and mean-square integrable random variable. Therefore, A_h maps E_h into F_h. See Section 3 for a complete statement of all assumptions.

Since the Itô-Taylor schemes are explicit, the operators A_h are bijective, i.e. there exists a unique solution $\tilde{X}_h$ to the equation $A_h(\tilde{X}_h) = Z_h$ for all $Z_h \in F_h$. In particular, the Itô-Taylor approximation X_h to (1) is equivalently written as the solution to the equation $A_h(X_h) = r_h^F Y$.

Next, we introduce our notion of consistency, bistability and convergence.

Definition 1 *Consider a one-step method given by a sequence of operators $(A_h)_h$. The method is called consistent of order $\gamma > 0$, if there exists a constant $C > 0$ and an upper step size bound $\overline{h} > 0$, such that the estimate*

$$\|A_h(r_h^E X) - r_h^F A(X)\|_{-1,h} \leq Ch^\gamma \tag{16}$$

holds for all grids τ_h with $h \leq \overline{h}$, where X denotes the analytic solution of (1).

The left hand side of (16) is called *local truncation error* or *consistency error*. Therefore, a one-step method is consistent if the diagram in Figure 1 commutes up to an error of order γ, that is $r_h^F \circ A \approx A_h \circ r_h^E$ for h small enough.

The second ingredient in the convergence theory is the concept of (numerical) stability. In [14] F. Stummel introduces the stronger notion of bistability and he proves that bistability of a numerical method can be characterized by the equicontinuity of the operators $(A_h)_h$ and $(A_h^{-1})_h$. In this sense the following definition is a sufficient condition for Stummel's notion of bistability.

Definition 2 *A one-step method defined by operators $(A_h)_h$ is called bistable, if there exist constants $C_1, C_2 > 0$ and an upper step size bound $\overline{h} > 0$ such that the operators A_h are bijective and the estimate*

$$C_1\|A_h(Z_h) - A_h(\tilde{Z}_h)\|_{-1,h} \leq \|Z_h - \tilde{Z}_h\|_{0,h} \leq C_2\|A_h(Z_h) - A_h(\tilde{Z}_h)\|_{-1,h}$$

holds for all $Z_h, \tilde{Z}_h \in E_h$ and for grids τ_h with $h < \overline{h}$.

Finally, we define the error of convergence in terms of the norm $\|\cdot\|_{0,h}$, the space E_h and the restriction operators r_h^E.

Definition 3 *A one-step method is called convergent of order $\gamma > 0$ if there exist an upper step size bound $\overline{h} > 0$ and a constant $C > 0$ such that the corresponding operators A_h are bijective and*

$$\|X_h - r_h^E X\|_0 \leq Ch^\gamma \tag{17}$$

for all $h \leq \overline{h}$. Here X_h denotes the solution to $A_h(X_h) = r_h^F Y$.

3 Main result

In this section we give a precise formulation of the underlying assumptions and our main result.

(A1) The initial value X_0 is an $\mathcal{F}_0$-measurable and $\mathbb{R}^d$-valued random variable satisfying $\mathbb{E}(|X_0|^2) < \infty$.

(A2) For all $\alpha \in \mathcal{A}_\gamma$ there exists a constant $L_\alpha > 0$ such that

$$|f_\alpha(t,x) - f_\alpha(t,y)| \leq L_\alpha|x-y| \quad \text{and} \quad |f_\alpha(t,x)| \leq L_\alpha(1+|x|)$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0,T]$.

(A3) For a given order γ the Itô-Taylor expansion of $X(t)$ with respect to $\mathcal{A}_\gamma$ exists for all $t \in [0,T]$.

(A4) For all $\alpha \in \mathcal{B}(\mathcal{A}_\gamma)$ we have

$$\int_0^T \mathbb{E}\left(|f_\alpha(s, X(s))|^2\right) ds < \infty.$$

The first two assumptions are used, for example, in [1] to assure the existence and uniqueness of the solution X on $[0,T]$, such that $X(t)$ is mean-square integrable for all $t \in [0,T]$. The assumption (A2) also assures that the operators A_h are well-defined and bistable. In (A3) we assume that the Itô-Taylor expansion exists up to a given order γ. Assumption (A4) is needed in order to prove the consistency of the Itô-Taylor schemes. There we use the notation of the remainder set $\mathcal{B}(\mathcal{A}_\gamma)$ of the Itô-Taylor expansion which is given by

$$\mathcal{B}(\mathcal{A}_\gamma) = \{\alpha = (j_1, j_2, \ldots, j_l) \in \mathcal{M} \,:\, (j_2, \ldots, j_l) \in \mathcal{A}_\gamma\} \subset \mathcal{M}$$

(c.f. [7]). Now we formulate our main result, which is proven in [8].

Theorem 1 *Let the assumptions (A1)-(A4) hold for $\gamma \in \{\frac{n}{2} \,|\, n \in \mathbb{N}\}$. Then the Itô-Taylor scheme of order γ is*

(i) consistent of order γ,

(ii) bistable with respect to the norms $\|\cdot\|_{0,h}$ and $\|\cdot\|_{-1,h}$,

(iii) convergent of order γ.

Moreover, there exists $\overline{h} > 0$ such that the two-sided error estimate

$$C_1 \|A_h(r_h^E X) - r_h^F Y\|_{-1,h} \leq \|r_h^E X - X_h\|_{0,h} \leq C_2 \|A_h(r_h^E X) - r_h^F Y\|_{-1,h}$$

holds for all grids τ_h with $|h| \leq \overline{h}$.

Remark 1 *Theorem* 1 *also holds for implicit methods like the stochastic theta method [3] and for stochastic multi-step methods [8].*

Remark 2 *The two-sided error estimate in Theorem* 1 *can be used to discuss the optimal order of convergence of the Itô-Taylor methods. J. M. C. Clark and R. J. Cameron [5] constructed the example*

$$dX(t) = \begin{pmatrix} 1 & 0 \\ 0 & X_1(t) \end{pmatrix} d \begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{18}$$

to show that, in general, the maximum order of convergence is equal to $\frac{1}{2}$ if the numerical method, like the Euler-Maruyama scheme, uses only the increments $W^k(t_i) - W^k(t_{i-1})$ of the driving Wiener processes. For this example the local trunction error of the Euler-Maruyama is exactly computed to be $\sqrt{\frac{1}{2}Th}$. Hence, the strong error of convergence is bounded from below by a term of order $\gamma = \frac{1}{2}$.

A suitable generalization of this example gives corresponding results for the higher order schemes [8].

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THE SUB-SUPERTRAJECTORY METHOD. APPLICATION TO THE NONAUTONOMOUS COMPETITION LOTKA-VOLTERRA MODEL

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Abstract

In this paper we study in detail the pullback and forwards attractions to non-autonomous competition Lotka-Volterra system. In particular, under some conditions on the parameters, we prove the existence of a unique non-degenerate global solution for these models, which attracts any other complete bounded trajectory. For that we present the sub-supertrajectory tool as a generalization of the now classical sub-supersolution method.

Key words: *Sub-supertrajectory method, Lotka-Volterra competition system, attracting complete trajectories.*
AMS subject classifications: *35B40, 35K55, 92D25, 37L05.*

1 Introduction

In this paper we collect some results from [6] and [7] to analyze the asymptotic dynamics of the following non-autonomous Lotka-Volterra competition model

$$\left\{\begin{array}{ll} u_t - \Delta u = u(\lambda(t,x) - a(t,x)u - b(t,x)v) & x \in \Omega,\ t > s \\ v_t - \Delta v = v(\mu(t,x) - c(t,x)u - d(t,x)v) & x \in \Omega,\ t > s \\ u = v = 0 & x \in \partial\Omega,\ t > s \\ u(s) = u_s,\ v(s) = v_s. & \end{array}\right. \tag{1}$$

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Here, u and v represent the population densities of two species within a habitat Ω, a bounded and smooth domain in $\mathbb{R}^N$, $N \geq 1$, which compete in the habitat. λ, μ are the growth rates of the species, b, c are the interaction rates between the species, a, d describe the limiting effects of crowding in each population. We are assuming that Ω is fully surrounded by inhospitable areas, since the population densities are subject to homogeneous Dirichlet boundary conditions. u_s, v_s are regular and positive functions which implies that the solution of (1) satisfies $u, v \geq 0$.

In this work we are interested in determining the asymptotic behaviour of solutions of the system (1). This is a very complicated task, and only partial results are known. For example in the autonomous case (all the coefficients in (1) are constants) and denoting by Λ_0 the principal eigenvalue associated to $-\Delta$, then if λ or $\mu \leq \Lambda_0$, then one of the two species (or both of them) will be driven to extinction. However, there exist two increasing maps $F, G : [\Lambda_0, \infty) \mapsto \mathbb{R}$ such that if

$$\lambda > G(\mu) \quad \text{and} \quad \mu > F(\lambda),$$

then (1) is permanent and moreover there exists a positive equilibrium solution (see Cantrell et al. [2] and López-Gómez [9]).

When non-autonomous terms are allowed in the equations, this is usually done under the assumption of periodicity, quasiperiodicity or almost periodicity, and in this case similar results can be obtained to those for autonomous equations (see Hess [4], Hetzer and Shen [5] and references there in).

Cantrell and Cosner [1] assume general non-autonomous terms that are bounded by periodic functions, and using a comparison method give conditions on λ and μ that guarantee that (1) is permanent.

In [6] we show that, under a smallness condition on the coupling coefficients bc, if there exists a bounded and bounded away from zero complete trajectories of (1), it is the unique such trajectory, and it also describes the unique pullback and forwards attracting for (1), i.e. (u^*, v^*) is a bounded trajectory such that, for any $s \in \mathbb{R}$ and for any positive solution $(u(t,s), v(t,s))$ of (1) defined for $t > s$, one has

$$(u(t,s) - u^*(t), v(t,s) - v^*(t)) \to (0,0) \qquad \text{as} \quad t \to \infty, \text{ or } \quad s \to -\infty. \tag{2}$$

In this work (see [7]) we show that this trajectory really exists. To this end we introduce the sub-supertrajectory method as a tool to get existence of intermediate complete trajectories associated to (1). Note that our construction is independent of whether or not (1) has monotonicity properties. Note also that the usual way in previous works (for instance [6], [11]) to get existence of complete trajectories associated to a particular system is by means of the pullback attractor. The sub-supertrajectory method adopts a different and, in this case, more fruitful strategy. Moreover, we also get the existence of minimal and maximal global bounded trajectories associated to ordered systems.

In Section 2 we present the sub-supertrajectory tool, Section 3 is devoted to the logistic equation which appears when one species is absent. Finally, in Section 4 we show the results of system (1).

2 The sub-supertrajectory method for complete solutions

Consider the general problem

$$\begin{cases} u_t - \Delta u = f(t,x,u,v) & x \in \Omega,\ t > s \\ v_t - \Delta v = g(t,x,u,v) & x \in \Omega,\ t > s \\ u = v = 0 & x \in \partial\Omega,\ t > s \\ u(s) = u_s,\ v(s) = v_s, & \end{cases} \tag{3}$$

where f, g are bounded on bounded sets of $\mathbb{R} \times \overline{\Omega} \times \mathbb{R}^2$ and are locally Hölder continuous in time. We denote the solutions of (3) as

$$u(t,s;u_s,v_s), \quad v(t,s;u_s,v_s), \quad \text{for } t > s.$$

Definition 1 *A pair of functions $(u,v) \in C^{1,2}_{t,x}(\mathbb{R} \times \overline{\Omega})$ is a complete trajectory of (3), if for all $s < t$ in $\mathbb{R}$, $(u(t),v(t))$ is the solution of (3) with initial data $u_s = u(s)$, $v_s = v(s)$.*

Definition 2 *A positive function $u(t,x)$ is non–degenerate at ∞ (respectively $-\infty$) if there exists $t_0 \in \mathbb{R}$ such that u is defined in $[t_0,\infty)$ (respectively $(-\infty,t_0]$) and there exists a $C^1_0(\overline{\Omega})$ function $\varphi_0(x) > 0$ in Ω, such that for all $x \in \overline{\Omega}$, $u(t,x) \geq \varphi_0(x)$ for all $t \geq t_0$ (respectively for all $t \leq t_0$).*

The use of sub-supertrajectory pairs to construct complete solutions can be found in Chueshov [3] or Langa and Suárez [8]. Both references use monotonicity properties of the equations, see Corollaries 2 and 3 below. In particular this applies to scalar equations. Here we use similar ideas to construct bounded complete trajectories, without such monotonicity assumptions.

Given $T_0 \leq \infty$ and two functions $w, z \in C((-\infty,T_0) \times \overline{\Omega})$ with $w \leq z$ we denote

$$[w,z] := \{u \in C((-\infty,T_0) \times \overline{\Omega}) : w \leq u \leq z\}.$$

Now we introduce the concept of complete sub-supertrajectory pair.

Definition 3 *Let $T_0 \leq \infty$ and $(\underline{u},\underline{v}), (\overline{u},\overline{v}) \in \mathcal{X} = C^{1,2}_{t,x}((-\infty,T_0) \times \overline{\Omega})$. We say that $(\underline{u},\underline{v}) - (\overline{u},\overline{v})$ is a complete sub-supertrajectory pair of (3) if*

1. *$\underline{u}(t) \leq \overline{u}(t)$ and $\underline{v}(t) \leq \overline{v}(t)$ in Ω, for all $t < T_0$.*

2. *$\underline{u} \leq 0 \leq \overline{u}$ and $\underline{v} \leq 0 \leq \overline{v}$ on $\partial\Omega$, for all $t < T_0$.*

3. *For all $x \in \Omega$, $t < T_0$*

$$\begin{aligned} &\underline{u}_t - \Delta\underline{u} - f(t,x,\underline{u},v) \leq 0 \leq \overline{u}_t - \Delta\overline{u} - f(t,x,\overline{u},v), \quad \forall v \in [\underline{v},\overline{v}], \\ &\underline{v}_t - \Delta\underline{v} - g(t,x,u,\underline{v}) \leq 0 \leq \overline{v}_t - \Delta\overline{v} - g(t,x,u,\overline{v}), \quad \forall u \in [\underline{u},\overline{u}]. \end{aligned}$$

Note that the concept of a sub-supersolution pair, defined for $t > s$, has been widely used and developed, see e.g. Pao [10], to construct solutions for the initial value problem (3). The main result of this section is:

Theorem 1 *Assume that there exists a complete sub-supertrajectory pair of (3), $(\underline{u},\underline{v})-(\overline{u},\overline{v})$, in the sense of Definition 3. Moreover, assume $\underline{u}$, $\underline{v}$, $\overline{u}$ and $\overline{v}$ are bounded at $-\infty$. Then, there exists a complete trajectory $(u^*,v^*)\in\mathcal{X}$ of (3) such that*

$$(u^*,v^*)\in\mathcal{I}:=[\underline{u},\overline{u}]\times[\underline{v},\overline{v}].$$

When f and g have some monotonicity properties, we can go further:

Corollary 2 *Under the assumptions of Theorem 1, assume moreover that f is increasing in v and g in u. Then, there exist two complete trajectories (u_*,v_*) and (u^*,v^*) of (3) with $(u_*,v_*),(u^*,v^*)\in\mathcal{I}:=[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ such that they are minimal and maximal in $\mathcal{I}$ in the following sense: for any other complete trajectory $(u,v)\in\mathcal{I}$ we have:*

$$\begin{array}{l}\underline{u}(t)\le u_*(t)\le u(t)\le u^*(t)\le\overline{u}(t),\\ \underline{v}(t)\le v_*(t)\le v(t)\le v^*(t)\le\overline{v}(t),\end{array}\quad \text{for all } t<T_0. \tag{4}$$

Corollary 3 *Under the assumptions of Theorem 1, assume moreover that f is decreasing in v and g in u. Then, there exist two complete trajectories (u_*,v^*) and (u^*,v_*) of (3) with $(u_*,v^*),(u^*,v_*)\in\mathcal{I}:=[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ and such that they are minimal-maximal and maximal-minimal in the following sense: for any other complete trajectory $(u,v)\in\mathcal{I}$ we have:*

$$\begin{array}{l}\underline{u}(t)\le u_*(t)\le u(t)\le u^*(t)\le\overline{u}(t),\\ \underline{v}(t)\le v_*(t)\le v(t)\le v^*(t)\le\overline{v}(t),\end{array}\quad \text{for all } t<T_0. \tag{5}$$

3 The non-autonomous logistic equation

Note that (1) always admits semi-trivial trajectories of the form $(u,0)$ or $(0,v)$. In this case, when one species is not present, the other one satisfies the logistic equation

$$\left\{\begin{array}{ll} u_t-\Delta u=h(t,x)u-g(t,x)u^2 & \text{in } \Omega,\ t>s\\ u=0 & \text{on } \partial\Omega,\\ u(s)=u_s\ge 0 & \text{in } \Omega.\end{array}\right. \tag{6}$$

It is well known that if

$$h_M:=\sup_{\overline{Q}}h(t,x)<\infty \quad \text{and} \quad g_L:=\inf_{\overline{Q}}g(t,x)>0, \tag{7}$$

then, for every non-trivial $u_s\in C(\overline{\Omega})$, $u_s\ge 0$, there exists a unique positive solution of (6) denoted by $\Theta_{[h,g]}(t,s;u_s)$.
On the other hand, for $m\in L^\infty(\Omega)$ we denote by $\Lambda(m)$, the first eigenvalue of

$$-\Delta u=\lambda u+m(x)u \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega.$$

In particular, we denote by $\Lambda_0:=\Lambda(0)$. It is well known that $\Lambda(m)$ is a simple eigenvalue with a positive eigenfunction, and a continuous and decreasing function of m.

Finally, for $h, g \in L^\infty(\Omega)$ with $g_L := \inf\{g(x),\ x \in \overline{\Omega}\} > 0$ consider the elliptic equation

$$\begin{cases} -\Delta u = h(x)u - g(x)u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

It is well known that (8) possesses a unique positive solution if, and only if, $\Lambda(h) < 0$, which we denote by $\omega_{[h,g]}(x)$.

In the following result (see [12], [11] and [7] for a complete study of (6)) we show the existence and properties of a complete nonnegative trajectory for (6). For this we will assume henceforth that $h(t,x)$ and $g(t,x)$ satisfy (7) and there exist bounded functions $h_0^\pm(x)$ and $H_0^\pm(x)$ defined in Ω such that

$$\limsup_{t\to\pm\infty} \sup_{x\in\Omega} \Big(h(t,x) - H_0^\pm(x)\Big) \le 0, \quad 0 \le \liminf_{t\to\pm\infty} \inf_{x\in\Omega} \Big(h(t,x) - h_0^\pm(x)\Big). \tag{9}$$

Proposition 4 *Assume (7) and (9). Then:*
i) There exists a maximal bounded complete trajectory, denoted by $\varphi_{[h,g]}(t)$, of (6), in the sense that, for any other non-negative complete bounded trajectory $\xi(t)$ of (6) we have

$$0 \le \xi(t) \le \varphi_{[h,g]}(t), \qquad t \in \mathbb{R}.$$

Moreover, if $\varphi_{[h,g]}(t,x)$ is nondegenerate at $-\infty$ then it is the only one of such solutions.
ii) If $\Lambda(H_0^-) > 0$, then $\varphi_{[h,g]}(t) = 0$ for all $t \in \mathbb{R}$. Therefore all non-negative solutions of (6) converge to 0, uniformly in Ω, in the pullback sense.
iii) If $\Lambda(h_0^-) < 0$ then $\varphi_{[h,g]}$ is the unique complete bounded and non-degenerate trajectory at $-\infty$ of (6), and for t in compact sets of $\mathbb{R}$, if $s \mapsto u_s \ge 0$ is bounded and non-degenerate, then

$$\Theta_{[h,g]}(t,s;u_s) - \varphi_{[h,g]}(t) \to 0 \quad \text{as } s \to -\infty \quad \text{uniformly in } \Omega.$$

iv) If $\Lambda(H_0^+) > 0$, then for all $u_s \in C(\overline{\Omega})$, $u_s \ge 0$, the positive solution of (6) satisfies $\Theta_{[h,g]}(t,s;u_s) \to 0$ uniformly in Ω as $t \to \infty$. In particular, $\varphi_{[h,g]}(t) \to 0$ uniformly in Ω as $t \to \infty$.
v) If $\Lambda(h_0^+) < 0$ and $\varphi_{[h,g]} \ne 0$, then $\varphi_{[h,g]}$ is non-degenerate at ∞ and for any s and any non-trivial initial data $u_s \ge 0$,

$$\Theta_{[h,g]}(t,s;u_s) - \varphi_{[h,g]}(t) \to 0 \quad \text{in } C^1(\overline{\Omega}) \text{ as } t \to \infty.$$

4 Applications to the Lotka-Volterra competition model

We assume from now on that $\lambda, \mu \in \mathbb{R}$ and

$$a_L, d_L, b_L, c_L > 0. \tag{10}$$

We will assume that there exist quantities $a_I^\pm \le a_S^\pm$, $b_I^\pm \le b_S^\pm$, $c_I^\pm \le c_S^\pm$ and $d_I^\pm \le d_S^\pm$ such that

$$\begin{array}{ll} 0 < a_I^\pm \le a(t,x) \le a_S^\pm, & 0 < b_I^\pm \le b(t,x) \le b_S^\pm, \\ 0 < c_I^\pm \le c(t,x) \le c_S^\pm, & 0 < d_I^\pm \le d(t,x) \le d_S^\pm, \end{array} \tag{11}$$

for all $x \in \Omega$ and for all $t \geq t_0$ or $t \leq t_0$. In the following result we show the existence of a complete trajectory of (1).

Proposition 5 (Competitive case) *There exists a complete trajectory (u^*, v^*) of (1) with*

$$\varphi_{[\lambda - b\varphi_{[\mu,d]},a]}(t) \leq u^*(t) \leq \varphi_{[\lambda,a]}(t),\ \varphi_{[\mu - c\varphi_{[\lambda,a]},d]}(t) \leq v^*(t) \leq \varphi_{[\mu,d]}(t), \quad t \in \mathbb{R}.$$

Moreover, if (11) is satisfied for very negative t and

$$\lambda > \Lambda(-b_S^- \omega_{[\mu,d_I^-]}) \qquad \text{and} \qquad \mu > \Lambda(-c_S^- \omega_{[\lambda,a_I^-]}), \tag{12}$$

then (u^*, v^*) is non-degenerate at $-\infty$.

If moreover (11) is satisfied for large and very negative t, (12) and

$$\lambda > \Lambda(-b_S^+ \omega_{[\mu,d_I^+]}) \qquad \text{and} \qquad \mu > \Lambda(-c_S^+ \omega_{[\lambda,a_I^+]}) \tag{13}$$

holds, then (u^*, v^*) is non-degenerate at ∞.

Proof. Note that in this case f is decreasing in v and g in u. It is enough to take

$$(\underline{u}, \overline{u}) = (\varphi_{[\lambda - b\varphi_{[\mu,d]},a]}, \varphi_{[\lambda,a]}) \quad \text{and} \quad (\underline{v}, \overline{v}) = (\varphi_{[\mu - c\varphi_{[\lambda,a]},d]}, \varphi_{[\mu,d]}).$$

Moreover, if λ and μ satisfy (12), resp. (13), then by Proposition 6 we obtain that $\underline{u}$ and $\underline{v}$ are non-degenerate at $-\infty$, resp. $+\infty$. □

Now, we can summarize the results for the system (1).

Theorem 6 (Competitive case)

1. *If $\lambda < \Lambda_0$ and $\mu < \Lambda_0$*

$$\lim_{s \to -\infty} (u(t,s;u_s,v_s), v(t,s;u_s,v_s)) = \lim_{t \to \infty} (u(t,s;u_s,v_s), v(t,s;u_s,v_s)) = (0,0).$$

2. *If $\lambda < \Lambda_0$ and $\mu > \Lambda_0$, then*

$$\lim_{t \to \infty} u(t,s;u_s,v_s) = 0,$$

and for every nonnegative nontrivial $\tilde{v}_s$ we have

$$\lim_{t \to \infty} \big(v(t,s;u_s,v_s) - \Theta_{[\mu,d]}(t,s;\tilde{v}_s)\big) = \lim_{t \to \infty} \big(v(t,s;u_s,v_s) - \varphi_{[\mu,d]}(t)\big) = 0.$$

3. *If $\lambda > \Lambda_0$ and $\mu < \Lambda_0$, then*

$$\lim_{t \to \infty} v(t,s;u_s,v_s) = 0,$$

and for every nonnegative nontrivial $\tilde{v}_s$ we have

$$\lim_{t \to \infty} \big(u(t,s;u_s,v_s) - \Theta_{[\lambda,a]}(t,s;\tilde{v}_s)\big) = \lim_{t \to \infty} \big(u(t,s;u_s,v_s) - \varphi_{[\lambda,a]}(t)\big) = 0.$$

4. If

$$\lambda > \Lambda(-b_S^- \omega_{[\mu, d_I^-]}) \qquad and \qquad \mu > \Lambda(-c_S^- \omega_{[\lambda, a_I^-]}), \tag{14}$$

there exists a complete bounded non-degenerate at $-\infty$ trajectory of (1) $(u^*(t), v^*(t))$. Moreover, if b or c are small at $-\infty$, that is,

$$\limsup_{t\to-\infty} \|b\|_{L^\infty(\Omega)} \limsup_{t\to-\infty} \|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$, then this is the unique bounded non-degenerate at $-\infty$ trajectory of (1) and it is pullback attracting, that is

$$\lim_{s\to-\infty} (u(t,s;u_s,v_s) - u^*(s), v(t,s;u_s,v_s) - v^*(s)) = (0,0).$$

If moreover

$$\lambda > \Lambda(-b_S^+ \omega_{[\mu, d_I^+]}) \qquad and \qquad \mu > \Lambda(-c_S^+ \omega_{[\lambda, a_I^+]}), \tag{15}$$

then $(u(t,s;u_s,v_s), v(t,s;u_s,v_s))$ is non-degenerate at ∞. If additionally b or c are small at ∞, that is,

$$\limsup_{t\to\infty} \|b\|_{L^\infty(\Omega)} \limsup_{t\to\infty} \|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$, then all solutions of (1) have the same asymptotic behavior as $t \to \infty$. If (14) is also satisfied, then $(u^*(t), v^*(t))$ is non-degenerate at ∞ and it is also forwards attracting, that is,

$$\lim_{t\to\infty} (u(t,s;u_s,v_s) - u^*(t), v(t,s;u_s,v_s) - v^*(t)) = (0,0).$$

Remark 1 *Similar results can be presented for the prey-predator and symbiosis cases.*

In Figure 1 we describe the asymptotic dynamical regimes (pullback -Case a)- and forwards -Case b)) when λ and μ are constant functions. Region A: extinction of both species; Regions B and C: stability of semitrivial complete trajectories; Regions D_P and D_F: permanence regions (existence of global non-degenerate global solutions). The limiting curves are given in (14) and (15).

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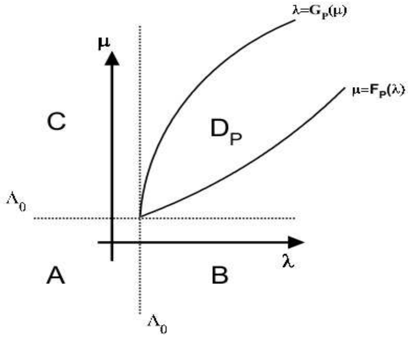
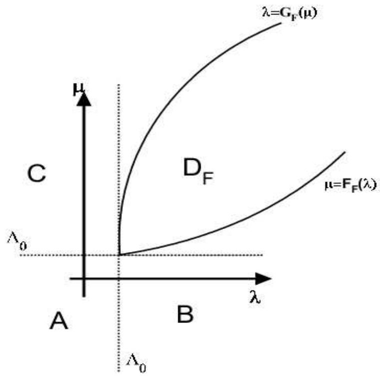


Case a)



Case b)

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NONLOCAL STOCHASTIC DIFFERENTIAL EQUATIONS: EXISTENCE AND UNIQUENESS OF SOLUTIONS

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Abstract

The focus of interest is the existence of strong solutions to stochastic functional differential equations which are not restricted to pathwise dependencies. Indeed, the evolution of the wanted random process may be prescribed by its current features being nonlocal with respect to the probability space — like the expected value and second moments. This result is concluded from Cauchy-Lipschitz Theorem for mutational equations (a form of generalized ODEs beyond vector spaces) and a new aspect of weakening their a priori requirements.

Key words: *Mutational equations, differential equations beyond metric spaces, dynamical systems with feedback, stochastic functional differential equations.*
AMS subject classifications: *60H10, 34A12, 34G99, 54H20, 54E50.*

1 Introduction

Let $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$ be a complete probability space with a filtration $(\mathcal{A}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{A}_0$ contains all P null sets in $\mathcal{A}$). $W = (W_t)_{t\geq 0}$ is a Wiener process on $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$. Set

$$E_{\mathcal{A}} := \left\{ (t, X) \;\middle|\; t \geq 0,\;\; X : \Omega \longrightarrow \mathbb{R} \text{ is } \mathcal{A}_t\text{–measurable},\;\; \mathbb{E}(|X|^2) < \infty \right\}.$$

$W^{1,\infty}(\mathbb{R})$ denotes the Sobolev space of bounded Lebesgue measurable functions $\mathbb{R} \longrightarrow \mathbb{R}$ whose weak derivative is also represented by a function in $L^\infty(\mathbb{R})$. The main result about stochastic differential equations (SDEs) states:

Theorem 1 *Suppose $f = (f_1, f_2) : E_{\mathcal{A}} \longrightarrow W^{1,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ to satisfy*

(i) $\sup_{(t,Y)\in E_{\mathcal{A}}} \left\| f(t,Y) \right\|_{W^{1,\infty}(\mathbb{R})} < \infty$,

(ii) for each $R > 0$, there are $L_R \geq 0$ and a modulus of continuity $\omega_R(\cdot)$ such that for all $(t_i, Y_i) \in E_{\mathcal{A}}$ with $|t_i| + \mathbb{E}(|Y_i|^2) \leq R$,

$$\left\| f(t_1, Y_1) - f(t_2, Y_2) \right\|^2_{L^\infty(\mathbb{R})} \;\leq\; L_R \cdot \mathbb{E}(|Y_1 - Y_2|^2) \;+\; \omega_R(|t_1 - t_2|).$$

Then for each initial $(0, X_0) \in E_{\mathcal{A}}$ and $T \in \,]0, \infty[$, there exists a unique curve

$[0,T] \longrightarrow E_{\mathcal{A}},\ t \longmapsto (t, X_t)$ *with* $(X_t)_{0 \le t \le T}$ *being a strong solution of*

$$d\,X_t(\omega) \;=\; f_1\big(t, X_t\big)\,(X_t(\omega))\;dt \;+\; f_2\big(t, X_t\big)\,(X_t(\omega))\;dW_t(\omega).$$

This type of initial value problem differs from what is usually investigated as a "stochastic functional differential equation" (as e.g. in [3, 7]) because its right-hand side can depend on nonlocal features of $X_t : \Omega \longrightarrow \mathbb{R}$ (instead of the more popular pathwise dependence). Indeed, the example

$$\begin{aligned} d\,X_t(\omega) \;=\; & \quad g_1\big(t,\ \mathbb{E}(X_t),\ \mathbb{E}(|X_t|^2)\big)\ \cdot\ \big(g_2(t) + g_3(X_t(\omega))\big)\;dt\ + \\ & + h_1\big(t,\ \mathbb{E}(X_t),\ \mathbb{E}(|X_t|^2)\big)\ \cdot\ \big(h_2(t) + h_3(X_t(\omega))\big)\;dW_t(\omega) \end{aligned}$$

with bounded and Lipschitz functions $g_1, h_1 : \mathbb{R}^3 \longrightarrow \mathbb{R},\ g_2, g_3, h_2, h_3 : \mathbb{R} \longrightarrow \mathbb{R}$ fulfils these assumptions $\big($with $f_1(t,Y) := g_1\big(t,\ \mathbb{E}(Y),\ \mathbb{E}(|Y|^2)\big) \cdot \big(g_2(t) + g_3(\cdot)\big)\big)$.

Theorem 1 can be easily extended to systems and finds its applications e.g. in dynamic cooperative games (with lacking or uncertain information about others). Moreover, it can be verified for Lipschitz coefficients of uniformly linear growth via $\|a_1 - a_2\| := \sup_{\mathbb{R}} \frac{a_1 - a_2}{1 + |\cdot|}$ instead of $\|a_1 - a_2\|_{L^\infty(\mathbb{R})}$ (as in [6, § 3.5]).

It results from the Cauchy-Lipschitz Theorem for mutational equations. Aubin introduced them for generalizing ordinary differential equations to metric spaces without linear structure [1, 2]. By means of conceptual extensions even beyond metric spaces, the author investigated such SDEs with *additive* noise in [6, § 3.5]. A very brief survey of the general theory is given in § 2 below. Now a new analytical trick (presented here in § 3, Proposition 4) enables us to weaken the a priori requirements for mutational equations and thus, it makes the restriction to additive noise redundant. In a word, the proof of Theorem 1 consists in the general Theorem 3 with Remark 2 below, the rather technical Proposition 4 and the particular preparations in Example 1.

2 Mutational equations beyond metric spaces: A survey

The main goal of mutational equations is to extend ordinary differential equations beyond vector spaces. As any linear structure is lacking, we still rely on the notion of first-order approximation, but use now a class of homotopies (instead of affine-linear maps) for comparing. The first essential questions focus on the distance functions and the additional properties of the homotopies which are to guarantee that Euler method provides solutions to initial value problems.

In [6, Ch. 3], mutational equations are presented beyond (pseudo-) metric spaces for the first time. The distance functions d, e are assumed to satisfy some conditions of continuity instead of the triangle inequality. Here we summarize some of the main results for a special case of distance functions and, all the proofs are given in [6, §§ 3.1 – 3.4].

General assumptions and notations for § 2.

(H1) $E \neq \emptyset$ is a set and, $d, e : E \times E \longrightarrow [0, \infty[$ are reflexive and symmetric. $\lfloor \cdot \rfloor : E \longrightarrow [0, \infty[$ is sequentially lower semicontinuous w.r.t. d.

(H2) There exist a metric $m_0 : E \times E \longrightarrow [0, \infty[$ on E and positive constants $C_1, C_2, C_3, C_4,\ p, q$ with $C_1 \cdot m_0^p \le d \le C_2 \cdot m_0^p$, $C_3 \cdot m_0^q \le e \le C_4 \cdot m_0^q$ such that d is continuous and e lower semicontinuous w.r.t. m_0.

(H3) A set $\Theta\big(E, d, e, \lfloor\cdot\rfloor\big) \neq \emptyset$ of so-called *transitions* $[0,1] \times E \longrightarrow E$ on the tuple $\big(E, d, e, \lfloor\cdot\rfloor\big)$ is given, i.e. by definition, each $\vartheta \in \Theta\big(E, d, e, \lfloor\cdot\rfloor\big)$ satisfies

1.) for every $x \in E :\quad \vartheta(0, x) = x$

2.) $\exists$ nondecreasing $\alpha(\vartheta; \cdot) : [0, \infty[\longrightarrow [0, \infty[:$ for any $x, y \in E$ $\big(\lfloor x\rfloor, \lfloor y\rfloor \le r\big)$,
$$\limsup_{h \downarrow 0} \frac{d(\vartheta(h,x),\ \vartheta(h,y)) - d(x,y)}{h} \le \alpha(\vartheta; r) \cdot d(x, y)$$

3.) $\exists$ nondecreasing $\beta(\vartheta; \cdot) : [0, \infty[\longrightarrow [0, \infty[:$ for any $s, t \in [0,1]$, x $\big(\lfloor x\rfloor \le r\big)$,
$$e\big(\vartheta(s, x),\ \vartheta(t, x)\big) \le \beta(\vartheta; r) \cdot |t - s|$$

4.) $\exists\, \gamma(\vartheta) \in [0, \infty[:\quad \lfloor\vartheta(t, x)\rfloor \le \big(\lfloor x\rfloor + \gamma(\vartheta)\ t\big) \cdot e^{\gamma(\vartheta)\, t}$ for any $t,\ x$,

(H4) $D : \Theta\big(E, d, e, \lfloor\cdot\rfloor\big) \times \Theta\big(E, d, e, \lfloor\cdot\rfloor\big) \times [0, \infty[\longrightarrow [0, \infty[$ fulfils for each $r \ge 0$

1.) $D(\,\cdot\,,\ \cdot\,;\ r)$ is reflexive and symmetric,

2.) $D(\,\cdot\,,\ \cdot\,;\ r)$ is sequentially continuous w.r.t. $\{D(\cdot, \cdot\,; R) \mid R \ge 0\}$,

3.) $D(\vartheta,\ \tau;\ \cdot\,)$ is nondecreasing for any ϑ, τ,

4.) $\displaystyle\limsup_{h \downarrow 0} \frac{d\big(\vartheta(t_1{+}h, x),\ \tau(t_2{+}h, y)\big) - d(\vartheta(t_1, x),\ \tau(t_2, y)) \cdot e^{\alpha(\tau; R) \cdot h}}{h} \le D(\vartheta, \tau; R)$

for any $\vartheta, \tau \in \Theta\big(E, d, e, \lfloor\cdot\rfloor\big)$, $x, y \in E$, $t_1, t_2 \in [0, 1[$ with $\lfloor x\rfloor$, $\lfloor y\rfloor \le r$ and $R := \big(r + \max\{\gamma(\vartheta), \gamma(\tau)\}\big) \cdot e^{\max\{\gamma(\vartheta), \gamma(\tau)\}}$.

Both the parameter α and the "distance" D between transitions are based on local information (w.r.t. time tending to 0), but they lay the basis for estimating the distance between two points evolving along two transitions – via Gronwall.

Proposition 2 ([6, Proposition 3.7]) *Let* $\vartheta, \tau \in \Theta\big(E, d, e, \lfloor\cdot\rfloor\big)$, $r \ge 0$ *and* $t_1, t_2 \in [0, 1[$ *be arbitrary. For any* $x, y \in E$ *suppose* $\lfloor x\rfloor \le r$, $\lfloor y\rfloor \le r$ *and set* $R := \big(r + \max\{\gamma(\vartheta),\ \gamma(\tau)\}\big) \cdot e^{\max\{\gamma(\vartheta),\ \gamma(\tau)\}} < \infty$.

Then the following estimate holds for each $h \in [0, 1[$ *with* $\max\{t_1{+}h,\ t_2{+}h\} \le 1$
$$d\big(\vartheta(t_1{+}h, x),\ \tau(t_2{+}h, y)\big) \le \Big(d\big(\vartheta(t_1, x),\ \tau(t_2, y)\big) + h \cdot D(\vartheta, \tau\,; R)\Big)\ e^{\alpha(\tau;\, R)\, h}.$$

The so-called mutation of a curve $x(\cdot) : [0, T] \longrightarrow E$ is the counterpart of the time derivative and, its definition reflects the notion of first-order approximation (for $h \downarrow 0$) in connection with the preceding structural inequality.

Definition 1 *Consider a curve* $x(\cdot) : [0, T] \longrightarrow E$ *with* $\sup \lfloor x(\cdot)\rfloor < \infty$. *The so-called* mutation *of* $x(\cdot)$ *at time* $t \in [0, T[$ *is defined as*
$$\overset{\circ}{x}(t) := \Big\{\vartheta \in \Theta\big(E, d, e, \lfloor\cdot\rfloor\big) \;\Big|\; \text{for each } R \ge \sup \lfloor x(\cdot)\rfloor, \text{ there is } \alpha_R > 0 \text{ s.t. for all } \tau \in \Theta\big(E, d, e, \lfloor\cdot\rfloor\big),\ y \in E,\ s \in [0, 1[\text{ with } \lfloor\tau(\cdot, y)\rfloor \le R : \limsup_{h \downarrow 0} \frac{d(x(t+h),\ \tau(s+h, y)) - d(x(t),\ \tau(s, y)) \cdot e^{\alpha_R \cdot h}}{h} \le D\big(\vartheta,\ \tau;\ R\big)\Big\}.$$

Remark 1 *If the set E has a separate real component indicating the respective time (as in Example 1 about SDEs below), then we can restrict all quantitative comparisons to "simultaneous" states $\tau(s,y)$, $x(t) \in E$. The resulting set of approximating transitions does not have to be identical to the mutation in Def. 1, but the relevant conclusions (in proofs of existence etc.) do not change [6, § 3.4].*

Definition 2 *Let a function $f : E \times [0,T] \longrightarrow \Theta\big(E, d, e, \lfloor\cdot\rfloor\big)$ be given. A curve $x(\cdot) : [0,T] \longrightarrow E$ is called a* solution *to the mutational equation $\overset{\circ}{x}(\cdot) \ni f\big(x(\cdot), \cdot\big)$ in $\big(E, d, e, \lfloor\cdot\rfloor, D\big)$ if it satisfies:*

1.) *$x(\cdot)$ is continuous with respect to e and bounded with respect to $\lfloor\cdot\rfloor$,*
2.) *for Lebesgue-almost every $t \in [0,T[$: $\; f(x(t),t) \in \overset{\circ}{x}(t)$,*

These terms do not use any linear structure explicitly and, they enable us to formulate the initial value problem for mutational equations. In particular, Peano's Theorem about existence of solutions (due to continuity and suitable compactness) has an analogue [6, § 3.3.3]. In regard to the SDEs, we present the counterpart of Cauchy-Lipschitz Theorem [6, Theorem 3.31] and the conclusion about uniqueness in [6, Proposition 3.11].

Definition 3 ([6, Definition 3.16]**)** *For any initial element $x_0 \in E$, time $T \in]0,\infty[$ and bounds $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} > 0$, let $\mathcal{N} = \mathcal{N}(x_0, T, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$ denote the (possibly empty) subset of all "Euler curves" $y(\cdot) : [0,T] \longrightarrow E$ constructed in the following piecewise way: Choosing any equidistant partition $0 = t_0 < t_1 < \ldots < t_n = T$ of $[0,T]$ (with $n > T$) and $\vartheta_1 \ldots \vartheta_n \in \Theta(E, d, e, \lfloor\cdot\rfloor)$ with*

$$\begin{cases} \sup_k \; \gamma\big(\vartheta_k\big) & \le \widehat{\gamma} \\ \sup_k \; \alpha\big(\vartheta_k;\, (\lfloor x_0\rfloor + \widehat{\gamma}\, T)\, e^{\widehat{\gamma}\, T}\big) & \le \widehat{\alpha}, \\ \sup_k \; \beta\big(\vartheta_k;\, (\lfloor x_0\rfloor + \widehat{\gamma}\, T)\, e^{\widehat{\gamma}\, T}\big) & \le \widehat{\beta}, \end{cases}$$

define $y(\cdot) : [0,T] \longrightarrow E$ as $y(0) := x_0$ and

$$y(t) := \vartheta_k\,(t - t_{k-1},\; y(t_{k-1})) \quad \text{for } \; t \in]t_{k-1}, t_k],\; k = 1, 2, \ldots, n.$$

It is related to piecewise constant $\vartheta(\cdot) : [0,T] \longrightarrow \Theta(E, d, e, \lfloor\cdot\rfloor)$ defined as $\vartheta(t) := \vartheta_j$ for $t \in [t_{j-1}, t_j[$ $\;(j = 1 \ldots n)$.

The tuple $(E, d, e, \lfloor\cdot\rfloor, \Theta)$ is called Euler equi-continuous *if for any $x_0 \in E$, $T \in]0,\infty[$, $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} > 0$, there exists a constant $L \in [0,\infty[$ such that every curve $y(\cdot) \in \mathcal{N}(x_0, T, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$ is L-Lipschitz continuous with respect to e.*

Theorem 3 (Extended Cauchy-Lipschitz Theorem, [6, § 3.3.7]**)** *Suppose the metric space (E, m_0) to be complete and the tuple $\big(E, d, e, \lfloor\cdot\rfloor,\; \Theta\big(E, d, e, \lfloor\cdot\rfloor\big)\big)$ to be Euler equi-continuous. For $f : E \times [0,T] \longrightarrow \Theta\big(E,\; d,\; e,\; \lfloor\cdot\rfloor\big)$ assume*

(1.) For each $R > 0$,

$$\begin{aligned} \widehat{\alpha}(R) &:= \sup_{x,t} \; \alpha(f(x,t);\; R) < \infty, \\ \widehat{\beta}(R) &:= \sup_{x,t} \; \beta(f(x,t);\; R) < \infty, \\ \widehat{\gamma} \quad &:= \sup_{x,t} \; \gamma(f(x,t)) \quad\;\; < \infty, \end{aligned}$$

(2.) *f is Lipschitz continuous w.r.t. state in the following sense: for each $r \geq 0$, there exist constants $\Lambda, \mu > 0$ and a modulus of continuity $\omega(\cdot)$ such that* $\delta : E \times E \longrightarrow [0, \infty[,\ (x,y) \longmapsto \inf \big\{ d(x,z) + \mu \cdot e(z,y) \;\big|\; z \in E,\ \lfloor z \rfloor \leq r \big\}$ *satisfies* $\quad D\big(f(x,s),\ f(y,t);\ r\big) \;\leq\; \Lambda \cdot \delta(x,y) \,+\, \omega(|t-s|)$ *whenever* $(x,s),\ (y,t) \in E \times [0,T]$ *fulfil* $\max\big\{\lfloor x \rfloor,\ \lfloor y \rfloor\big\} \leq r$.

Then for every initial element $x_0 \in E$, there exists a solution $x(\cdot) : [0,T] \longrightarrow E$ to the mutational equation $\overset{\circ}{x}(\cdot) \ni f\big(x(\cdot), \cdot\big)$ with $x(0) = x_0$.
Any other solution $y(\cdot)$ to this initial value problem satisfies $\delta\big(x(\cdot), y(\cdot)\big) \equiv 0$.

Remark 2 *In the special case of square-integrable random variables on $\mathbb{R}$, we prefer the square deviation for $d = e$ to the L^2 norm (see details in Example 1 below). Then it is sufficient to make the Lipschitz assumption w.r.t. $\mathbb{E}(|\cdot - \cdot|^2)$ instead of the auxiliary distance δ because for all square-integrable X, Y, Z,*

$$\mathbb{E}(|X-Z|^2) + \mathbb{E}(|Z-Y|^2) \;\geq\; \tfrac{1}{2}\, \mathbb{E}(|X-Y|^2).$$

3 The new aspect of weakening a priori assumptions

The definition of transition in (H3) implies the restriction that the initial distance between two points may grow (at most) exponentially while evolving along the same transition ϑ, i.e. for any $x, y \in E$ and $h \in [0,1]$,

$$d\,(\vartheta(h,x),\ \vartheta(h,y)) \;\leq\; d(x,y) \,\cdot\, e^{\alpha\, h}$$

with a constant $\alpha \in [0, \infty[$ depending on ϑ and $\max\{\lfloor x \rfloor, \lfloor y \rfloor\} < \infty$. The key goal of this section is some way out if the candidates for transitions only satisfy

$$d\,(\vartheta(h,x),\ \vartheta(h,y)) \;\leq\; C \,\cdot\, d(x,y) \,\cdot\, e^{\alpha\, h}$$

with a constant $C > 1$. In a very broad sense, we apply the same notion as for the step from Hille-Yosida Theorem (about contractive C^0 semigroups) to the Theorem of Feller, Miyadera and Phillips (about arbitrary C^0 semigroups) (see e.g. [4, Theorem II.3.8]). Indeed, we introduce a suitable auxiliary distance $\widehat{d}$ being "equivalent" to d, but beyond vector spaces now, there is no linear resolvent operator available as in the standard proof of the Theorem of Feller et al.

General assumptions and notations for § 3.

(A1) $\check{\Theta}(E, d, e, \lfloor\cdot\rfloor)$ is a nonempty set of functions $\vartheta : [0,1] \times E \longrightarrow E$ satisfying

(1.) for every $x \in E$: $\quad \vartheta(0,x) = x$

(3.) there is $\beta(\vartheta;\cdot) : [0,\infty[\longrightarrow [0,\infty[$ such that for any $s, t \in [0,1]$, $x \in E$ with $\lfloor x \rfloor \leq r$: $\quad e\big(\vartheta(s,x),\ \vartheta(t,x)\big) \;\leq\; \beta(\vartheta; r) \,\cdot\, |t-s|$

(4.) there is $\widehat{\gamma} \in [0, \infty[$ (not depending on ϑ) such that for any $t \in [0,1]$ and $x \in E$: $\quad \lfloor \vartheta(t,x) \rfloor \;\leq\; \big(\lfloor x \rfloor \,+\, \widehat{\gamma}\; t\big) \,\cdot\, e^{\widehat{\gamma}\, t}$

Moreover, a parameter function $\alpha : \check{\Theta}(E, d, e, \lfloor\cdot\rfloor) \times [0, \infty[\longrightarrow [0, \infty[$ is nondecreasing w.r.t. its second argument. (Its purpose is clarified in (A4) below.)

(A2) For any initial element $x_0 \in E$, time $T \in]0,\infty[$ and bounds $\widehat{\alpha}, \widehat{\beta} > 0$, $\mathcal{N} = \mathcal{N}(x_0, T, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$ consists of all "Euler curves" related to functions in $\check{\Theta}(E, d, e, \lfloor\cdot\rfloor)$ as in Definition 3 (but with the global bound $\widehat{\gamma}$ instead of $\gamma(\vartheta_k)$).

(A3) $D : \check{\Theta}(E, d, e, \lfloor\cdot\rfloor) \times \check{\Theta}(E, d, e, \lfloor\cdot\rfloor) \times [0,\infty[\longrightarrow [0,\infty[$ satisfies (H4) (1)–(3).

(A4) There is a nondecreasing function $\check{C} : [0,\infty[\longrightarrow]0,\infty[$ satisfying: Choose the bounds $\widehat{\alpha}, \widehat{\beta}, R, T > 0$ and initial points $x_0, y_0 \in E$ arbitrarily with $\max\{\lfloor x_0\rfloor, \lfloor y_0\rfloor\} < R$ and set $\rho(t) := (R + \widehat{\gamma}\, t)\, e^{\widehat{\gamma}\, t}$. Then any curves $x(\cdot) \in \mathcal{N}(x_0, T, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$, $y(\cdot) \in \mathcal{N}(y_0, T, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$ and the related piecewise constant functions $\vartheta, \tau : [0, T] \longrightarrow \check{\Theta}(E, d, e, \lfloor\cdot\rfloor)$ (as in Definition 3) fulfil

$$d\big(x(T), y(T)\big) \le \Big(\check{C}(0) \cdot d(x_0, y_0) + \check{C}(T) \cdot \int_0^T D\,(\vartheta(s), \tau(s); \rho(s)) \cdot e^{-\check{\alpha}_\rho(s)}\, ds\Big) \cdot e^{\check{\alpha}_\rho(T)}$$

with the abbreviation $\check{\alpha}_\rho(t) := \displaystyle\int_0^t \alpha(\tau(s); \rho(s))\, ds.$

In comparison with the summary in § 2, the essential new aspect is specified in assumption (A4). Indeed, the details about $\alpha(\vartheta; \cdot)$ and $D(\cdot, \cdot; r)$ are now reduced and, we *assume* the structural inequality (of Prop. 2) with three modifications:

(i) the initial error is now multiplied by a constant $\check{C}(0)$ (possibly > 1),
(ii) we suppose this modified inequality for all "Euler curves" related to *piecewise constant* curves in $\check{\Theta}(E, d, e, \lfloor\cdot\rfloor)$ in a finite time interval $[0, T]$,
(iii) there is an additional factor $e^{-\check{\alpha}_\rho(s)}$ in the integral – for technical reasons, but this is no severe restriction because we can usually adapt $\check{C}(T)$.

As $T > 0$ is arbitrary, restrictions imply immediately that the estimate in (A4) holds at even *every* point of time in $[0, T]$.

Example 1 *Let $(\Omega, \mathcal{A}, P)$ be a probability space. $W = (W_t)_{t\ge 0}$ is a Wiener process and, $(\mathcal{A}_t)_{t\ge 0}$ denotes an increasing family of sub–σ–algebras of $\mathcal{A}$ such that for all $0 \le s \le t$, W_t is $\mathcal{A}_t$-measurable with $\mathbb{E}(W_t - W_s \,|\, \mathcal{A}_s) = 0$, $\mathbb{E}(W_t \,|\, \mathcal{A}_0) = 0$ almost surely.*
Consider the stochastic differential equation $d\,X_t = a(X_t)\, dt + b(X_t)\, dW_t$ with Λ-Lipschitz continuous coefficients $a, b : \mathbb{R} \longrightarrow \mathbb{R}$. Then for every initial $\mathcal{A}_0$–measurable $X_0 : \Omega \longrightarrow \mathbb{R}$ with $\mathbb{E}\big(|X_0|^2\big) < \infty$, there exists a pathwise unique strong solution $(X_t)_{0\le t\le T}$ on $[0, T]$ with $\sup_{0\le t\le T} \mathbb{E}\big(|X_t|^2\big) < \infty$.
Moreover, at every time $t \in [0, T]$, it fulfils following estimates with constants C_1, C_2, C_3 depending only on $|a(0)|, |b(0)|, \Lambda$

$$\begin{aligned} \mathbb{E}\big(|X_t|^2\big) &\le \big(\mathbb{E}\big(|X_0|^2\big) + C_2\, t\big)\; e^{C_1\, t} \\ \mathbb{E}\big(|X_t - X_0|^2\big) &\le C_3\,(1 + T)\, \big(\mathbb{E}\big(|X_0|^2\big) + 1\big)\; e^{C_1\, t} \cdot t\,. \end{aligned}$$

[5, Theorems 4.5.3, 4.5.4]. This observation lays the foundations for choosing

these strong solutions (parametrized by the two Λ-Lipschitz coefficients a, b) as candidates for transitions. The aspect of suitable $\mathcal{A}_t$-measurability motivates us to take time t as additional real component into consideration (as in [6, § 3.5.3]):

$$\begin{array}{llll} E_{\mathcal{A}} := \big\{(t, X) \;\big|\; t \geq 0,\;\; X : \Omega \longrightarrow \mathbb{R} \textit{ is } \mathcal{A}_t\text{–}\textit{measurable},\;\; \mathbb{E}(|X|^2) < \infty \,\big\}, \\ d, e: \quad E_{\mathcal{A}} \times E_{\mathcal{A}} \longrightarrow [0, \infty[, & \big((s, X),\; (t, Y)\big) \longmapsto |t - s| + \mathbb{E}\big(|X - Y|^2\big) \\ \lfloor\cdot\rfloor: \quad E_{\mathcal{A}} \longrightarrow [0, \infty[, & (t, X) \longmapsto |t| + \mathbb{E}\big(|X|^2\big), \\ \vartheta_{a,b}: \quad [0,1] \times E_{\mathcal{A}} \longrightarrow E_{\mathcal{A}}, & \big(h,\; (t_0, Y_0)\big) \longmapsto (t_0 + h,\; X_{t_0+h}) \end{array}$$

with $(X_t)_{t \geq t_0}$ denoting here the strong solution of $d\,X_t = a(X_t)\,dt + b(X_t)\,dW_t$ and $X_{t_0} = Y_0$. Euler equi-continuity is ensured (due to [5, § 4.5]).
In regard to assumption (A4), we consider two nonautonomous *stochastic differential equations whose coefficients $\widetilde{a}_i, \widetilde{b}_i : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ $(i = 1, 2)$ are piecewise constant w.r.t. time and Λ-Lipschitz continuous w.r.t. the second argument. Then the corresponding strong solutions $\widetilde{X}^{(1)}, \widetilde{X}^{(2)}$ are known to exist pathwise uniquely [5, § 4.5] and, they satisfy*

$$\begin{aligned} &\mathbb{E}\big(|\widetilde{X}^{(1)}_T - \widetilde{X}^{(2)}_T|^2\big) \;\leq \\ &\leq\; e^{C_5\,(1+T)\,T} \cdot \Big(3\,\mathbb{E}\big(|\widetilde{X}^{(1)}_0 - \widetilde{X}^{(2)}_0|^2\big) + \\ &\qquad + C_4\,(1+T) \cdot \int_0^T \Big(\big\|\widetilde{a}_1(s,\cdot) - \widetilde{a}_2(s,\cdot)\big\|^2_{\sup} + \big\|\widetilde{b}_1(s,\cdot) - \widetilde{b}_2(s,\cdot)\big\|^2_{\sup}\Big)\, ds\Big) \end{aligned}$$

due to Gronwall's inequality (see the proof of [5, Theorem 4.5.6, page 139 f.]). The suitable choice of scaling factors implies assumption (A4).

Now we bridge the gap between functions in $\check{\Theta}(E, d, e, \lfloor\cdot\rfloor)$ and transitions (in the sense of hypothesis (H3) in § 2) by means of an auxiliary distance function. Additionally, a further real component is introduced for technical reasons. It is just to record properly to which "ball" $\{\lfloor\cdot\rfloor \leq r\} \subset E$ we have to refer for α, D. (Indeed, the tuple $(x, \rho) \in E \times [0, \infty[$ is related to $\{\lfloor\cdot\rfloor \leq \rho \cdot e^\rho\}$. This separate exponential factor is just to facilitate updating the radius along transitions.)

Proposition 4 *Consider $\widehat{E} := \{(x, \rho) \in E \times \mathbb{R} \mid \lfloor x\rfloor \leq \rho \cdot e^\rho\} \subset E \times [0, \infty[$ with $E \longrightarrow \widehat{E},\; x \longmapsto (x, \lfloor x\rfloor)$ and $\pi_2 : \widehat{E} \longrightarrow [0, \infty[,\; (x, \rho) \longmapsto \rho$. Moreover define the extensions of $d(\cdot,\cdot)$, $e(\cdot,\cdot)$, $\lfloor\cdot\rfloor$ and each $\vartheta \in \check{\Theta}(E, d, e, \lfloor\cdot\rfloor)$ as*

$$\begin{array}{llll} d: & \widehat{E} \times \widehat{E} \longrightarrow [0, \infty[, & \big((x_1, \rho_1),\; (x_2, \rho_2)\big) \longmapsto d(x_1, x_2), \\ e: & \widehat{E} \times \widehat{E} \longrightarrow [0, \infty[, & \big((x_1, \rho_1),\; (x_2, \rho_2)\big) \longmapsto e(x_1, x_2), \\ \lfloor\cdot\rfloor: & \widehat{E} \longrightarrow [0, \infty[, & (x, \rho) \longmapsto \lfloor x\rfloor, \\ \vartheta: & [0,1] \times \widehat{E} \longrightarrow \widehat{E}, & \big(h,\; (x, \rho)\big) \longmapsto \big(\vartheta(h, x),\; \rho + \widehat{\gamma}\, h\big). \end{array}$$

There exist some $T > 1$ and a function $\widehat{d} : \widehat{E} \times \widehat{E} \longrightarrow [0, \infty[$ satisfying for any $\vartheta, \tau \in \check{\Theta}(E, d, e, \lfloor\cdot\rfloor)$, $\widehat{x}, \widehat{y} \in \widehat{E}$, $t_1, t_2, h \geq 0$ with $t_1 + h, t_2 + h \leq 1$ and the abbreviation $\rho_1 := \big(\max\{\pi_2\,\widehat{x},\; \pi_2\,\widehat{y}\} + \widehat{\gamma}\big) \cdot e^{\max\{\pi_2\,\widehat{x},\; \pi_2\,\widehat{y}\} + \widehat{\gamma}}$

(1.) $d(\cdot,\cdot) \le \widehat{d}(\cdot,\cdot) \le \check{C}(0) \cdot d(\cdot,\cdot)$,

(2.) $\widehat{d}\big(\vartheta(t_1+h,\widehat{x}),\ \vartheta(t_2+h,\widehat{y})\big) \le \widehat{d}\big(\vartheta(t_1,\widehat{x}),\ \vartheta(t_2,\widehat{y})\big) \cdot e^{h\,(1+\alpha(\tau;\rho_1))}$,

(3.) $\widehat{d}\big(\vartheta(t_1+h,\widehat{x}),\ \tau(t_2+h,\widehat{y})\big)$
$\le \Big(\widehat{d}\big(\vartheta(t_1,\widehat{x}),\ \tau(t_2,\widehat{y})\big) + h \cdot \check{C}(T)\ D(\vartheta,\tau;\rho_1)\Big) \cdot e^{h\,(1+\alpha(\tau;\rho_1))}$.

In particular, each function $\vartheta \in \check{\Theta}(E,d,e,\lfloor\cdot\rfloor)$ *induces a unique transition on the tuple* $(\widehat{E},\widehat{d},e,\lfloor\cdot\rfloor)$ *in the sense of hypothesis (H3) in § 2.*

Proof. Fix some $T > 1$ with $C(0)\,e^{-(T-1)} \le \frac{1}{2}$ and set $\widehat{d} : \widehat{E} \times \widehat{E} \longrightarrow [0,\infty[$

$$\widehat{d}(\widehat{x}_0,\widehat{y}_0) := \sup \Big\{ e^{-t}\Big(d\big(\widehat{x}(t),\widehat{y}(t)\big) \cdot e^{-\check{\alpha}_\rho(t)} - \check{C}(T) \cdot \int_0^t D(\vartheta(s),\tau(s);\rho(s)) \cdot e^{-\check{\alpha}_\rho(s)}\ ds\Big) \ \Big|$$
$$t \in [0,T], \quad \widehat{\alpha},\widehat{\beta} \ge 0,$$
$$\widehat{x}(\cdot) \in \mathcal{N}(\widehat{x}_0,\ t,\ \widehat{\alpha},\widehat{\beta},\widehat{\gamma}) \text{ related to piecewise constant } \vartheta(\cdot) : [0,t] \longrightarrow \check{\Theta},$$
$$\widehat{y}(\cdot) \in \mathcal{N}(\widehat{y}_0,\ t,\ \widehat{\alpha},\widehat{\beta},\widehat{\gamma}) \text{ related to piecewise constant } \tau(\cdot) : [0,t] \longrightarrow \check{\Theta},$$
$$\rho(t') := \big(\max\{\pi_2\,\widehat{x}_0,\ \pi_2\,\widehat{y}_0\} + \widehat{\gamma}\,t'\big) \cdot e^{\max\{\pi_2\,\widehat{x}_0,\ \pi_2\,\widehat{y}_0\} + \widehat{\gamma}\,t'},$$
$$\check{\alpha}_\rho(t') := \int_0^{t'} \alpha(\tau(s);\ \rho(s))\ ds \quad \text{for each } t' \in [0,t] \Big\}.$$

(1.) $\widehat{d}(\widehat{x}_0,\widehat{y}_0) \ge d(\widehat{x}_0,\widehat{y}_0)$ is obvious for all $\widehat{x}_0,\widehat{y}_0 \in \widehat{E}$ (due to the option $t=0$). $\widehat{d}(\cdot,\cdot) \le \check{C}(0) \cdot d(\cdot,\cdot) < \infty$ results directly from assumption (A4).

(2.) This claim is a special case of statement (3.) because $D(\cdot,\cdot;\rho)$ is assumed to be reflexive in hypothesis (A3).

(3.) Choose any $\vartheta_0,\tau_0 \in \check{\Theta}(E,d,e,\lfloor\cdot\rfloor)$, $\widehat{x}_0,\widehat{y}_0 \in \widehat{E}$, $t_1,t_2,h \ge 0$ with $t_1+h \le 1$, $t_2+h \le 1$ and for $s \ge -h$, define the abbreviation

$$\rho(s) := \big(\max\{\pi_2\,\vartheta_0(t_1,\widehat{x}_0),\ \pi_2\,\tau_0(t_2,\widehat{y}_0)\} + \widehat{\gamma}\cdot(s+h)\big) \cdot e^{\max\{\pi_2\,\vartheta_0(t_1,\widehat{x}_0),\ \pi_2\,\tau_0(t_2,\widehat{y}_0)\} + \widehat{\gamma}\cdot(s+h)}.$$

In regard to an upper bound of $\widehat{d}\big(\vartheta_0(t_1+h,\widehat{x}_0),\ \tau_0(t_2+h,\widehat{y}_0)\big)$, let $t \in [0,T]$, $\widehat{\alpha},\widehat{\beta} \ge 0$ be arbitrary with $\alpha(\tau_0;\rho_1) \le \widehat{\alpha}$ (without loss of generality) and select any two "Euler curves" $\widehat{x}(\cdot) \in \mathcal{N}\big(\vartheta_0(t_1+h,\widehat{x}),t,\ \widehat{\alpha},\widehat{\beta},\widehat{\gamma}\big)$, $\widehat{y}(\cdot) \in \mathcal{N}\big(\tau_0(t_2+h,\widehat{y}),t,\ \widehat{\alpha},\widehat{\beta},\widehat{\gamma}\big)$ related to piecewise constant functions $\vartheta(\cdot),\tau(\cdot)$: $[0,t] \longrightarrow \check{\Theta}$ respectively.

Extend $\widehat{x}(\cdot),\widehat{y}(\cdot)$ and $\vartheta(\cdot),\tau(\cdot)$ to $[-h,t]$ according to $\widehat{x}(\cdot) := \vartheta_0(t_1+h+\ \cdot,\widehat{x}_0)$, $\widehat{y}(\cdot) := \tau_0(t_2+h+\ \cdot,\widehat{y}_0)$ and $\vartheta(\cdot) := \vartheta_0$, $\tau(\cdot) := \tau_0$ in $[-h,0[$. Then,

$$
\begin{aligned}
& d\big(\widehat{x}(t),\widehat{y}(t)\big)\cdot e^{-\check{\alpha}_\rho(t)} \qquad - \check{C}(T)\int_0^t D(\vartheta,\tau;\rho)\, e^{-\check{\alpha}_\rho}\big|_s\, ds \\
\le\ & e^{h\,\alpha(\tau_0;\rho_1)}\Big(d\big(\widehat{x}(t),\widehat{y}(t)\big)\cdot e^{-\int_{-h}^t \alpha(\tau;\,\rho)\,ds} - \check{C}(T)\int_{-h}^t D(\vartheta,\tau;\rho)\, e^{-\int_{-h}^s \alpha(\tau;\,\rho)\ dr}\, ds \\
& \qquad + \check{C}(T)\int_{-h}^0 D(\vartheta,\tau;\rho)\, e^{-\int_{-h}^s \alpha(\tau;\,\rho)\ dr}\, ds\Big)^+
\end{aligned}
$$

and if we now assume $t+h \le T$ in addition,

$$
\begin{aligned}
\le\ & e^{h\,\alpha(\tau_0;\rho_1)}\Big(\widehat{d}\big(\widehat{x}(-h),\ \widehat{y}(-h)\big) \qquad e^{t+h} + \check{C}(T)\int_{-h}^0 D(\vartheta(s),\tau(s);\rho(s))\ \cdot\ 1\ ds\Big) \\
\le\ & e^{h\,\alpha(\tau_0;\rho_1)}\Big(\widehat{d}\big(\vartheta_0(t_1,\widehat{x}_0),\ \tau_0(t_2,\widehat{y}_0)\big)e^{t+h} + \check{C}(T)\ \ h\ \cdot\ D(\vartheta_0,\tau_0;\rho_1)\Big).
\end{aligned}
$$

If $t+h > T$ (i.e. $0 \le T-1 \le T-h < t \le T$), we conclude from assumption (A4)

$$
\begin{aligned}
& e^{-\,t}\ \Big(d\big(\widehat{x}(t),\widehat{y}(t)\big)\cdot e^{-\check{\alpha}_\rho(t)}\ -\ \check{C}(T)\int_0^t D(\vartheta(s),\tau(s);\rho(s))\ \cdot\ e^{-\check{\alpha}_\rho(s)}\ ds\Big) \\
\le\ & e^{-\,t}\quad \check{C}(0)\cdot d\big(\widehat{x}(0),\ \widehat{y}(0)\big)\ \ \le\ \tfrac{1}{2}\ \cdot\ \widehat{d}\big(\vartheta_0(t_1{+}h,\widehat{x}_0),\ \tau_0(t_2{+}h,\widehat{y}_0)\big)
\end{aligned}
$$

and so, this case is not relevant for estimating $\widehat{d}\big(\vartheta_0(t_1{+}h,\widehat{x}_0),\ \tau_0(t_2{+}h,\widehat{y}_0)\big)$ as a supremum. Finally, the upper bound for $t+h \le T$ leads to the claim. □

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PROBABILISTIC REPRESENTATION OF SOLUTIONS FOR QUASI-LINEAR PARABOLIC PDE VIA FBSDE WITH REFLECTING BOUNDARY CONDITIONS

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Abstract

A probabilistic representation of the solution (in the viscosity sense) of a quasi-linear parabolic PDE system with non-lipschitz terms and a Neumann boundary condition is given via a fully coupled forward-backward stochastic differential equation with a reflecting term in the forward equation. The extension of previous results consists on the relaxation on the Lipschitz assumption on the drift coefficient of the forward equation, using a previous result of the authors.

Key words: *Probabilistic formulae for PDE, Forward backward stochastic differential equations, Skorokhod problem, Reflected Stochastic Differential Equations.*
AMS subject classifications: *60H10, 35K55, 60J60, 60K25.*

Introduction

Deeper relations between stochastic differential equations and systems of PDE have been established since [4] developed the theory of backward stochastic differential equations. Roughly speaking, combining a forward stochastic differential equation with a BSDE, the Feyman-Kac formula can be extended to nonlinear PDE, and not only in a classical sense, but also via viscosity solutions.

Usually, the deterministic problems treated in this way are posed in the whole domain $\mathbb{R}^d$, or in a bounded domain of $\mathbb{R}^d$ with Dirichlet boundary condition. With a Neumann boundary condition, the problem was studied by Y. Hu using local time around the boundary of the domain. This technique is closely related to a stochastic version of the Skorokhod problem (see e.g. [6], for a direct application in this sense). We extend these studies and relations to the case of fully coupled systems of FBSDER in which the open set is not necessarily convex but still smooth (this restriction is for commodity and may be removed), and the drift coefficient of the forward equation is monotone in x, instead of Lipschitz. In this way, we generalize some results from [5] and [1].

In this paper we give a probabilistic representation of the solution of a quasi-linear PDE system extending some results of those given in [5] and [1] on a system of a fully coupled forward-backward stochastic differential equations with a reflecting term in the forward equation (FBSDER) and its relation with a system of quasi-linear partial differential equations, in short PDE. Preceding works on this line were due to Y. Hu and to E. Pardoux and S. Zhang (cf. [6]). In our case, the drift satisfies the monotonicity condition introduced before, and the domain $\mathcal{O}$ is not necessarily convex. Existence of solution under such conditions was proved in a precedent paper by the authors (cf. [3]).

In Section 1 we start giving the suitable framework for the reflected problem and recall a previous result which will be used later on. In Section 2, we state the general framework for the study of a fully coupled FBSDER, and provide a probabilistic interpretation for a system of quasi-linear PDE with homogeneous Neumann boundary condition.

1 Statement of the "reflected" problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing and right continuous family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and $\{W_t;\ t \geq 0\}$ an m-dimensional standard $\{\mathcal{F}_t\}$-Wiener process.

Let $\mathcal{O}$ be an open connected bounded subset of $\mathbb{R}^d$ given by $\mathcal{O} = \{\phi > 0\}$, with $\phi \in C^2(\mathbb{R}^d)$, and such that $\partial\mathcal{O} = \{\phi = 0\}$, with $|\nabla\phi(x)| = 1$ for all $x \in \partial\mathcal{O}$. Observe that in particular ϕ, $\nabla\phi$ and $D^2\phi$ are bounded in $\bar{\mathcal{O}}$. Then there exists a constant $C_0 > 0$ such that

$$2(x' - x, \nabla\phi(x)) + C_0|x' - x|^2 \geq 0, \qquad \forall\, x \in \partial\mathcal{O},\ \forall\, x' \in \bar{\mathcal{O}}. \tag{1}$$

We are also given a final time $T > 0$, and two random functions:

$$b : \Omega \times [0,T] \times \bar{\mathcal{O}} \to \mathbb{R}^d, \qquad \sigma : \Omega \times [0,T] \times \bar{\mathcal{O}} \to \mathbb{R}^{d\times m},$$

such that

(i) b and σ are uniformly bounded;

(ii) for all $x \in \bar{\mathcal{O}}$ the processes $b(\cdot,\cdot,x)$ and $\sigma(\cdot,\cdot,x)$ are $\{\mathcal{F}_t\}$-progressively measurable;

(iii) for all $t \in [0,T]$ and a.s. ω, the function $b(\omega,t,\cdot)$ is continuous on $\bar{\mathcal{O}}$;

(iv) there exist two constants $L_{b_x} \in \mathbb{R}$ and $L_{\sigma_x} \geq 0$ such that for all $t \in [0,T]$ and all $x, x' \in \bar{\mathcal{O}}$,

$$(x - x', b(\omega,t,x) - b(\omega,t,x')) \leq L_{b_x}|x - x'|^2, \quad a.s.,$$

$$\|\sigma(\omega,t,x) - \sigma(\omega,t,x')\| \leq L_{\sigma_x}|x - x'|, \quad a.s.,$$

where $|\cdot|$ and $\|\cdot\|$ denote the usual Euclidean and trace norm for vectors and matrices respectively.

From now on, we will omit the explicit dependence of the processes on ω.

Consider the following problem:

$$X_t = x_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s - k_t, \tag{2}$$

$$k_t = -\int_0^t \nabla\phi(X_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s \in \partial\mathcal{O}\}}\, d|k|_s, \quad t \in [0,T], \tag{3}$$

where $x_0 \in \bar{\mathcal{O}}$ is given, and $|k|_t$ stands for the total variation of k on $[0,t]$.

Definition 1 *A strong solution to the above problem is a pair of $\{\mathcal{F}_t\}$-adapted and continuous processes (X,k) defined on $\Omega \times [0,T]$, the first one with values in $\bar{\mathcal{O}}$, the second one with values in $\mathbb{R}^d$ and paths of bounded variation in $[0,T]$, satisfying the equations (2)-(3) a.s. for all $t \in [0,T]$.*

Main result stated in [3], which generalizes a result by Lions and Sznitman when b is Lipschitz, is the following:

Theorem 1 *Under the assumptions* (i)-(iv)*, for each $x_0 \in \bar{\mathcal{O}}$ given there exists a unique pair (X,k), strong solution of (2)-(3).*

2 Forward-Backward Stochastic Differential Equations with Reflection and representation of a PDE system

We continue considering the complete probability space $(\Omega, \mathcal{F}, P)$, and the m-dimensional standard $\{\mathcal{F}_t\}$-Wiener process $\{W_t;\ t \geq 0\}$ given in Section 1, but now we suppose that, for each $t \geq 0$, $\mathcal{F}_t$ coincides with the σ-algebra $\sigma(W_s;\, 0 \leq s \leq t)$ augmented with all the P-null sets of $\mathcal{F}$.

Let $T > 0$ be fixed, and consider the open set $\mathcal{O}$ introduced in Section 1.

For each integer $l \geq 1$, we shall denote by $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^l)$ the Hilbert subspace of $L^2(\Omega \times (0,T);\mathbb{R}^l)$ formed by those elements that are $\{\mathcal{F}_t\}$-progressively measurable, and we will write $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^l))$ to denote the space of the elements of $L^2(\Omega; C([0,T];\mathbb{R}^l))$ that are $\{\mathcal{F}_t\}$-progressively measurable. Thus, $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^l))$ is a Banach subspace of $L^2(\Omega; C([0,T];\mathbb{R}^l))$.

Similarly, we denote by $M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}})$ the complete metric subspace of the space $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^d)$ constituted by the elements $X \in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^d)$ such that a.e. $t \in (0,T)$, $X_t \in \bar{\mathcal{O}}$ a.s.; we shall also use $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\bar{\mathcal{O}}))$ to denote the complete metric subspace of $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^l))$ formed by those elements X in the last space such that a.s. $X_t \in \bar{\mathcal{O}}$ for all $t \in [0,T]$. Finally, we shall denote by $L^2(\Omega, \mathcal{F}_T;\bar{\mathcal{O}})$ the complete metric subspace of $L^2(\Omega, \mathcal{F}_T;\mathbb{R}^d)$ formed by the $\mathcal{F}_T$-measurable random variables $\xi \in L^2(\Omega;\mathbb{R}^d)$ such that a.s. $\xi \in \bar{\mathcal{O}}$.

We are given four random functions:

$$b : \Omega \times [0,T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}^d,\ f : \Omega \times [0,T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}^n,$$

$$\sigma : \Omega \times [0,T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}^{d\times m},\ h : \Omega \times \bar{\mathcal{O}} \to \mathbb{R}^n,$$

such that

(i') b and σ are uniformly bounded;

(ii') for all $(x,y,z) \in \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$ the processes $b(\cdot,x,y,z)$, $f(\cdot,x,y,z)$ and $\sigma(\cdot,x,y,z)$ are $\{\mathcal{F}_t\}$-progressively measurable, and the random variable $h(\cdot,x)$ is $\mathcal{F}_T$-measurable;

(iii') for all $(t,x,y,z) \in [0,T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$ the functions $b(t,\cdot,y,z)$ and $f(t,x,\cdot,z)$ are a.s. continuous on $\bar{\mathcal{O}}$ and $\mathbb{R}^n$ respectively;

(iv') there exist real constants L_{b_x} and L_{f_y}, and nonnegative constants L_{b_y}, L_{b_z}, L_{f_x}, L_{f_z}, L_{σ_x}, L_{σ_y}, L_{σ_z}, L_h and l_0 such that for all $t \in [0,T]$, all $x,x' \in \bar{\mathcal{O}}$, all $y,y' \in \mathbb{R}^n$, all $z,z' \in \mathbb{R}^{n\times m}$, and a.s.,

$$(x-x', b(t,x,y,z) - b(t,x',y,z)) \le L_{b_x}|x-x'|^2,$$

$$|b(t,x,y,z) - b(t,x,y',z')| \le L_{b_y}|y-y'| + L_{b_z}\|z-z'\|,$$

$$\|\sigma(t,x,y,z) - \sigma(t,x',y',z')\|^2 \le L_{\sigma_x}^2|x-x'|^2 + L_{\sigma_y}^2|y-y'|^2 + L_{\sigma_z}^2\|z-z'\|^2,$$

$$(y-y', f(t,x,y,z) - f(t,x,y',z)) \le L_{f_y}|y-y'|^2,$$

$$|f(t,x,y,z) - f(t,x',y,z')| \le L_{f_x}|x-x'| + L_{f_z}\|z-z'\|,$$

$$|f(t,x,y,z)| \le |f(t,x,0,z)| + l_0(1+|y|),$$

$$|h(x) - h(x')| \le L_h|x-x'|;$$

(v') $\quad E\displaystyle\int_0^T |f(t,0,0,0)|^2\,dt + E|h(0)|^2 < \infty.$

We want to study the following problem:

$$X_t = x_0 + \int_0^t b(s,X_s,Y_s,Z_s)\,ds + \int_0^t \sigma(s,X_s,Y_s,Z_s)\,dW_s - k_t, \quad (4)$$

$$Y_t = h(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)\,ds - \int_t^T Z_s\,dW_s, \quad (5)$$

$$k_t = -\int_0^t \nabla\phi(X_s)\,d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s\in\partial\mathcal{O}\}}\,d|k|_s, \quad t \in [0,T], \quad (6)$$

where $x_0 \in \bar{\mathcal{O}}$ is given.

Definition 2 *A solution to the problem (4)-(6) is a set (X,Y,Z,k) of four $\{\mathcal{F}_t\}$-progressively measurable processes defined on $\Omega \times [0,T]$, such that X is continuous with values in $\bar{\mathcal{O}}$, k is continuous with values in $\mathbb{R}^d$ and paths of bounded variation in $[0,T]$, $(Y,Z) \in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$, and the equations (4)-(6) are satisfied a.s. for all $t \in [0,T]$.*

For the resolution of the above fully coupled FBSDER, we will use the following result, that is a direct consequence of Theorem 1:

Corollary 2 *Under the assumptions* (i')-(iv'), *if $(Y,Z) \in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$ is fixed, there exists a unique pair (X,k) of $\{\mathcal{F}_t\}$-progressively measurable processes defined on $\Omega\times[0,T]$, such that X is continuous with values* *in* $\bar{\mathcal{O}}$, *k is continuous with values in* $\mathbb{R}^d$ *and paths of bounded variation in* $[0,T]$, *and they satisfy a.s. for all* $t \in [0,T]$ *that*

$$X_t = x_0 + \int_0^t b(s, X_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)\, dW_s - k_t, \quad (7)$$

$$k_t = -\int_0^t \nabla\phi(X_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s \in \partial\mathcal{O}\}}\, d|k|_s. \quad (8)$$

We will also need the following well-known result (see for instance Pardoux's notes at Geilo, 1996) for the backward equation:

Theorem 3 *Under the assumptions* (ii')-(v'), *let be given* $X \in M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}})$ *and* $\xi \in L^2(\Omega, \mathcal{F}_T; \bar{\mathcal{O}})$. *Then, there exists a unique pair* $(Y,Z) \in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$ *such that*

$$Y_t = h(\xi) + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad (9)$$

a.s. for all $t \in [0,T]$. *Moreover, we have that* $Y \in L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^n))$.

Using the two results above, it is not difficult to prove existence and uniqueness of solution of problem (4)-(6) if T is small enough. More exactly, we have the following result, whose proof we will omit for the sake of brevity:

Theorem 4 *Suppose the assumptions* (i')-(v'), *and that moreover* σ *does not depend on* z. *Then, there exists a* $T_* > 0$ *such that if* $T \le T_*$, *the application* Φ *defined from*

$$L^2_{\mathcal{F}_t}(\Omega; C([0,T];\bar{\mathcal{O}})) \times L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^n)) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$$

on itself by $\Phi(X,Y,Z) = (\bar{X},\bar{Y},\bar{Z})$, *with* $(\bar{X},\bar{Y},\bar{Z})$ *the unique solution of*

$$\bar{X}_t = x_0 + \int_0^t b(s, \bar{X}_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, \bar{X}_s, Y_s)\, dW_s - \bar{k}_t,$$

$$\bar{k}_t = -\int_0^t \nabla\phi(\bar{X}_s)\, d|\bar{k}|_s, \quad |\bar{k}|_t = \int_0^t 1_{\{\bar{X}_s \in \partial\mathcal{O}\}}\, d|\bar{k}|_s,$$

$$\bar{Y}_t = h(\bar{X}_T) + \int_t^T f(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s)\, ds - \int_t^T \bar{Z}_s\, dW_s,$$

a.s. for all $t \in [0,T]$, *is a contraction. So, if* $T \le T_*$, *the problem (4)-(6) has a unique solution.*

For the resolution of the above fully coupled FBSDER for any $T > 0$, we follow [5] and [1].

We shall denote by Γ_1 the mapping

$$\Gamma_1 : M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m}) \to M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m}),$$

defined by $\Gamma_1(Y,Z) = (\bar{Y}, \bar{Z})$, with $(\bar{X}, \bar{Y}, \bar{Z}, \bar{k})$ the unique solution of

$$\begin{aligned} \bar{X}_t &= x_0 + \int_0^t b(s, \bar{X}_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, \bar{X}_s, Y_s, Z_s)\, dW_s - \bar{k}_t, \\ \bar{k}_t &= -\int_0^t \nabla\phi(\bar{X}_s)\, d|\bar{k}|_s, \quad |\bar{k}|_t = \int_0^t 1_{\{\bar{X}_s \in \partial\mathcal{O}\}}\, d|\bar{k}|_s, \\ \bar{Y}_t &= h(\bar{X}_T) + \int_t^T f(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s)\, ds - \int_t^T \bar{Z}_s\, dW_s, \end{aligned}$$

a.s. for all $t \in [0,T]$.

We will denote by Γ_2 the mapping

$$\Gamma_2 : M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}}) \times L^2(\Omega, \mathcal{F}_T; \bar{\mathcal{O}}) \to M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}}) \times L^2(\Omega, \mathcal{F}_T; \bar{\mathcal{O}}),$$

defined by $\Gamma_2(X,\xi) = (\bar{X}, \bar{X}_T)$, with $\bar{X}$ such that $(\bar{X}, \bar{Y}, \bar{Z}, \bar{k})$ is the unique solution of

$$\begin{aligned} \bar{Y}_t &= h(\xi) + \int_t^T f(s, X_s, \bar{Y}_s, \bar{Z}_s)\, ds - \int_t^T \bar{Z}_s\, dW_s, \\ \bar{X}_t &= x_0 + \int_0^t b(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s)\, ds + \int_0^t \sigma(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s)\, dW_s - \bar{k}_t, \\ \bar{k}_t &= -\int_0^t \nabla\phi(\bar{X}_s)\, d|\bar{k}|_s, \quad |\bar{k}|_t = \int_0^t 1_{\{\bar{X}_s \in \partial\mathcal{O}\}}\, d|\bar{k}|_s, \end{aligned}$$

a.s. for all $t \in [0,T]$.

By Corollary 2 and Theorem 3, under the conditions (i')-(v') the maps Γ_1 and Γ_2 are well defined. Also, it is clear that to solve the problem (4)-(6) is equivalent to finding a fixed point for Γ_1 or Γ_2. Thus, in order to prove existence and uniqueness of solution to problem (4)-(6), it is enough to find a Hilbert norm in $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$, such that Γ_1 is a contraction for this norm. Analogously, it is enough to find a complete metric in $M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}}) \times L^2(\Omega, \mathcal{F}_T; \bar{\mathcal{O}})$, for which the map Γ_2 is a contraction.

From now on, for $l \geq 1$ integer, and $\lambda \in \mathbb{R}$, we will denote by $\|\cdot\|_\lambda$ the norm on $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^l)$, equivalent to the usual one, given by

$$\|\zeta\|^2_\lambda = E\int_0^T e^{-\lambda s}|\zeta|^2 ds.$$

For the sake of brevity on these notes we omit here the estimates on the difference of two solutions (X,k) and (X',k') associated respectively to processes (Y,Z) and (Y',Z'), or the inverse. If we combine these estimates in the two possible orders, to obtain estimations for Γ_1 and Γ_2, we have two possibilities.

On the one hand, one can search for a $\lambda \in \mathbb{R}$ such that, with the norm on $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$ defined by

$$\|(Y,Z)\|^2_\lambda = \|Y\|^2_\lambda + \|Z\|^2_\lambda,$$

the mapping Γ_1 is a contraction.

On the other hand, one can search for a λ such that, with the metric on $M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}}) \times L^2(\Omega,\mathcal{F}_T;\bar{\mathcal{O}})$ induced by the norm on $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^d) \times L^2(\Omega,\mathcal{F}_T;\mathbb{R}^d)$ defined by

$$\|(X,\xi)\|^2_\lambda = \exp(-\lambda T)E|\xi|^2 + \lambda_1\|X\|^2_\lambda,$$

the mapping Γ_2 is a contraction.

Then, one obtains existence and uniqueness for (4)-(6) that generalize to b monotone and $\mathcal{O}$ not necessarily convex some of the results in [5] and [1].

For example, existence and uniqueness of solution for (4)-(6) hold when its coupling is weak, that is, when dependence of b and σ respect to their variables y and z is small, or, analogously for the backward equation, when the dependence of f and h with respect to x is small. More exactly, we have:

Theorem 5 *Let conditions* (i')-(v') *hold. Then there exists an* $\varepsilon_0 > 0$ *depending on* L_{σ_x}, L_{b_x}, L_{f_x}, L_{f_y}, L_{f_z}, L_h *and* T *such that if* L_{b_y}, L_{b_z}, L_{σ_y}, $L_{\sigma_z} \in [0,\varepsilon_0)$, *then there exists* λ *such that* Γ_1 *is a contraction, and thus there exists a unique solution to (4)-(6). On the other hand, the same thesis holds for* Γ_2, *changing roles of* L_{b_y}, L_{b_z}, L_{σ_y}, *and* L_{σ_z}, *with* L_h *and* L_{f_x}.

Also, using Γ_2, and reasoning as in [1] or [2], one can prove

Theorem 6 *Let conditions* (i')-(v') *hold, and suppose one of the following two conditions:*
a) *If h is independent of x, there exists $\alpha \in (0,1)$ such that $\mu(\alpha,T)L_{f_x}C_3 < \lambda_1$.*
b) *If h does depend on x, there exists $\alpha \in (k_1 L^2_{\sigma_z} L^2_h, 1)$ such that $\mu(\alpha,T)L^2_h < 1$.*
Then, there exists a unique solution for (4)-(6).

Remark 1 *Reasoning as in [2], one can make some (technical) improvements. Namely, it is possible to consider that σ can depend on z, but introducing compatibility conditions. On other hand, if L_{f_y} is negative enough, then (4)-(6) has a unique solution for all final time $T > 0$.*

Finally, as in [5], and in [1], with the previous results on the problem (4)-(6), one can prove existence of viscosity solution to a homogeneous Neumann problem for an associated system of quasi-linear parabolic PDE. We briefly recall here how this can be done.

For each $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, consider the problem

$$X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr + \int_t^s \sigma(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dW_r - k^{t,x}_s,$$

$$Y^{t,x}_s = h(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr - \int_s^T Z^{t,x}_r\, dW_r,$$

$$k^{t,x}_s = -\int_t^s \nabla\phi(X^{t,x}_r)\, d|k^{t,x}|_r, \quad |k^{t,x}|_s = \int_t^s 1_{\{X^{t,x}_r \in \partial\mathcal{O}\}}\, d|k^{t,x}|_r, \quad s \in [t,T].$$

It is immediate to extend to this family of problems the previous theorems on existence and uniqueness of solution for problem (4)-(6).

To establish the relation with PDE, we assume now that b, σ, f and h are deterministic, moreover, we suppose that σ does not depend on z. Also, for simplicity, we consider $n = 1$. For short, we introduce the following notation:

$$(L\varphi)(s,x,y,z) = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(s,x,y)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(s,x) + (b(s,x,y,z), \nabla\varphi(s,x)),$$

and consider the homogeneous Neumann problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + (Lu)(t,x,u(t,x),(\nabla u(t,x))^*\sigma(t,x,u(t,x))) \\ &+f(t,x,u(t,x),(\nabla u(t,x))^*\sigma(t,x,u(t,x))) = 0, \quad (t,x) \in (0,T)\times\mathcal{O}, \\ &\frac{\partial u}{\partial n}(t,x) = 0, \quad (t,x) \in (0,T)\times\partial\mathcal{O}, \\ &u(T,x) = h(x), \quad x \in \mathcal{O}. \end{aligned} \tag{10}$$

Then, we have, for example, the following result, that can be proved as Theorem 3.8 in [1], and actually can also be adapted to deal with a system.

Theorem 7 *Under the assumptions of Theorem* 6*, suppose, moreover,* $n = 1$*. Suppose also that* b*,* σ*,* f *and* h *are deterministic, continuous in all its variables, and* σ *does not depend on* z*. Then, the function* u *defined by* $u(t,x) = Y_t^{x,t}$*,* $(t,x) \in [0,T]\times\bar{\mathcal{O}}$*, is a viscosity solution of (10).*

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EXISTENCE AND UNIQUENESS OF SOLUTIONS, AND PULLBACK ATTRACTOR FOR A SYSTEM OF GLOBALLY MODIFIED 3D–NAVIER-STOKES EQUATIONS WITH FINITE DELAY

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Abstract

We first study the existence and uniqueness of strong solutions of a three dimensional system of globally modified Navier-Stokes equations with finite delay in the locally Lipschitz case. The asymptotic behaviour of solutions, and the existence of pullback attractor are also analyzed.

Key words: *3-dimensional Navier-Stokes equations, Galerkin approximations, weak solutions, existence and uniqueness of strong solutions, pullback attractors.*
AMS subject classifications: *35Q30, 35K90, 37L30.*

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary Γ, and consider the following system of *globally modified Navier-Stokes equations (GMNSE)* on Ω with a homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N\left(\|u\|\right)[(u\cdot\nabla)u] + \nabla p = f(t), \qquad \text{in } \ (0,+\infty)\times\Omega, \\ \nabla\cdot u \ = 0 \quad \text{in } \ (0,+\infty)\times\Omega, \\ u = 0 \ \text{ on } (0,+\infty)\times\Gamma, \\ u(0,x) = u^0(x), \quad x\in\Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u^0 the initial velocity field, $f(t)$ a given external force field, and F_N : $\mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r\in\mathbb{R}^+,$$

for some $N \in \mathbb{R}^+$.

The GMNSE (1) has been introduced and studied in [1] (see also [2], [3], [8] and [9]). In this paper we are interested in the case in which terms containing finite delays appear. We consider the following version of GMNSE (we will refer to it as GMNSED):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|)[(u\cdot\nabla)u] + \nabla p \\ = G(t, u(t-\rho(t))) \quad \text{in} \quad (\tau, +\infty)\times\Omega, \\ \nabla\cdot u \ = 0 \quad \text{in} \quad (\tau,+\infty)\times\Omega, \\ u = 0 \ \text{ on } (\tau,+\infty)\times\Gamma, \\ u(\tau,x) = u^0(x), \quad x\in\Omega, \\ u(t,x) = \phi(t-\tau,x), \ \text{in } (\tau-h,\tau)\times\Omega, \end{cases} \tag{2}$$

where $\tau \in \mathbb{R}$ is an initial time, the term $G(t, u(t-\rho(t)))$ is an external force depending eventually on the value $u(t-\rho(t))$, $\rho(t) \geq 0$ is a delay function and ϕ is a given velocity field defined in $(-h, 0)$, with $h > 0$ a fixed time such that $\rho(t) \leq h$.

The aim of this paper is to report on some recent results concerning the existence, uniqueness and asymptotic behaviour of solutions of (2). The detailed proofs of these results can be found in [4]. In the next section we state some preliminaries, establish the framework for our problem, and the existence and uniqueness of weak and strong solutions. In Section 3 we analyze the asymptotic behaviour of solutions, obtaining finally the existence of pullback attractor for our model.

2 Preliminaries

To set our problem in the abstract framework, we consider the following usual abstract spaces (see [12] and [14, 15]):

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^3$,

$$(u, v) = \sum_{j=1}^{3} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^3$,

$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. Finally, we will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V'.

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad \forall\, u,v,w \in V,$$

and we denote

$$b_N(u,v,w) = F_N(\|v\|) b(u,v,w), \quad \forall\, u,v,w \in V,$$

and

$$\langle B_N(u,v), w\rangle = b_N(u,v,w), \quad \forall\, u,v,w \in V.$$

The form b_N is linear in u and w, but it is nonlinear in v.

We also consider $A : V \to V'$ defined by $\langle Au, v\rangle = ((u,v))$. Denoting $D(A) = (H^2(\Omega))^3 \cap V$, then $Au = -P\Delta u, \forall\, u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H). Moreover, we assume $G : \mathbb{R} \times H \longrightarrow H$, is such that

c1) $G(\cdot, u) : \mathbb{R} \longrightarrow H$ is measurable, $\forall u \in H$,

c2) there exists a nonnegative function $g \in L^p_{loc}(\mathbb{R})$ for some $1 \le p \le +\infty$, and a nondecreasing function $L : (0,\infty) \to (0,\infty)$, such that for all $R > 0$ if $|u|, |v| \le R$, then

$$|G(t,u) - G(t,v)| \le L(R) g^{1/2}(t) |u - v|,$$

for all $t \in \mathbb{R}$, and

c3) there exists a nonnegative function $f \in L^1_{loc}(\mathbb{R})$, such that for any $u \in H$,

$$|G(t,u)|^2 \le g(t)|u|^2 + f(t), \quad \forall\, t \in \mathbb{R}.$$

Finally, we suppose $\phi \in L^{2p'}(-h,0;H)$ and $u^0 \in H$, where $\frac{1}{p} + \frac{1}{p'} = 1$. In this situation, we consider a delay function $\rho \in C^1(\mathbb{R})$ such that $0 \le \rho(t) \le h$ for all $t \in \mathbb{R}$, and there exists a constant ρ_* satisfying

$$\rho'(t) \le \rho_* < 1 \quad \forall\, t \in \mathbb{R}.$$

Definition 1 *Let $\tau \in \mathbb{R}$, $u^0 \in H$ and $\phi \in L^{2p'}(-h,0;H)$ be given. A weak solution of (2) is a function*

$$u \in L^{2p'}(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \quad \textit{for all } T > \tau,$$

such that

$$\begin{cases} \dfrac{d}{dt}u(t) + \nu Au(t) + B_N(u(t), u(t)) = G(t, u(t - \rho(t))) \ \textit{in } \mathcal{D}'(\tau, +\infty; V'), \\ u(\tau) = u^0, \\ u(t) = \phi(t - \tau) \quad t \in (\tau - h, \tau). \end{cases}$$

Remark 1 *We suppose u is a weak solution of (2) and we define $\tilde{g}(t) = g \circ \theta^{-1}(t)$, where $\theta : [\tau, +\infty) \longrightarrow [\tau - \rho(\tau), +\infty)$ is the differentiable and strictly increasing function given by $\theta(s) = s - \rho(s)$. Then, taking into account that $\tilde{g} \in L^p(\tau - \rho(\tau), T)$ for all $T > \tau$, we have that $G(t, u(t - \rho(t)))$ belongs to $L^2(\tau, T; H)$ for all $T > \tau$.*

Then, $\dfrac{d}{dt} u(t) \in L^2(\tau, T; V')$, and consequently (see [15]) $u \in C([\tau, +\infty); H)$ and satisfies the energy equality, for all $\tau \le s \le t$,

$$|u(t)|^2 - |u(s)|^2 + 2\nu \int_s^t \|u(r)\|^2 \, dr = 2 \int_s^t (G(r, u(r - \rho(r))), u(r)) \, dr. \quad (3)$$

The following theorem, which improves Theorem 3 in [5], states the existence and uniqueness of weak and/or strong solutions.

Theorem 1 *Under the conditions c1)-c3) in the previous section, assume that $\tau \in \mathbb{R}$, $u^0 \in H$ and $\phi \in L^{2p'}(-h, 0; H)$ are given. Then, there exists a unique weak solution u of (2) which is, in fact, a strong solution in the sense that*

$$u \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A)), \quad \text{for all } T - \tau > \varepsilon > 0. \quad (4)$$

Moreover, if $u^0 \in V$, then

$$u \in C([\tau, T]; V) \cap L^2(\tau, T; D(A)), \quad \text{for all } T > \tau. \quad (5)$$

3 Asymptotic behaviour of solutions

In this section we first establish a result about the asymptotic behavior of the solutions of problem (2) when t goes to $+\infty$.

Theorem 2 *Let us suppose that c1)-c3) hold with $g \in L^\infty(\mathbb{R})$, and assume also that $\nu^2 \lambda_1^2 (1 - \rho_*) > |g|_\infty$, where $|g|_\infty := \|g\|_{L^\infty(\mathbb{R})}$.*

Let us denote $\varepsilon > 0$ the unique root of $\varepsilon - \nu\lambda_1 + \frac{|g|_\infty e^{\varepsilon h}}{\nu\lambda_1(1-\rho_*)} = 0$. Then, for any $(u^0, \phi) \in V \times L^2(-h, 0; H)$, and any $\tau \in \mathbb{R}$, the corresponding solution $u(t; \tau, u^0, \phi)$ of problem (2) satisfies

$$\begin{aligned} \left|u(t; \tau, u^0, \phi)\right|^2 \quad &\le \quad \left(\left|u^0\right|^2 + \frac{|g|_\infty e^{\varepsilon h}}{\nu\lambda_1(1 - \rho_*)} \int_{-h}^0 e^{\varepsilon s} \left|\phi(s)\right|^2 ds \right) e^{\varepsilon(\tau - t)} \\ &\quad + \frac{e^{-\varepsilon t}}{\nu\lambda_1} \int_\tau^t e^{\varepsilon s} f(s) \, ds, \quad \text{for all } \ t \ge \tau. \end{aligned}$$

In particular, if $\int_\tau^\infty e^{\varepsilon s} f(s)\, ds < \infty$, then every solution $u(t; \tau, u^0, \phi)$ of (2) converges exponentially to 0 as $t \to +\infty$.

Now, we study the existence of global attractor for the dynamical system generated by our problem. As this model is non-autonomous, our analysis requires of the theory of pullback attractor which we will introduced below (see [7], [10] and [11]).

Let X be a metric space.

Definition 2 *A family of mappings* $\{U(t,\tau) : X \to X :\ t,\tau \in \mathbb{R}, t \geq \tau\}$ *is said to be a process (or a two–parameter semigroup, or an evolution semigroup) in* X *if*

$$\begin{aligned} U(t,r)U(r,\tau) &= U(t,\tau) \quad \textit{for all } t \geq r \geq \tau, \\ U(\tau,\tau) &= Id \quad\quad\quad \textit{for all } \tau \in \mathbb{R}. \end{aligned}$$

The process $U(\cdot,\cdot)$ *is said to be continuous if the mapping* $x \to U(t,\tau)x$ *is continuous on* X *for all* $t,\tau \in \mathbb{R}, t \geq \tau$*.*

Recall that $dist(A,B)$ denotes the Hausdorff semidistance between the sets A and B, which is given by

$$dist(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b), \quad \text{for } A,B \subset X.$$

Definition 3 *Let* $U(\cdot,\cdot)$ *be a process in the metric space* X*. A family of compact sets* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *is said to be a (global) pullback attractor for* $U(\cdot,\cdot)$ *if, for every* $t \in \mathbb{R}$*, if follows*

(i) $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ *for all* $\tau \leq t$ *(invariance), and*

(ii) $\lim_{\tau\to-\infty} dist(U(t,\tau)D, \mathcal{A}(t)) = 0$ *(pullback attraction) for all bounded subset* $D \subset X$*.*

The concept of pullback attractor is related to that of pullback absorbing set.

Definition 4 *The family of subsets* $\{B(t)\}_{t\in\mathbb{R}}$ *of* X *is said to be pullback absorbing with respect to the process* $U(\cdot,\cdot)$ *if, for every* $t \in \mathbb{R}$ *and all bounded subset* $D \subset X$*, there exists* $\tau_D(t) \leq t$ *such that*

$$U(t,\tau)D \subset B(t), \quad \textit{for all} \quad \tau \leq \tau_D(t).$$

In fact, as happens in the autonomous case, the existence of compact pullback attracting sets is enough to ensure the existence of pullback attractors. The following result can be found in [7] and [13] (see also [6]).

Theorem 3 *Let* $U(\cdot,\cdot)$ *be a continuous process on the metric space* X*. If there exists a family of compact pullback attracting sets* $\{B(t)\}_{t\in\mathbb{R}}$*, then there exists a pullback attractor* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$*, with* $\mathcal{A}(t) \subset B(t)$ *for all* $t \in \mathbb{R}$*, given by*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D\subset X \\ bounded}} \Lambda_D(t)}, \quad \textit{where} \quad \Lambda_D(t) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{\tau\leq t-n} U(t,\tau)D}.$$

Now we will establish the existence of the pullback attractor for our GMNSED model (2).

First we construct the associated process. To this end, assume that $G : \mathbb{R} \times H \to H$ satisfies c1), c2) and c3) with $g \in L^\infty(\mathbb{R})$. Thus, without loss of generality we can assume that G satisfies c2) with $g \equiv 1$, and there exists a nonnegative constant a such that

$$|G(t,u)|^2 \leq a|u|^2 + f(t) \quad \forall\,(t,u) \in \mathbb{R} \times H. \tag{6}$$

We will assume moreover that

$$f \in L^\infty_{loc}(\mathbb{R}). \tag{7}$$

Under these assumptions, for each initial time $\tau \in \mathbb{R}$, and any $\phi \in C(-h,0;H)$, Theorem 1 ensures that if we take $u^0 = \phi(0)$, problem (2) possesses a unique solution $u(\cdot;\tau,\phi) = u(\cdot;\tau,\phi(0),\phi)$, which belongs to the space $C([\tau-h,T];H) \cap L^2(\tau,T;V) \cap C([\tau+\epsilon,T];V) \cap L^2(\tau+\epsilon,T;D(A))$ for all $T > \tau+\epsilon > \tau$.

Then, we define a process in the phase space $C_H = C([-h,0];H)$ with sup norm, $\|\phi\|_{C_H} = \sup_{s\in[-h,0]} |\phi(s)|$, as the family of mappings $U(t,\tau) : C_H \to C_H$ given by

$$U(t,\tau)\phi = u_t(\cdot;\tau,\phi), \tag{8}$$

for any $\phi \in C_H$, and any $\tau \leq t$, where $u_t(\cdot;\tau,\phi) \in C_H$ is defined by

$$u_t(s;\tau,\phi) = u(t+s;\tau,\phi) \quad \forall\, s \in [-h,0]. \tag{9}$$

Proposition 4 *It is easy to check that if G satisfies c1), c2) with $g = 1$, (6) and (7), then the family of mappings $U(\tau,t)$, $\tau \leq t$, defined by (8) and (9) is a continuous process on C_H.*

Now, we will obtain that, under suitable assumptions, there exists a family of bounded pullback absorbing sets in C_H and then, another one in C_V, for the process $U(t,\tau)$.

Theorem 5 *Assume that G satisfies c1), c2) with $g = 1$, (6), (7), and $\nu^2\lambda_1^2(1-\rho_*) > a$.*
Let $\varepsilon > 0$ denote the unique solution of $\varepsilon - \nu\lambda_1 + \frac{ae^{\varepsilon h}}{\nu\lambda_1(1-\rho_*)} = 0$.
Let us suppose that $\int_{-\infty}^0 e^{\varepsilon r} f(r)\,dr < \infty$, and define

$$\rho_H(t) = 1 + \frac{e^{\varepsilon(1+h-t)}}{\nu\lambda_1} \int_{-\infty}^t e^{\varepsilon r} f(r)\,dr \qquad t \in \mathbb{R}.$$

Then, for every bounded subset $D \subset C_H$ there exists a $T_D > 1+h$ such that for any $t \in \mathbb{R}$ and all $\phi \in D$ one has

$$|u(s;\tau,\phi)|^2 \leq \rho_H(t) \quad \forall\, s \in [t-h-1,t], \quad \text{for all } \tau \leq t - T_D.$$

As a direct consequence of the preceding result, we get the existence of the family of bounded absorbing sets in C_H.

In fact, one can prove the following result of existence of an absorbing family of bounded sets in $C_V = C([-h,0];V)$ and a necessary bound on the term $\int_{t+\theta_1}^{t+\theta_2} |Au(r)|^2\, dr$.

Theorem 6 *Under the assumptions in Theorem 5, there exist two positive functions $\rho_V, F \in C(\mathbb{R})$ such that for any bounded set $D \subset C_H$ and for any $t \in \mathbb{R}$,*

$$\|u(t;\tau,\phi)\|^2 \le \rho_V(t) \ \ \forall\, \tau \le t - T_D, \ \forall\, \phi \in D,$$

and

$$\int_{t+\theta_1}^{t+\theta_2} |Au(r;\tau,\phi)|^2\, dr \le F(t), \quad \forall\, \tau \le t - T_D - h, \ \forall\, \theta_1 \le \theta_2 \in [-h,0], \quad \forall\, \phi \in D,$$

where T_D is given in Theorem 5.

Finally, under an additional assumption, we can ensure the existence of the pullback attractor.

Theorem 7 *Under the assumptions in Theorem 5, suppose moreover that*

$$\sup_{s \le 0} e^{-\varepsilon s} \int_{-\infty}^{s} e^{\varepsilon r} f(r)\, dr < \infty.$$

Then, there exists a pullback attractor $\{\mathcal{A}_{C_H}(t)\}_{t\in\mathbb{R}}$ for the process $U(\cdot,\cdot)$ in C_H defined by (8) *and* (9)*. Moreover, $\mathcal{A}_{C_H}(t)$ is a bounded subset of C_V for any $t \in \mathbb{R}$.*

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UPPER AND LOWER SOLUTIONS METHOD FOR FUZZY DIFFERENTIAL EQUATIONS

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Abstract

Key words: *Fuzzy differential equations, Existence of solution, Upper and lower solutions, Monotone method.*
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1 Introduction

Factors which influence the behavior of real phenomena are very often imprecise due to inexact measurements or imprecise data. Fuzzy mathematics can be used to handle this kind of uncertainty and make predictions about the changes produced in a certain process.

We consider a model consisting on a first-order fuzzy differential equation and illustrate how the existence of solutions can be deduced considering the existence of an adequate pair of upper and lower solutions. We mention that our approach considers the concept of differentiability of fuzzy functions in the sense of Hukuhara.

Upper and lower solutions method is shown to be effective for the study of the initial value problems for fuzzy differential equations, but also for other situations, such as periodic boundary value problems.

For the development of the method of upper and lower solutions for fuzzy differential equations, we consider a partial ordering in the space E^1 of normal, upper semicontinuous, fuzzy-convex and compact-supported mappings $u : \mathbb{R} \longrightarrow [0, 1]$. The expression of the differential problem on its equivalent integral form and the application of some fixed point results allow to deduce the existence of solution, and even uniqueness of solutions.

On the other hand, for the development of the monotone iterative technique for first-order fuzzy differential equations, it is required to prove some additional results concerning preservation of ordering in convergence and the application of (relative) compactness criteria in the space E^1.

The limitations of the solutions to fuzzy differential equations from the point of view of Hukuhara differentiability are skipped by the introduction of impulses.

2 Some basic concepts

In the space of n-dimensional fuzzy sets E^n, we consider the metric

$$d_\infty(x,y) = \sup_{a\in[0,1]} d_H([x]^a,[y]^a),\ \ x,\, y \in E^n,$$

where d_H represents the Hausdorff distance in the space of nonempty compact and convex subsets of $\mathbb{R}^n$, denoted by $\mathcal{K}_C^n$.

A fuzzy differential equation $u'(t) = f(t,u(t))$, $t \in [t_0,+\infty)$, where $t_0 \in \mathbb{R}$ and $f : [t_0,+\infty) \times E^n \longrightarrow E^n$, can be written on its integral representation $x(t) = x(t_0) + \int_{t_0}^t f(s,x(s))\,ds$, $t \geq t_0$. In the sequel, for simplicity, we assume that $t_0 = 0$.

If we consider the set of fuzzy intervals E^1, then we use the following notation: for $x \in E^1$, the level sets of x are denoted by

$$[x]^a = [x_{al}, x_{ar}],\ \text{for all}\ a \in [0,1],$$

but we also use the parametric functions to represent fuzzy intervals $x_L : [0,1] \longrightarrow \mathbb{R}$, $x_L(a) = x_{al}$, $a \in [0,1]$, $x_R : [0,1] \longrightarrow \mathbb{R}$, $x_R(a) = x_{ar}$, $a \in [0,1]$.

3 Upper and lower solutions

We consider the first-order fuzzy differential equation

$$u'(t) = f(t,u(t)),\ t \in [0,+\infty), \tag{1}$$

where $f : [0,+\infty) \times E^n \longrightarrow E^n$ is a fuzzy-valued function.

First, consider $n = 1$. In order to define the concepts of upper and lower solutions to problem (1), we define the following ordering relations in the space of fuzzy intervals E^1.

Definition 1 *Given $x, y \in E^1$, we say that $x \leq y$ if and only if $x_{al} \leq y_{al}$ and $x_{ar} \leq y_{ar}$, for every $a \in [0,1]$.*

Definition 2 *Given $x, y \in E^1$, we say that $x \preceq y$ if and only if $x_{al} \geq y_{al}$ and $x_{ar} \leq y_{ar}$, for every $a \in [0,1]$, that is, $[x]^a \subseteq [y]^a, \forall\, a \in [0,1]$.*

Remark 1 *In terms of the parametric functions, $x \leq y$ is equivalent to $x_L \leq y_L$ and $x_R \leq y_R$ in $[0,1]$, and $x \preceq y$ is equivalent to $y_L \leq x_L$ and $x_R \leq y_R$ in $[0,1]$.*

Note that the ordering relation $\preceq$ makes sense also for $n > 1$, that is, given $x, y \in E^n$, we say that $x \preceq y$ if and only if $[x]^a \subseteq [y]^a, \forall\, a \in [0,1]$.

These ordering relations can be also extended to the space of fuzzy-valued functions defined on a certain real interval.

Definition 3 *Given $f, g : [a,b] \to E^1$, we say that $f \leq g$ if $f(t) \leq g(t)$, for every $t \in I$. A similar concept can be given for the ordering relation $\preceq$.*

Next, we are in conditions to define the concepts of upper and lower solution of (1).

Definition 4 *A function $\alpha \in C^1([0,+\infty), E^1)$ is a $\leq$-lower solution for (1) if*

$$\alpha'(t) \leq f(t, \alpha(t)),\ t \in [0,+\infty).$$

We define an $\leq$-upper solution $\beta \in C^1([0,+\infty), E^1)$ as a function satisfying the reversed inequality.

Analogous concepts can be defined for the partial ordering $\preceq$.

We see how the existence of solution for problem (1) can be derived from the existence of appropriate upper and lower solutions.

4 Existence of solution to fuzzy differential equations via upper and lower solutions method

First, we recall some extensions of the Banach fixed point theorem to partially ordered sets which are applicable in the study of the existence and uniqueness of solution for fuzzy differential and fuzzy integral equations.

The following fixed point result from [14] is useful to obtain the existence of solution to differential and integral equations assuming the existence of a lower solution.

Theorem 1 *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that f is continuous or X satisfies that*

$$\text{if a nondecreasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } x_n \leq x,\ \forall n. \tag{2}$$

Let $f : X \longrightarrow X$ be a monotone nondecreasing mapping such that there exists $k \in [0, 1)$ with $d(f(x), f(y)) \leq kd(x, y)$, $\forall x \geq y$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

This result extends some results in [22], and the addition of the hypothesis

$$\text{every pair of elements in } X \text{ has a lower bound or an upper bound,} \tag{3}$$

provides uniqueness of the fixed point.

In Theorem 2.4 [14], assumptions of Theorem 1 are adapted in order to obtain the existence of a fixed point of f, replacing the existence of x_0 under the conditions of Theorem 1 by the existence of $x_0 \in X$ with $x_0 \geq f(x_0)$, and also replacing condition (2) by

$$\text{if a nonincreasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } x \leq x_n,\ \forall n. \tag{4}$$

In [17], similar results are proved for nonincreasing functions f.

As illustrated in [16], conditions (2), (3) and (4) are satisfied for the spaces $(E^1, \leq)$, $(E^1, \preceq)$, $(C(J, E^1), \leq)$, and $(C(J, E^1), \preceq)$ (J a real compact interval).

In this reference, some results are given on the existence (or existence of a unique solution) for the fuzzy equation $F(x) = x$, where $F : E^1 \longrightarrow E^1$ or $F : C(J, E^1) \longrightarrow C(J, E^1)$, for J a real compact interval, in presence of an upper (or lower) solution. In the case of higher dimensional fuzzy sets (base space E^n), the same results hold, considering the partial ordering $\preceq$. Besides, the initial value problem for fuzzy differential equations is studied in E^n, $n \geq 1$, deriving results on the existence of a (unique) solution. Moreover, the fuzzy differential equation with finite delay

$$\begin{cases} u'(t) = f(t, u_t),\ t \in J = [0, T], \\ u_0 = \varphi \in C_0, \end{cases} \tag{5}$$

where $f \in C(J \times C_0, E^n)$, $C_0 = C([-\tau, 0], E^n)$, and $\tau > 0$, is also considered in [16] in relation with the method of upper and lower solutions.

Another fixed point result which allows to deduce interesting properties on the existence of solutions to problem (1) is Tarski's Fixed Point Theorem [28], in relation with the existence of fixed points for a nondecreasing function F which maps a complete lattice X into itself and such that there exists $x_0 \in X$ with $F(x_0) \geq x_0$. In the reference [18], complete lattices are analyzed in the spaces of fuzzy intervals and fuzzy-interval-valued functions and, hence, application of Tarski's Theorem allows to deduce the existence of solutions for fuzzy equations and fuzzy differential and integral equations in E^1 in presence of a lower or an upper solutions. The corresponding solutions are localized in the region delimited by the upper or/and the lower solutions.

On the other hand, consider the boundary value problem

$$\begin{cases} u'(t) = f(t, u(t)),\ t \in J = [0, T], \\ \lambda u(0) = u(T), \end{cases} \tag{6}$$

where $T > 0$, $f : J \times E^1 \longrightarrow E^1$, and $\lambda > 0$. We remark that, for a function which is differentiable in the sense of Hukuhara, the diameter of the level sets is nondecreasing in time, thus the study of periodicity presents more difficulties in the context of fuzzy differential equations.

In [8], and considering the approach of Hukuhara-differentiability, some aspects of the boundary value problem (6) are considered.

Besides, in [19], different fixed point theorems are applied to the boundary value problem (6) obtaining some results on the existence of solutions in presence of upper and lower solutions. We recall, as an example, Theorem 4.6 [19].

Theorem 2 (Theorem 4.6 [19]) *Let $M > 0$ and $\lambda > e^{MT}$. Suppose that f is continuous, the existence of a $\leq$-lower solution α to problem (6), and that the Hukuhara differences $f(t, x) -_H Mx$, exist for every (t, x) with $x \geq \alpha(t)$. Also assume the validity of the following monotonicity property*

$$f(t, x) -_H Mx \leq f(t, y) -_H My,\ \forall t \in J,\ x,\ y \in E^1,\ \alpha(t) \leq x \leq y,$$

and there exists $k > 0$ such that

$$d_\infty(f(t, x) -_H Mx, f(t, y) -_H My) \leq k d_\infty(x, y),\ \forall t \in J,\ x,\ y \in E^1,$$

with $x \geq y \geq \alpha(t)$, *where* $\frac{kT}{\ln \lambda - MT} < 1$. *Then there exists a unique solution* u *to the BVP (6) with* $u \geq \alpha$.

References [23, 24, 25] are devoted to the analysis of the existence of periodic solutions to fuzzy differential equations from the point of view of Hukuhara differentiability, solving the inconvenience of the increasing character of the diameter of the level sets of the solution by the introduction of impulses in the equation.

In particular, the problem studied in [25] is the following

$$\left\{ \begin{array}{l} u'(t) = f(t, u(t)),\ t \in [0,T],\ t \neq t_k,\ k = 1, 2, \ldots, p, \\ u(t_k^+) = I_k(u(t_k)),\ k = 1, \ldots, p, \\ u(0) = u(T), \end{array} \right. \tag{7}$$

where $T > 0$, $J = [0,T]$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $J_k = [t_{k-1}, t_k]$, for $k = 1, \ldots, p+1$, $I_k : E^1 \longrightarrow E^1$ continuous for $k = 1, 2, \ldots, p$, and $f : J \times E^1 \longrightarrow E^1$ continuous in $(J \setminus \{t_1, \ldots, t_p\}) \times E^1$ is such that there exist the limits $\lim_{t \to t_k^-} f(t,x) = f(t_k, x)$, $\lim_{t \to t_k^+} f(t,x)$ for $k = 1, \ldots, p$ and $x \in E^1$.

To complete the study of the existence and the approximation of solutions for problem (7) through the monotone iterative technique, it is necessary to study the main properties of fuzzy sets in relation with

- Calculus of the exact solution for some fuzzy 'linear' problems which are taken as auxiliary problems in the study of a nonlinear equation.

- Study of comparison results valid in the fuzzy context but extending well-known comparison results in the field of ordinary differential equations.

- Relative compactness criteria in spaces of fuzzy functions, interesting to approximate solutions by iteration taking a lower solution or an upper solution as the starting point.

- Study of the properties concerning the preservation of ordering in convergence of sequences of fuzzy sets and fuzzy functions.

With more detail, we specify the problems which are addressed in references [23, 24, 25] in order to obtain sequences of functions which approximate the extremal solutions to (7) in the functional interval delimited by a pair of well-ordered lower and upper solutions.

In [25], we calculate the exact solution for some auxiliary 'linear' problems with impulses from the point of view of Hukuhara differentiability, solutions which are given by an integral expression. Indeed, we analyze the solutions to

$$\left\{ \begin{array}{l} u'(t) + Mu(t) = \sigma(t),\ t \in (t_k, t_{k+1}),\ k = 0, 1, \ldots, p, \\ u(t_k^+) = c_k,\ k = 1, \ldots, p, \\ u(0) = u(T), \end{array} \right. \tag{8}$$

where $M > 0$, $T > 0$, $J = [0, T]$, $\sigma \in PC(J, E^1)$, and $c_k \in E^1$, $k = 1, 2, \ldots, p$. Taking into account that in the fuzzy case these problems are not equivalent, we also study

$$\begin{cases} u'(t) = -Mu(t) + \sigma(t),\ t \in (t_k, t_{k+1}),\ k = 0, 1, \ldots, p, \\ u(t_k^+) = c_k,\ k = 1, \ldots, p, \\ u(0) = u(T). \end{cases} \tag{9}$$

The study of the solvability of these problems is closely connected with the results in [5, 20], where initial value problems for non impulsive fuzzy 'linear' differential equations are considered.

Reference [23] includes comparison results which are useful to compare the solutions to different initial value problems for fuzzy 'linear' differential equations by comparing the independent term and the initial condition, in such a way that the 'sign' of the independent term and the initial condition determines the 'sign' of the solution, understanding the 'sign' as the relationship between the specific function and $\chi_{\{0\}}$, with respect to some partial ordering in E^1.

We consider two functions which are, respectively, an upper and a lower solution to the nonlinear problem (7). In order to obtain two sequences which approximate the extremal solutions to (7) between the lower and the upper solutions, we iterate starting, respectively, at the lower solution and the upper solution. To follow this procedure, we need some results from [24] which guarantee the following properties:

- If a sequence of functions with values in E^1 is pointwise convergent and all the terms are bounded by a fixed function with respect to some partial ordering, the same relation holds for the pointwise limit of the sequence.

- If a sequence of (E^1) fuzzy-valued functions defined on a real compact interval is monotone and it has a convergent subsequence, then the whole sequence is convergent to the same limit.

Besides, also in [24], it is proved a relative compactness criteria for subsets of $C(J, E^1)$, based on the relative compactness of the sets of the left- (resp., right-) branches of their elements.

Applying the tools of [23, 24], the monotone method is finally developed in [25] for the periodic boundary value problem (7), under the appropriate hypotheses on f, and functions I_k.

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PATHWISE STATIONARY SOLUTIONS FOR STOCHASTIC NEURAL NETWORKS WITH DELAY

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Abstract

In this paper, a nontrivial stationary solution for a stochastic neural network with delay is studied. The analysis is done in the context of the theory of random dynamical system and the idea of M-matrices.

Key words: *stochastic neural networks, delay equations, random dynamical systems, M-matrices, random fixed points.*
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1 Introduction

The analysis of neural networks is an interesting and very important mathematical field due to its wide range of applications. They include, for example, the construction of artificial intelligence, models for neurobiology and image recognition. Many of those applications can be described by a neural network that was introduced in [4] by Micheal A. Cohen and Stephen Grossberg. It is referred to as Cohen-Grossberg neural network. In this article we consider the neural network with delay of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j^t) + I_i(t) \tag{1}$$

for $i = 1, \ldots, n$ and $t \geq 0$ (cf. [7]) with initial condition $x_i(t) = \xi_i(t)$ for $t \in [-h, 0]$ with $h > 0$. The whole mathematical background will be given in the next section. As described in [7], n denotes the number of neurons in the network and $x_i(t)$ the state of the ith neuron at time t. The functions f_j and g_j are called activation functions of the jth neuron. $I_i(t)$ is the external bias on the ith neuron at time t. a_{ij} and b_{ij} represent the connection weight of the jth neuron on the ith neuron and c_i denotes the rate with which the ith neuron

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will reset its potential to the resting state in isolation when disconnected from the network and external inputs.

In this article we are interested in equilibrium states of such a network. These states play an important role in the behavior of neural networks which can be represented clearly by using the example of image recognition: Some kind of input stimulates neurons that pass on the stimulations until the neural network evaluates to a stable state (the output) that should represent the input. In our case the input is, for instance, a blurry picture which is transformed to its sharp original by the network. Thereby the original picture represents the equilibrium state.

One can imagine that the forwarding of the stimulations can be randomly influenced. That is the reason why we consider a stochastic neural network, i.e. we replace the external bias by white noise (cf. [10]), so that (1) becomes

$$\mathrm{d}x_i(t) = \left[-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j^t) \right] \mathrm{d}t + \mathrm{d}W_i(t) \qquad (2)$$

where W_i is a two-sided Wiener process for $i = 1, \dots, n$ defined on a probability space that will be introduced in section 2.3. Our intention is to investigate the dynamics of such a system with respect to the theory of random dynamical systems in which equilibrium states are represented by so called random fixed points. The main goal of this article is to find such a fixed point in the *pathwise* sense rather than in the mean square sense (cf. [10]).

When dealing with delay differential equations one often assumes a condition to estimate the delay by a term which is independent of that delay (cf. [3, p. 275] and the citations therein). We try to overcome such an assumption by using the concept of M-matrices and a *general Gronwall inequality* which is based on the so called Halanay inequality (cf. [5, p. 378]).

2 Preliminaries

2.1 Random dynamical systems

In this paper we need some basic definitions concerning the theory of random dynamical systems (RDS) such as, for example, the concept of temperedness and random fixed points. But due to the interest of brevity we refer to [3, pp. 282–283] where all of these definitions have been introduced.

2.2 M-matrices

We denote by $Z^{n\times n}$ the class of Z-matrices which consists of matrices with nonpositive off-diagonal elements. In particular, we are interested in nonsingular M-matrices which are elements of $Z^{n\times n}$. As described in [2, p. 132], M-matrices often occur in physical and biological science and, for example, play a role in finite difference methods for PDEs and in Markov processes.

For convenience we introduce the notation $|M| = (\|m_{ij}\|_{\mathbb{R}})_{i,j}$ for a matrix $M = (m_{ij})_{i,j}$ with real-valued entries where $\|\cdot\|_{\mathbb{R}}$ denotes the absolute value of a real number. Furthermore, we interpret the notation $A \leq B$ resp. $A < B$ for two matrices $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ componentwise as $a_{ij} \leq b_{ij}$ resp. $a_{ij} < b_{ij}$ for all i, j.

Definition 1 *$A \in Z^{n\times n}$ is called a nonsingular M-matrix if there exists a vector $x > 0$ such that $Ax > 0$.*

The concept of M-matrices will be used below to prove a *generalized Gronwall lemma* which is the main ingredient to show the existence of a stationary solution to the neural network.

Remark 1 *There are a lot of equivalent definitions for matrices to be a M-matrix. 50 of them can be found in [2, pp. 134].*

2.3 Stochastic neural network

We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The $\mathbb{R}^n$-valued two-sided Wiener process $W = (W_1, \dots, W_n)^\top$ generates a canonical probability space where Ω is the space $C_0(\mathbb{R}, \mathbb{R}^n)$, which consists of continuous functions that are zero at zero, $\mathcal{F}$ the associated Borel-σ-algebra with respect to the compact open topolgy of the Fréchet space Ω and $\mathbb{P}$ the Wiener measure. The metric dynamical system (cf. [3, p. 273]) is defined by the so called *Wiener shift* operators $\theta_t : \Omega \to \Omega, \omega \mapsto \omega(\cdot + t) - \omega(t)$ for $t \in \mathbb{R}$, $\omega \in \Omega$. Notice that $\mathbb{P}$ is ergodic w.r.t. θ. What we have introduced is called the *Wiener space*.

We want to denote the neural network (2) in a more comprehensive vector form. We denote by $C([-h, 0]; \mathbb{R}^n)$ the space of continuous functions from $[-h, 0]$ into $\mathbb{R}^n$ equipped with the supremum norm. For $j = 1, \dots, n$ and $t \geq 0$ the function $x_j^t \in C([-h, 0]; \mathbb{R})$ is defined by $x_j^t(s) = x_j(t + s)$ for $s \in [-h, 0]$. By using the notations

- $x(t) = (x_1(t), \dots, x_n(t))^\top$, $\xi(t) = (\xi_1(t), \dots, \xi_n(t))^\top$, $W(t) = (W_1(t), \dots, W_n(t))^\top$
- $A = (a_{ij})_{i,j=1,\dots,n}$, $B = (b_{ij})_{i,j=1,\dots,n}$, $C = \operatorname{diag}(c_1, \dots, c_n)$
- $f(x(t)) = (f_1(x_1(t)), \dots, f_n(x_n(t)))^\top$, $g(x^t) = (g_1(x_1^t), \dots, g_n(x_n^t))^\top$

we can rewrite (2) into the vector form

$$\begin{cases} \mathrm{d}x(t) = [-Cx(t) + Af(x(t)) + Bg(x^t)]\,\mathrm{d}t + \mathrm{d}W(t) & , t \geq 0 \\ x(t) = \xi(t) & , t \in [-h, 0]. \end{cases} \tag{3}$$

We assume c_i to be a positive constant and a_{ij}, b_{ij} to be nonnegative constants for $i, j = 1, \dots, n$. In addition, we suppose the operators $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : C([-h, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ to satisfy the Lipschitz condition, i.e.

$$\begin{aligned} |f(u) - f(v)| &\leq L^f |u - v| && \text{for } u, v \in \mathbb{R}^n, \\ |g(x) - g(y)| &\leq L^g |x - y|_{C([-h,0];\mathbb{R}^n)} && \text{for } x, y \in C([-h, 0]; \mathbb{R}^n) \end{aligned}$$

where

$$|x|_{C([-h,0];\mathbb{R}^n)} := \sup_{-h\le s\le 0} |x(s)| = \left(\sup_{-h\le s\le 0} \|x_1(s)\|_{\mathbb{R}}, \dots, \sup_{-h\le s\le 0} \|x_n(s)\|_{\mathbb{R}} \right)^\top$$

for $x \in C([-h,0];\mathbb{R}^n)$.

For a pathwise investigation of the neural network we transform the stochastic differential equation (SDE) into a random differential equation (RDE). Using the stationary solution $(\omega, t) \mapsto z^*(\theta_t\omega)$ (known as the Ornstein-Uhlenbeck process or OU process) of the SDE $\mathrm{d}z(t) = -Cz(t)\,\mathrm{d}t + \mathrm{d}W(t)$ we can rewrite (3) into the RDE

$$\begin{cases} \dot{u}(t) = -Cu(t) + Af(u(t) + z^*(\theta_t\omega)) + Bg(u^t + z^*(\theta_{\cdot+t}\omega)) & ,t \ge 0 \\ u(t) = \xi(t) - z^*(\theta_t\omega) =: \mu(t) & ,t \in [-h,0] \end{cases} \tag{4}$$

with $u(t) := x(t) - z^*(\theta_t\omega)$. We define

$$F : C([-h,0];\mathbb{R}^n) \to \mathbb{R}^n, \quad x^t \mapsto Af(x^t(0) + z^*(\theta_t\omega)) + Bg(x^t + z^*(\theta_{\cdot+t}\omega)).$$

As a result of

$$\begin{aligned} |F(x^t) - F(y^t)| &\le AL^f|x(t) - y(t)| + BL^g|x^t - y^t|_{C([-h,0];\mathbb{R}^n)} \\ &\le AL^f \sup_{-h\le s\le 0} |x(t+s) - y(t+s)| + BL^g|x^t - y^t|_{C([-h,0];\mathbb{R}^n)} \\ &= (AL^f + BL^g)|x^t - y^t|_{C([-h,0];\mathbb{R}^n)} \end{aligned}$$

F is Lipschitz continuous. In addition, $\mu \to \varphi(t, \omega, \mu)$ is continuous (cf. (12)). Hence there exists a unique solution to (4) which generates an RDS φ given by

$$\varphi : \mathbb{R}_0^+ \times \Omega \times C([-h,0];\mathbb{R}^n) \to C([-h,0];\mathbb{R}^n), \quad (t, \omega, \mu) \mapsto u^t(\cdot, \omega, \mu) \tag{5}$$

for $t \ge 0$, $\omega \in \Omega$ and $\mu \in C([-h,0];\mathbb{R}^n)$ (cf. [3, p. 286]).

3 Stationary solution

Gronwall's lemma resp. inequality plays an important role in many topics studying the qualitative behavior of differential equations. We will use a special kind of such an inequality which we call *generalized Gronwall inequality*. It is mainly based on the *Halanay inequality* introduced in [5, p. 378]. A generalization can already be found in [9, p. 111]. However, the inequality to the generalization presented below is extended by a function added to the right handside which is why refer to it as the *strong version*. The generalization in [9] will be called the *weak version*.

We recall that the inequalities are to be understood in the componentwise sense. D^+ denotes the usual Dini derivative and E the $n \times n$ identity matrix.

Lemma 1 (Generalized Gronwall lemma) *Let $P = (p_{ij})$ with $p_{ij} \geq 0$ for $i \neq j$ and $Q = (q_{ij}) \geq 0$ be two $n \times n$ matrices such that $-(P+Q)$ is a nonsingular M-matrix. Consider $T > 0$ and assume that $u \in C([0,T];\mathbb{R}^n)$ satisfies the differential inequality*

$$\mathrm{D}^+ u(t) \leq Pu(t) + Q \sup_{-h \leq s \leq 0} u(t+s) + \operatorname{diag}(K_1, \ldots, K_n) v(t), \qquad t \geq 0 \quad (6)$$

where v is a nonnegative function, the initial condition $\mu \in C([-h,0];\mathbb{R}^n)$ fulfills

$$\mu(s) \leq K \mathrm{e}^{-\lambda s} \qquad \text{for} \quad s \in [-h, 0]. \quad (7)$$

$K = (K_1, K_2, \ldots, K_n)^\top > 0$ and $\lambda > 0$ are determined by

$$(P + Q\mathrm{e}^{\lambda h} + \lambda E)K < 0.$$

Then we have

$$u(t) \leq K\mathrm{e}^{-\lambda t} + \operatorname{diag}(K_1, \ldots, K_n) \int_0^t \mathrm{e}^{-\lambda(t-r)} v(r)\,\mathrm{d}r \qquad \text{for} \quad t \geq 0. \quad (8)$$

Proof. Since $-(P+Q)$ is a nonsingular M-matrix there exists a constant vector $K = (K_1, K_2, \ldots, K_n)^\top > 0$ such that $(P+Q)K < 0$. Hence, by continuity, there is a constant $\lambda > 0$ with $(\lambda E + P + Q\mathrm{e}^{\lambda h})K < 0$.

Next we want to show that for any $t \geq 0$ it holds

$$u_i(t) \leq K_i \mathrm{e}^{-\lambda t} + K_i \int_0^t \mathrm{e}^{-\lambda(t-r)} v_i(r)\,\mathrm{d}r =: \Phi_i(t) \qquad \text{for} \quad i = 1, \ldots, n. \quad (9)$$

Assume that (9) is false. Then there exist $t^* > 0$ and $m \in \{1, \ldots, n\}$ such that $u_m(t^*) > \Phi_m(t^*)$. Because of (7), the continuity of u and the nonnegativity of v there also exists a $t_0 \geq 0$ such that $u_m(t_0) = \Phi_m(t_0)$ and

$$\mathrm{D}^+ u_m(t_0) \geq \mathrm{D}^+ \Phi_m(t_0). \quad (10)$$

In addition, we can choose t_0 such that for all $i \in \{1, \ldots, n\}$ it holds $u_i(t) \leq \Phi_i(t)$ for $t \in [-h, t_0]$. This is justified by (7) once again. Note that the last inequality implies $\sup_{-h \leq s \leq 0} u_i(t_0 + s) \leq \sup_{-h \leq s \leq 0} \Phi_i(t_0 + s)$ for $i \in \{1, \ldots, n\}$. But then we also have

$$\begin{aligned}
&\mathrm{D}^+ u_m(t_0) \\
&\leq \sum_{i=1}^n \left\{ p_{mi} u_i(t_0) + q_{mi} \sup_{-h \leq s \leq 0} u_i(t_0 + s) \right\} + K_m v_m(t_0) \\
&\leq \sum_{i=1}^n \left\{ p_{mi} K_i \mathrm{e}^{-\lambda t_0} + p_{mi} K_i \mathrm{e}^{-\lambda t_0} \int_0^{t_0} \mathrm{e}^{\lambda r} v_i(r)\,\mathrm{d}r + q_{mi} K_i \sup_{-h \leq s \leq 0} \left[\mathrm{e}^{-\lambda(t_0 + s)} \right] \right. \\
&\qquad \left. + q_{mi} K_i \sup_{-h \leq s \leq 0} \left[\mathrm{e}^{-\lambda(t_0+s)} \int_0^{t_0+s} \mathrm{e}^{\lambda r} v_i(r)\,\mathrm{d}r \right] \right\} + K_m v_m(t_0)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} \left\{ p_{mi} K_i \mathrm{e}^{-\lambda t_0} + p_{mi} K_i \mathrm{e}^{-\lambda t_0} \int_0^{t_0} \mathrm{e}^{\lambda r} v_i(r)\,\mathrm{d}r + q_{mi} K_i \mathrm{e}^{-\lambda(t_0-h)} \right. \\
&\quad \left. + q_{mi} K_i \mathrm{e}^{-\lambda(t_0-h)} \int_0^{t_0} \mathrm{e}^{\lambda r} v_i(r)\,\mathrm{d}r \right\} + K_m v_m(t_0) \\
&= \sum_{i=1}^{n} \left\{ \left(p_{mi} + q_{mi}\mathrm{e}^{\lambda h}\right) K_i \mathrm{e}^{-\lambda t_0} + \left(p_{mi} + q_{mi}\mathrm{e}^{\lambda h}\right) K_i \mathrm{e}^{-\lambda t_0} \int_0^{t_0} \mathrm{e}^{\lambda r} v_i(r)\,\mathrm{d}r \right\} \\
&\quad + K_m v_m(t_0) \\
&< (-\lambda) K_m \mathrm{e}^{-\lambda t_0} + (-\lambda) K_m \mathrm{e}^{-\lambda t_0} \int_0^{t_0} \mathrm{e}^{\lambda r} v_m(r)\,\mathrm{d}r + K_m v_m(t_0) \\
&= -\lambda K_m \left(\mathrm{e}^{-\lambda t_0} + \mathrm{e}^{-\lambda t_0} \int_0^{t_0} \mathrm{e}^{\lambda r} v_m(r)\,\mathrm{d}r \right) + K_m v_m(t_0) \\
&= -\lambda \Phi_m(t_0) + K_m v_m(t_0) \\
&= \mathrm{D}^+ \Phi_m(t_0)
\end{aligned}
$$

which contradicts (10). Hence (9) is true and the lemma is proved. □

Remark 2 *If we replace $v(t)$ in (6) by $\operatorname{diag}(K_1^{-1}, \ldots, K_n^{-1})v(t)$ the diagonal matrix in (8) can be taken as the identity matrix. Hence the lemma above is still valid if we think of the term $\operatorname{diag}(K_1, \ldots, K_n)$ in (6) and (8) to be the identity matrix.*

Next, we prove that two solutions to (4) approach each other exponentially fast as the time tends to infinity. We denote

$$
\|x\| := \max_{i=1,\ldots,n} \left(\sup_{-h \leq s \leq 0} \|x_i(s)\|_{\mathbb{R}} \right).
$$

Note that $\|\cdot\|$ is equivalent to the norm in $C([-h,0];\mathbb{R}^n)$ given by $\|\cdot\|_{C([-h,0];\mathbb{R}^n)} := \sup_{-h \leq s \leq 0} \|\cdot(s)\|_{\mathbb{R}^n}$.

Lemma 2 *Let $\tilde{u}$ and $\hat{u}$ be two solutions to (4) with initial conditions $\tilde{\mu}$ resp. $\hat{\mu}$ and assume that $-(L^f|A| + L^g|B| - C)$ is a nonsingular M-matrix. Then it holds*

$$
|\tilde{u}(t) - \hat{u}(t)| \leq c\|\tilde{\mu} - \hat{\mu}\|\mathrm{e}^{-\lambda t} K
$$

for $t \geq -h$ where c is a constant and K and λ are given by Lemma 1.

Proof. We denote by $\triangle u(t)$ and $\triangle \mu(t)$ the difference between the two solutions and initial conditions, respectively. We need to show the assumptions (6) and (7) to apply the generalized Gronwall Lemma.

`Step 1:` Let $t \geq 0$ and sgn denote the usual signum function. For $\triangle u(t) \neq 0$ it holds

$$\begin{aligned} \mathrm{D}^+|\triangle u(t)| &= \operatorname{sgn}(\triangle u(t)) \frac{\mathrm{d}\triangle u}{\mathrm{d}t}(t) \\ &\leq -C|\triangle u(t)| + |A||f(\tilde{u}(t) + z^*(\theta_t\omega)) - f(\hat{u}(t) + z^*(\theta_t\omega))| \\ &\quad + |B||g(\tilde{u}^t + z^*(\theta_{\cdot+t}\omega)) - g(\hat{u}^t + z^*(\theta_{\cdot+t}\omega))| \\ &\leq \left(L^f|A| - C\right)|\triangle u(t)| + L^g|B| \sup_{-h\leq s\leq 0} |\triangle u(t+s)| \end{aligned}$$

and for $\triangle u(t) = 0$ we derive (cf. [6, p. 87])

$$\mathrm{D}^+|\triangle u(t)| \leq L^g|B| \sup_{-h\leq s\leq 0} |\triangle u(t+s)|$$

based on the continuous trajectories of the OU process (cf. [3, p. 285]). Hence condition (6) is fulfilled.
`Step 2:` For $t \in [-h, 0]$ we have

$$|\triangle u(t)| \leq \max_{i=1,\ldots,n} \left(\sup_{-h\leq s\leq 0} \|\triangle \mu_i(s)\|_{\mathbb{R}} \right) \mathbf{1} \leq c\|\triangle\mu\| \mathrm{e}^{-\lambda t} K \tag{11}$$

where $\mathbf{1} := (1, \ldots, 1)^\top$ and $c := \frac{1}{\min_{i=1,\ldots,n} K_i} > 0$. Hence assumption (7) is satisfied whereby K is replaced by $c\|\triangle\mu\|K$.

Therefore we can apply Lemma 1 (in its weak sense) and receive $|\triangle u(t)| \leq c\|\triangle\mu\|\mathrm{e}^{-\lambda t}K$ for $t \geq 0$ which is also true for $t \geq -h$ since (11) holds. □

In particular, we have $\|\triangle u_i(t)\|_{\mathbb{R}} \leq c\|\triangle\mu\|\mathrm{e}^{-\lambda t}K_i$ for $i \in \{1, 2, \ldots, n\}$ and $t \geq -h$. Hence $\sup_{-h\leq s\leq 0} \|\triangle u_i(t+s)\|_{\mathbb{R}} \leq c\|\triangle\mu\|\mathrm{e}^{-\lambda(t-h)}K_i$ for $t \geq 0$. Taking the maximum on both hand sides and defining $C := c\mathrm{e}^{\lambda h} \max_{i=1,\ldots,n} K_i$ yield $\|\triangle u^t\| \leq C\|\triangle\mu\|\mathrm{e}^{-\lambda t}$. Due to the definition of the RDS given in (5) we get

$$\|\triangle\varphi(t, \omega, \mu)\| \leq C\|\triangle\mu\|\mathrm{e}^{-\lambda t}. \tag{12}$$

Now we can show the existence of a nontrivial stationary solution to the neural network.

Theorem 3 *Assume that $-(L^f|A| + L^g|B| - C)$ is a nonsingular M-matrix. Then the neural network* (4) *has a unique exponentially attracting random fixed point $u^*(\omega)$ where $\|u^*(\omega)\|$ is tempered.*

Proof. The proof follows the method described by Schmalfuss (cf. [8, pp. 95–96]). We do not give the full details here but note that the existence of a Cauchy sequence is based on (12) and similar calculations used in the proof of Lemma 2. We emphasize that for these calculations, however, we need the strong version of the generalized Gronwall lemma. In addition, the temperedness of the OU process w.r.t. $C([-h, 0]; \mathbb{R})$ is necessary. It bases upon the temperedness of the

OU process w.r.t. to $\mathbb{R}$ (cf. [3, pp. 284–285]) and is therefore shown by

$$\lim_{t\to\pm\infty} \frac{\log^+ \|z_i^*(\theta_t\omega)\|_{C([-h,0],\mathbb{R})}}{\|t\|_{\mathbb{R}}} = \lim_{t\to\pm\infty} \frac{\log^+ \sup_{-h\le s\le 0} \|z_i^*(\theta_{s+t}\omega)\|_{\mathbb{R}}}{\|t\|_{\mathbb{R}}}$$

$$= \lim_{t\to\pm\infty} \underbrace{\frac{\log^+ \|z_i^*(\theta_{s_0(t)+t}\omega)\|_{\mathbb{R}}}{\|s_0(t)+t\|_{\mathbb{R}}}}_{\to 0} \underbrace{\frac{\|s_0(t)+t\|_{\mathbb{R}}}{\|t\|_{\mathbb{R}}}}_{\to 1} = 0$$

where $i \in \{1,\dots,n\}$ and $s_0(t) \in [-h,0]$ is defined by $\sup_{-h\le s\le 0} \|z_i^*(\theta_{s+t}\omega)\|_{\mathbb{R}} = \|z_i^*(\theta_{s_0(t)+t}\omega)\|_{\mathbb{R}}$. □

Remark 3 *Due to the stationarity of the OU process a similar result is valid for the SDE* (3)*. This can be shown by transforming the solution to* (4) *back.*

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AN INVERSE PROBLEM ON VAKONOMIC MECHANICS

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Abstract

We study a version of the inverse problem of Calculus of Variations in the context of Vakonomic Mechanics.

Key words: *Vakonomic Mechanics, Inverse Problem of the Calculus of Variations*
AMS subject classifications: *49N45, 37J60.*

1 Introduction

The classical inverse problem of the Calculus of Variations consists in finding conditions under which a given system of differential equations derives from a variational principle. The origins of this problem date back to Helmholtz ([5]), in the end of the 19th century, who sought new applications of the powerful Hamilton-Jacobi method to integrate the equations of Mechanics. Two major contributions to solve this problem were made in the last century: the first, by Douglas in the 1930s in his classic paper ([3], [4]); the second, in the 1980s, by Vinogradov ([14], [15]), Tulczyjew ([13]), Anderson ([1]), Tsujishita ([12]) , among others (see references in [1]), who geometrized the problem through the introduction of the so-called *variational bicomplex*. That is a double complex of differential forms in a Fréchet manifold of infinite jets of sections of a fibered manifold, one of the coboundary operators of which is the classical Euler-Lagrange operator of the Calculus of Variations.

We consider a variation of this problem related to *Vakonomic Mechanics*: given a smooth finite dimensional manifold M and a system of mixed first- and second-order differential equations, we study conditions under which these equations are the Vakonomic equations induced by a non-autonomous Lagrangian L defined on $\mathbb{R} \times \mathsf{TM}$.

This work is developed in [8].

2 Notations and Definitions

In this section we fix some notation for jet bundles and we define the variational bicomplex on the bundle of infinite jets of sections of a fibration. We particularize the definitions for a fibration $\pi : \mathsf{E} \to \mathbb{R}$ over $\mathbb{R}$, where E is a smooth n-dimensional manifold. The reader is referred to [9] and [1] for more details on the bundle of infinite jets and the variational bicomplex.

For all $k \in \mathbb{N}$, $\pi_k : \mathsf{J}^k\pi \to \mathbb{R}$ denotes the bundle of k-jets of sections of π and, for $0 \leqslant l < k$, $\pi_{k,l} : \mathsf{J}^k\pi \to \mathsf{J}^l\pi$ denotes the natural projections (where $\mathsf{J}^0\pi \doteq \mathsf{E}$). We call π_1 the *source projection* and $\pi_{1,0}$ the *target projection* of the 1st jet bundle $\mathsf{J}^1\pi$. Let $(t, u^\alpha)_{1\leqslant\alpha\leqslant n}$ be coordinates on an open set $\mathcal{U} \subset \mathsf{E}$ adapted to the fibration $\mathsf{E} \to \mathbb{R}$. This coordinate system induce coordinates $(t, u^\alpha_{(i)})_{1\leqslant\alpha\leqslant n, 0\leqslant i\leqslant k}$ on $\mathcal{U}^k \doteq \pi_{k,0}^{-1}\mathcal{U} \subset \mathsf{J}^k\pi$. We use the notation $\dot{u}^\alpha \doteq u^\alpha_{(1)}$ and $\ddot{u}^\alpha \doteq u^\alpha_{(2)}$, so that $(t, u^\alpha, \dot{u}^\alpha)_{1\leqslant\alpha\leqslant n}$ and $(t, u^\alpha, \dot{u}^\alpha, \ddot{u}^\alpha)_{1\leqslant\alpha\leqslant n}$ are coordinates on $\mathcal{U}^1 \subset \mathsf{J}^1\pi$ and $\mathcal{U}^2 \subset \mathsf{J}^2\pi$, respectively.

Let $s : I \subset \mathbb{R} \to \mathsf{E}$ be a smooth section of π. Given $t \in I$, we denote by $\mathrm{j}^\infty_t\, s$ the equivalence class of all sections of π defined on a neighborhood of t whose derivatives of all orders at t coincide with that of s. Such an equivalence class is called an *infinite order jet*. We denote by $\mathsf{J}^\infty\pi$ the Fréchet manifold of infinite order jets of sections of π; it is a smooth manifold modelled on the Fréchet space $\mathbb{R}^\infty$. We denote by $(\forall\, k \in \mathbb{Z}_+)$ $\pi_{\infty,k} : \mathsf{J}^\infty\pi \to \mathsf{J}^k\pi$ and $\pi_\infty : \mathsf{J}^\infty\pi \to \mathbb{R}$ the natural projections. The chart $(t, u^\alpha)_{1\leqslant\alpha\leqslant n}$ on $\mathcal{U} \subset \mathsf{E}$ adapted to the fibration $\mathsf{E} \to \mathbb{R}$ induces a chart $(t, u^\alpha_{(i)})_{1\leqslant\alpha\leqslant n, 0\leqslant i\leqslant\infty}$ on $\mathcal{U}^\infty \doteq \pi_{\infty,0}^{-1}\mathcal{U} \subset \mathsf{J}^\infty\pi$.

We say that a smooth function on $\mathsf{J}^\infty\pi$ has order $k \in \mathbb{Z}_+$ if it is the pullback by $\pi_{\infty,k}$ of a smooth function on $\mathsf{J}^k\pi$. We say that a smooth function on $\mathsf{J}^\infty\pi$ is of *finite order* if, for some $k \in \mathbb{Z}_+$, it has order k. Differential forms on $\mathsf{J}^\infty\pi$ of finite order are similarly defined. In this paper, all smooth functions or differential forms on $\mathsf{J}^\infty\pi$ are assumed to be of finite order.

There exists a natural bigrading on the $\mathbb{R}$-vector space of differential forms on $\mathsf{J}^\infty\pi$:

$$\Omega_*(\mathsf{J}^\infty\pi) = \bigoplus_{0\leqslant i\leqslant 1,\, 0\leqslant j\leqslant\infty} \Omega_{i,j}(\mathsf{J}^\infty\pi).$$

A differential form belongs to $\Omega_{i,j}(\mathsf{J}^\infty\pi)$ if, locally, on the charts described above, it is a sum of terms of the form $f\, \mathrm{d}t^i \wedge \delta u^{\alpha_1}_{(k_1)} \wedge \cdots \wedge \delta u^{\alpha_j}_{(k_j)}$, where f is a smooth function on $\mathsf{J}^\infty\pi$ and $\delta u^\alpha_{(j)} \doteq \mathrm{d}u^\alpha_{(j)} - u^\alpha_{(j+1)}\mathrm{d}t$. Such a form is said to be of *type* (i, j), or *i-horizontal and j-vertical*. Given $\omega \in \Omega_{i,j}(\mathsf{J}^\infty\pi)$, its exterior derivative $\mathrm{d}\omega$ belongs to $\Omega_{i+1,j}(\mathsf{J}^\infty\pi) \oplus \Omega_{i,j+1}(\mathsf{J}^\infty\pi)$; we denote by $\mathrm{d_h}\omega$ its projection on the first factor and by $\mathrm{d_v}\omega$ its projection on the second one. We extend $\mathrm{d_h}$ and $\mathrm{d_v}$ to $\Omega_*(\mathsf{J}^\infty\pi)$ by linearity; $\mathrm{d_h}$ is called *horizontal derivative* and $\mathrm{d_v}$ is called *vertical derivative*. They are both anti-derivations on $\Omega_*(\mathsf{J}^\infty\pi)$ of degree $+1$ and satisfy ${\mathrm{d_h}}^2 = 0$, ${\mathrm{d_v}}^2 = 0$, $\mathrm{d_h}\mathrm{d_v} = -\mathrm{d_v}\mathrm{d_h}$. Therefore, for each fixed i we obtain a cochain complex $\big(\Omega_{i,j}(\mathsf{J}^\infty\pi), \mathrm{d_v}\big)_{j\geqslant 0}$ — the *columns* of the so-called *variational bicomplex* — and for each fixed j we obtain a cochain complex $\big(\Omega_{i,j}(\mathsf{J}^\infty\pi), \mathrm{d_h}\big)_{i\geqslant 0}$ — the *lines* of the variational bicomplex. The horizontal and vertical derivatives can be easily computed in coordinates: if

$f = f(t, u^\alpha, u^\alpha_{(1)}, \dots, u^\alpha_{(k)})$ is a smooth function on $\mathsf{J}^\infty\pi$, we have:

$$\mathrm{d_h} f = \mathrm{D_t} f \, \mathrm{d}t$$
$$\mathrm{d_v} f = \sum_{\alpha=1}^{n} \sum_{j=0}^{k} \frac{\partial f}{\partial u^\alpha_{(j)}} \, \delta u^\alpha_{(j)},$$

where $\mathrm{D_t} f = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{n} \sum_{j=0}^{k} \frac{\partial f}{\partial u^\alpha_{(j)}} \, u^\alpha_{(j+1)}$ is the *total derivative* of f; moreover, $\mathrm{d_h}\mathrm{d}t = 0 = \mathrm{d_v}\mathrm{d}t$ and $\mathrm{d_h}\delta u^\alpha_{(k)} = \mathrm{d}t \wedge \delta u^\alpha_{(k)}$, $\mathrm{d_v}\delta u^\alpha_{(k)} = 0$.

The differential forms in $\Omega_{1,0}(\mathsf{J}^\infty\pi)$ are called *Lagrangian forms.* In coordinates, a Lagrangian form may be written as $\mathrm{L}\,\mathrm{d}t$, where L is a smooth function on $\mathsf{J}^\infty\pi$; the Lagrangian is said to be of order k if L is of order k, i.e. if $\mathrm{L} = \mathrm{L}(t, u^\alpha, u^\alpha_{(1)}, \dots, u^\alpha_{(k)})$. Such a Lagrangian induces a functional on compactly supported smooth sections of π: $s \mapsto \int_{\mathbb{R}} (\mathrm{j}^\infty s)^* \mathrm{L}\,\mathrm{d}t$. A smooth section s of π is said to be an *extremal* of the functional induced by $\mathrm{L}\mathrm{d}t$ if, for any compactly supported variation s_τ of s, we have $\frac{d}{d\tau}_{|\tau=0} \int_{\mathbb{R}} (\mathrm{j}^\infty s_\tau)^* \mathrm{L}\,\mathrm{d}t = 0$. A necessary and sufficient condition for s to be an extremal of the functional induced by $\mathrm{L}\mathrm{d}t$ is that the Euler-Lagrange form Ω_L of L be null along $\mathrm{j}^\infty s$. The Euler-Lagrange form Ω_L is a form in $\Omega_{1,1}(\mathsf{J}^\infty\pi)$ which, in coordinates, is written as:

$$\Omega_\mathrm{L} = \sum_{\alpha=1}^{n} \mathrm{E}_\alpha(\mathrm{L}) \delta u^\alpha \wedge \mathrm{d}t, \tag{1}$$

where $\mathrm{E}_\alpha(\mathrm{L}) = \sum_{i=0}^{k} (-\mathrm{D_t})^i \frac{\partial \mathrm{L}}{\partial u^\alpha_{(i)}}$ for a Lagrangian of order k.

For $s \geqslant 1$, we call the quotient $\mathcal{F}^s(\mathsf{J}^\infty\pi) \doteq \Omega_{1,s}(\mathsf{J}^\infty\pi)/\mathrm{d_h}\Omega_{0,s}(\mathsf{J}^\infty\pi)$ space of *type s functional forms.* It may be identified with the subspace of $\Omega_{1,s}(\mathsf{J}^\infty\pi)$ which is the image of the *interior Euler operator* $I : \Omega_{1,s}(\mathsf{J}^\infty\pi) \to \Omega_{1,s}(\mathsf{J}^\infty\pi)$. In coordinates, we have $I(\omega) = \frac{1}{s} \sum_{\alpha=1}^{n} \delta u^\alpha \wedge \mathrm{F}_\alpha(\omega)$, where $\mathrm{F}_\alpha(\omega) \doteq \sum_{i=0}^{k} (-\mathrm{D_t})^i [\delta u^\alpha_{(i)} \lrcorner\, \omega]$ if ω has order k. For $s = 1$, a differential form in $\Omega_{1,1}(\mathsf{J}^\infty\pi)$ is a type 1 functional form, also called a *source form*, if, and only if, it is locally of the form $\sum_{\alpha=1}^{n} P_\alpha \, \mathrm{d}t \wedge \delta u^\alpha$, where the P_α's are smooth functions on $\mathsf{J}^\infty\pi$. Thus, from (1), the Euler-Lagrange form associated to a Lagrangian is a source form, i.e. it belongs to $\mathcal{F}^1(\mathsf{J}^\infty\pi)$. We think of a source form $\omega \in \mathcal{F}^1(\mathsf{J}^\infty\pi)$ of order k as an intrinsic definition of a system of n differential equations of order k; its solutions are the sections s of π such that ω vanishes along $\mathrm{j}^\infty s$.

The spaces $\mathcal{F}^s$ are part of a cochain complex, the so-called *Euler-Lagrange complex* of the fibration $\pi : \mathsf{E} \to \mathbb{R}$:

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega_{0,0} \xrightarrow{\mathrm{d_h}} \Omega_{1,0} \xrightarrow{\mathrm{E}} \mathcal{F}^1 \xrightarrow{\delta_\mathrm{v}} \mathcal{F}^2 \xrightarrow{\delta_\mathrm{v}} \cdots \tag{2}$$

where E is the *Euler operator* $\mathrm{L}\mathrm{d}t \mapsto \Omega_\mathrm{L}$ given by (1) and $\delta_\mathrm{v} : \mathcal{F}^i \to \mathcal{F}^{i+1}$ is the vertical derivative induced on functional forms, given by $\delta_\mathrm{v} \doteq I \circ \mathrm{d_v}$.

3 Vakonomic Mechanics

We denote by $\xi : \mathbb{R} \times \mathsf{M} \to \mathbb{R}$ the trivial bundle and $\xi_1 : \mathsf{J}^1\xi \to \mathbb{R}$ its first jet bundle, which is identified with $\mathbb{R} \times \mathsf{TM}$, where $\tau_\mathsf{M} : \mathsf{TM} \to \mathsf{M}$ denotes the tangent bundle of M. We consider a smooth time-dependent Lagrangian $\mathrm{L} : \mathsf{J}^1\xi \to \mathbb{R}$ and a smooth vector sub-bundle $\Pi : \mathscr{D} \to \mathbb{R} \times \mathsf{M}$ of the target projection $\xi_{1,0} : \mathbb{R} \times \mathsf{TM} \to \mathbb{R} \times \mathsf{M}$ (which stands for the reonomic linear constraints) and we denote its annihilator in $\mathbb{R} \times \mathsf{T}^*\mathsf{M}$ by $\Pi^* : \mathscr{D}^\perp \to \mathbb{R} \times \mathsf{M}$. We denote by $\pi : \mathscr{D}^\perp \to \mathbb{R}$ the natural projection. We call $(\mathsf{M}, \mathscr{D}, \mathrm{L})$ a *(linearly) constrained mechanical system.* We say that a smooth section $\gamma : I \subset \mathbb{R} \to \mathbb{R}\times\mathsf{M}$ of ξ is *compatible with* $\mathscr{D}$ or *horizontal with respect to* $\mathscr{D}$ if its prolongation $\mathrm{j}^1\gamma$ lies in $\mathscr{D}$. There are two natural approaches to formulate equations of motion for constrained mechanical systems yielding solutions which are compatible with $\mathscr{D}$: (1)*nonholonomic mechanics* (see [7], [6], [10] and references therein), known as "mechanics of the straightest paths", based on *d'Alembert-Chetaev's principle*; (2) *Vakonomic Mechanics*(see [2], [7], [6], [11] and references therein), known as "mechanics of the shortest paths", based on the *Hamilton's principle of the stationary action.* A particular case of the latter is the so-called *sub-Riemannian geometry.*

We shall briefly recall how the equations of motion in Vakonomic Mechanics are formulated and we show how these equations may be obtained as the Euler-Lagrange equations of a modified Lagrangian $\mathscr{L}$ defined on $\mathsf{J}^1\pi$, the total space of the first jet bundle of $\pi : \mathscr{D}^\perp \to \mathbb{R}$.

The action functional induced by L on compactly supported sections $\gamma : I \subset \mathbb{R} \to \mathbb{R} \times \mathsf{M}$ of ξ, I an open interval, is defined by $\gamma \mapsto \int_I \mathrm{L} \circ \mathrm{j}^1\gamma$. We say that a section $\gamma : I \subset \mathbb{R} \to \mathbb{R} \times \mathsf{M}$ of π is a *vakonomic trajectory* of $(\mathsf{M}, \mathscr{D}, \mathrm{L})$ if it is a critical point of the action functional on compactly supported variations of γ compatible with $\mathscr{D}$. By a compactly supported variation of γ we mean a smooth map $\Gamma : (-\delta, \delta) \times I \to \mathbb{R} \times \mathsf{M}$ such that, for all $s \in (-\delta, \delta)$, $\Gamma_s \doteq \Gamma(s, \cdot) : I \to \mathbb{R} \times \mathsf{M}$ is a section of ξ, $\Gamma_0 = \gamma$ and there exists a compact set $K \subset I$ such that for all $s \in (-\delta, \delta)$ and all t outside K, $\gamma_s(t) = \gamma(t)$; we say that such a variation is compatible with $\mathscr{D}$ if, for all $s \in (-\delta, \delta)$, Γ_s is compatible with $\mathscr{D}$. There are two types of vakonomic trajectories (see [6], [11]): the *normal*, which are associated to a certain system of Euler-Lagrange equations, and the *abnormal*, which correspond to the critical points of the so-called *endpoint map.* We propose the following characterization of the normal vakonomic trajectories:

Proposition 1 *Let* $\mathscr{L} : \mathsf{J}^1\pi \to \mathbb{R}$ *be defined by:*

$$\mathrm{j}^1_t\,\Theta \mapsto \mathrm{L}(\mathrm{j}^1_t\,\gamma) + \langle \Theta(t), \mathrm{j}^1_t\,\gamma \rangle, \tag{3}$$

where $\gamma \doteq \Pi^* \circ \Theta$ *and* $\langle \cdot, \cdot \rangle$ *is the canonical pairing between* $\mathbb{R}\times\mathsf{TM}$ *and* $\mathbb{R}\times\mathsf{T}^*\mathsf{M}$. *Then the normal vakonomic trajectories are projections on* $\mathbb{R}\times\mathsf{M}$ *of the solutions of the Euler-Lagrange equations of* $\mathscr{L}$.

Definition 1 *We say that* L *is* $\mathscr{D}$-regular *if* $\mathbb{F}\mathrm{L}|_\mathscr{D} : \mathscr{D} \to \mathscr{D}^*$ *is a local diffeomorphism (where* $\mathbb{F}\mathrm{L}$ *denotes the* fiber derivative *of* L, *i.e.* $\forall\,(t, v_q) \in \mathbb{R} \times \mathsf{TM}, \mathbb{F}\mathrm{L}(t, v_q) \doteq \mathsf{D}(\mathrm{L}|_{\mathbb{R}\times\mathsf{T}_q\mathsf{M}})(t, v_q) \in \mathbb{R} \times \mathsf{T}^*_q\mathsf{M}$*).*

Definition 2 *The* mixed bundle *is the Whitney sum* $\mathscr{D} \oplus_{\mathbb{R}\times\mathsf{M}} \mathscr{D}^\perp$*. We define* $F : \mathsf{J}^1\xi \oplus_{\mathbb{R}\times\mathsf{M}} \mathscr{D}^\perp \to \mathbb{R}\times\mathsf{T}^*\mathsf{M}$ *by* $(v,\Theta)_{(t,q)} \mapsto \mathbb{F}\mathrm{L}(v) + \Theta$ *and* $\overline{F} \doteq F|_{\mathscr{D}\oplus_{\mathbb{R}\times\mathsf{M}}\mathscr{D}^\perp}$.

It is an immediate consequence of the above definitions that L is $\mathscr{D}$-regular if, and only if, $\overline{F}$ is a local diffeomorphism. Let θ be the canonical contact form on $\mathbb{R}\times\mathsf{T}^*\mathsf{M}$, $\theta_{\mathrm{L}} \doteq \overline{F}^*\theta$, $\omega_{\mathrm{L}} \doteq -\mathrm{d}\theta_{\mathrm{L}} + \mathrm{d}t \wedge \mathrm{dH}$, where $\mathrm{H} : \mathscr{D} \oplus_{\mathbb{R}\times\mathsf{M}} \mathscr{D}^\perp \to \mathbb{R}$ is given by $(v,\Theta)_{(t,q)} \mapsto \mathrm{L}(v) - \mathbb{F}\mathrm{L}(v)\cdot v$. Then, if L is $\mathscr{D}$-regular, $(\omega_{\mathrm{L}}, \mathrm{d}t)$ is a *cosymplectic structure* on $\mathscr{D} \oplus_{\mathbb{R}\times\mathsf{M}} \mathscr{D}^\perp$, where $\mathrm{d}t$ is the canonical volume form on $\mathbb{R}$. Moreover, we have the following:

Theorem 2 *If* L *is* $\mathscr{D}$*-regular, the normal vakonomic trajectories are in 1-1 correspondence with the integral curves of the Reeb vector field of the cosymplectic structure* $(\omega_{\mathrm{L}}, dt)$.

Remark 1 *This is a generalization of the well known fact that, in the autonomous case and under the same regularity hypothesis, the normal vakonomic trajectories are integral curves of a Hamiltonian vector field (see [6], [11]).*

4 The Inverse Problem

Let $\pi : \mathscr{D}^\perp \to \mathbb{R}$ be as in the previous section and $(t, q^i)_{1\leqslant i\leqslant n}$ be coordinates on an open set $\mathbb{R}\times\mathcal{U}$ in $\mathbb{R}\times\mathsf{M}$, where $n = \dim\mathsf{M}$. Let $(\theta^\alpha)_{1\leqslant\alpha\leqslant n-k}$ be a basis of $\mathscr{D}^\perp|_{\mathbb{R}\times\mathcal{U}}$, where $k = \mathrm{rk}\,\mathscr{D}$. Let (t, q^i, λ_α) be the induced coordinates on $\mathscr{D}^\perp|_{\mathbb{R}\times\mathcal{U}}$. This coordinate system induces, as described in section 2, coordinates in $\mathsf{J}^k\pi$, $1 \leqslant k \leqslant \infty$.

We consider a source form Ω on $\mathcal{F}^1(\mathsf{J}^\infty\pi)$ of order 2 which, on the coordinates above, is of the form:

$$\sum_i P_i(t, q^i, \dot{q}^i, \ddot{q}^i, \lambda_\alpha, \dot{\lambda}_\alpha)\delta q^i \wedge \mathrm{d}t + \sum_{i,\alpha} \theta^\alpha_i \dot{q}^i \delta\lambda_\alpha \wedge \mathrm{d}t, \tag{4}$$

where $\theta^\alpha_i \doteq \langle \theta^\alpha, \frac{\partial}{\partial q^i} \rangle$.

Definition 3 *With the notation above, we call* Ω *a* $\mathscr{D}$-source form*. The integral curves of* Ω *are the sections* Θ *of* π *such that* Ω *vanishes along* $\mathrm{j}^\infty\Theta$.

Proposition 3 *The definition above is intrinsic, i.e. independent of the coordinate system.*

Proof. Let $(t, \overline{q}^i)_{1\leqslant i\leqslant n}$ be another coordinate system on $\mathbb{R}\times\mathcal{U} \subset \mathbb{R}\times\mathsf{M}$; we assume these coordinate systems to be related by $(t, q^i) \mapsto \big(t, \overline{q}^i(q^1, \dots, q^n)\big)$. Let $(\overline{\theta}^\alpha)_{1\leqslant\alpha\leqslant n-k}$ be another basis of $\mathscr{D}^\perp|_{\mathbb{R}\times\mathcal{U}}$; we assume $\overline{\theta}^\alpha = \sum_\beta A^\alpha_\beta \theta^\beta$, where $A^\alpha_\beta = A^\alpha_\beta(t,q)$ is a smooth function on $\mathbb{R}\times\mathcal{U}$ for $1 \leqslant \alpha, \beta \leqslant n-k$. The matrix $A = (A^\alpha_\beta)$ is, then, invertible. Let $(t, \overline{q}^i, \overline{\lambda}_\alpha)$ be the induced coordinates on $\mathscr{D}^\perp|_{\mathbb{R}\times\mathcal{U}}$. This coordinate system induces coordinates in $\mathsf{J}^k\pi$, $1 \leqslant k \leqslant \infty$. A direct computation then shows that, in this new coordinate system, Ω given by

(4) transforms into $\sum_i \overline{P}_i(t,\overline{q}^i,\dot{\overline{q}}^i,\ddot{\overline{q}}^i,\overline{\lambda}_\alpha,\dot{\overline{\lambda}}_\alpha)\delta\overline{q}^i\wedge \mathrm{d}t+\sum_{i,\alpha}\overline{\theta}_i^\alpha\dot{\overline{q}}^i\delta\overline{\lambda}_\alpha\wedge\mathrm{d}t$, where $\overline{\theta}_i^\alpha \doteq \langle \overline{\theta}^\alpha, \frac{\partial}{\partial \overline{q}^i}\rangle$ and:

$$\overline{P}_i(t,\overline{q}^i,\dot{\overline{q}}^i,\ddot{\overline{q}}^i,\overline{\lambda}_\alpha,\dot{\overline{\lambda}}_\alpha) = \sum_j P_j \frac{\partial q^j}{\partial \overline{q}^i} + \sum_{j,\alpha,\beta,\gamma} \dot{\overline{q}}^j \overline{\theta}_j^\beta (A^{-1})_\beta^\alpha \frac{\partial A_\alpha^\gamma}{\partial \overline{q}^i}\overline{\lambda}_\gamma \tag{5}$$

□

Note that, if Θ is an integral curve of Ω, then the projection of Θ on $\mathbb{R}\times\mathsf{M}$ is compatible with $\mathscr{D}$ and, locally, $\Theta : t \mapsto \big(t, q^i(t), \lambda_\alpha(t)\big)$ in the above coordinates is a solution of the system of mixed first- and second-order equations $P_i(t,q^i,\dot{q}^i,\ddot{q}^i,\lambda_\alpha,\dot{\lambda}_\alpha)=0$.

We now consider the following problem: to find necessary and sufficient conditions for a given $\mathscr{D}$-source form Ω to be the Euler-Lagrange form of a Lagrangian of the form (3). If that is the case, the integral curves of Ω are the solutions of the Vakonomic equations of the Lagrangian L.

Definition 4 *We say that a $\mathscr{D}$-source form Ω is* 2-1-affine *if, written in coordinates as* (4)*, for $1\leqslant i\leqslant n$:* (i) *the functions P_i are affine in the variables $\ddot{q}^k$, $1\leqslant k\leqslant n$ and $\dot{\lambda}_\alpha$, $1\leqslant\alpha\leqslant n-k$;* (ii) $\frac{\partial P_i}{\partial \ddot{q}^k}$ *is a function of $(t,q,\dot{q})$ and* $\frac{\partial P_i}{\partial \dot{\lambda}^k}$ *is a function of (t,q).*

It follows from (5) that the above definition does not depend on the coordinate system.

Definition 5 *Let Ω be a $\mathscr{D}$-source form. We say that Ω is a* locally variational $\mathscr{D}$-source form *if, locally, it is the Euler-Lagrange form of a Lagrangian of the form* (3)*. We say that Ω is* globally variational *if the latter condition holds globally on $\mathsf{J}^\infty\pi$.*

Our main results are stated in the following theorems: in the first one we describe the $\mathscr{D}$-source forms which are locally variational; in the second one we show that the topological obstruction for a locally variational $\mathscr{D}$-source form to be globally variational lies in $\mathsf{H}^2(\mathsf{M})$.

Theorem 4 *Let Ω be a $\mathscr{D}$-source form. Then Ω is locally variational if, and only if, Ω is 2-1-affine and $\delta_{\mathrm{v}}\Omega=0$, where δ_{v} is defined in* (2)*. In coordinates, if Ω is given by* (4)*, the latter condition reads:*

$$\begin{aligned}
\frac{\partial P_j}{\partial q^i} &= \frac{\partial P_i}{\partial q^j} - \mathrm{D_t}\frac{\partial P_i}{\partial \dot{q}^j} + \mathrm{D_t^2}\frac{\partial P_i}{\partial \ddot{q}^j}\\
-\frac{\partial P_j}{\partial \dot{q}^i} &= \frac{\partial P_i}{\partial \dot{q}^j} - 2\mathrm{D_t}\frac{\partial P_i}{\partial \ddot{q}^j}\\
\frac{\partial P_j}{\partial \ddot{q}^i} &= \frac{\partial P_i}{\partial \ddot{q}^j}\\
\sum_j \frac{\partial \theta_j^\alpha}{\partial q^i}\dot{q}^j &= \frac{\partial P_i}{\partial \lambda_\alpha} - \mathrm{D_t}\frac{\partial P_i}{\partial \dot{\lambda}_\alpha}\\
-\theta_i^\alpha &= \frac{\partial P_i}{\partial \dot{\lambda}_\alpha}
\end{aligned} \tag{6}$$

Theorem 5 *If* $\mathsf{H}^2(\mathsf{M}) = 0$*, every locally variational* $\mathscr{D}$*-source form is globally variational. On the other hand, if* $\mathsf{H}^2(\mathsf{M}) \neq 0$*, there exist locally variational* $\mathscr{D}$*-source forms which are not globally variational.*

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SOME REMARKS ABOUT EXTINCTION IN NONAUTONOMOUS KOLMOGOROV SYSTEMS

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Abstract

This paper gives a recent results on extinction in nonautonomous Kolmogorov systems.

Key words: *Kolmogorov system, Lotka - Volterra system, upper average, lower average, logistic equation, permanence, global attractivity.*

AMS subject classifications: *primary 34D05; secondary 34C12; 34C29; 34D23; 92D40.*

1 Introduction

It is well known that the long-term coexistence problem of species is a basic one in population dynamics. One of the famous models for dynamics of population is the Lotka - Volterra competition system

$$u_i' = u_i \left(a_i(t) - \sum_{j=1}^{N} b_{ij} u_j(t) \right), \tag{LV}$$

where $a_i, b_{ij} > 0$. Gopalsamy [2], [3] and Alvarez and Tineo [8] showed that if for $i = 1, \dots, N$

$$a_{iL} > \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{b_{ijM} a_{jM}}{b_{jjL}} \quad \text{for} \quad i = 1, \dots, N$$

where g_L (resp. g_M) denotes the infimum (resp. the supremum) of the function g, then system (LV) is permanent and globally attractive. In [1] Ahmad and Lazer showed that permanence and global attractivity hold under weaker conditions, which they called *average conditions* or *conditions A*. The authors applied the notion of the *upper* and *lower averages* of a function; namely,

$$m[g] := \liminf_{t-s\to\infty} \frac{1}{t-s} \int_s^t g(\tau) d\tau,$$

$$M[g] := \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t g(\tau)d\tau.$$

With the help of the upper and lower average of a function they obtained a condition which guarantees the permanence and global attractivity in a Lotka - Volterra system. The average conditions for system (LV) are

$$m[a_i] > \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{b_{ijM} M[a_j]}{b_{jjL}} \quad \text{for} \quad i = 1, \ldots, N$$

At the same time Francisco Montes de Oca and Mary Lou Zeeman dealt with extinction (see for example [4]). They considered competing system (LV), where $a_i, b_{ij} : \mathbb{R} \to (0, \infty)$ are continuous functions bounded by positive reals. In [4] they gave algebraic criteria on the parameters which guarantee that all but one of the species are driven to extinction; namely, for each $k > 1$ there exists $i_k < k$ such that for any $j \leqslant k$ the inequality

$$a_{kM} b_{i_k jM} < b_{i_k L} a_{kjL} \tag{E}$$

holds. They proved that under conditions (E) the species $u_2, \ldots, u_N$ are driven to extinction whilst the species u_1 stabilizes at the unique bounded solution u_1^* of the logistic equation on the u_1 axis. Moreover, they showed convergence of trajectories to u_1^*.

For each $r \leqslant N$, let H^r denote the r - dimensional coordinate subspace on which $x_{r+1}, \ldots, x_N$ vanish. We use the variable v to denote the restriction of system (LV) to H^r,

$$v_i'(t) = v_i(t)\Big(a_i(t) - \sum_{j=1}^{r} b_{ij}(t) v_j(t)\Big), \quad i = 1, \ldots, r. \tag{LV$_r$}$$

In [4], Montes de Oca and Zeeman showed that given $r \leqslant N$, if for each $k > r$ there exists $i_k < k$ such that for any $j \leqslant k$ the inequality

$$a_{iL} > \sum_{\substack{j=1\\ j\neq i}}^{N} b_{ijM}\left(\frac{a_{jM}}{b_{jjL}}\right)$$

holds, then the system $(\mathrm{LV})_r$ has a unique bounded strictly positive solution $v^*(t) = (v_1^*(t), \ldots, v_r^*(t))$ and every other positive solution $u(t) = (u_1(t), \ldots, u_N(t))$ of system (LV) has the property that

$$\lim_{t\to\infty}(u_j(t) - v_j^*(t)) = 0, \quad j = 1, \ldots, r,$$
$$\lim_{t\to\infty} u_j(t) = 0, \quad j = r+1, \ldots, N.$$

In this paper we present results which we proved in [5, 6, 7].

We consider an N- species nonautonomous competitive Kolmogorov system

$$u_i' = u_i f_i(t,u) \tag{K}$$

on the nonnegative cone

$$C = \{u = (u_1, \dots, u_N) : u_i \geqslant 0, 1 \leqslant i,j \leqslant N\},$$

where

(1) $f = (f_1, \dots, f_N) : [0,\infty) \times C \to \mathbb{R}^N$ together with its first derivatives $\frac{\partial f_i}{\partial u_j}$ are continuous,

(2) for each compact $\tilde{C} \subset C$, $\frac{\partial f_i}{\partial u_j}(t,u)$ are bounded and uniformly continuous on $[0,\infty) \times \tilde{C}$ with respect to u,

(3) there exist $a_i^{(1)}, a_i^{(2)} > 0$ such that $a_i^{(1)} \leqslant f_i(t,0,\dots,0) \leqslant a_i^{(2)}, \quad t \geqslant 0, \quad 1 \leqslant i \leqslant N$,

(4) $\frac{\partial f_i}{\partial u_j}(t,u) \leqslant 0$, for all $t \geqslant 0$ $u \in C$, $\quad i,j = 1, \dots, N$,

(5) there exist $b_{ii}^{(1)} > 0$ such that $\frac{\partial f_i}{\partial u_i}(t,u) \leqslant -b_{ii}^{(1)}$ for all $t \geqslant 0$, $u \in C$, $i = 1, \dots, N$.

Definition 1 *A solution $u(t)$ of system (K) is* positive *if $u_i(t) > 0$ for all $t \geqslant 0$.*

2 Preliminaries

We begin with the following result.

Lemma 1 *If $u : [t_0, \tau_{\max}) \to C$, $t_0 \geqslant 0$, is a maximally defined positive solution of (K) then*

(i) $\tau_{\max} = \infty$,

(ii) $\limsup_{t\to\infty} u_i(t) \leqslant \frac{a_i^{(2)}}{b_{ii}^{(1)}}$ *for $i = 1, \dots, N$.*

Proof. See for example [5, 6, 7].

□

Define

$$B := \left[0, \frac{a_1^{(2)}}{b_{11}^{(1)}}\right] \times \cdots \times \left[0, \frac{a_N^{(2)}}{b_{NN}^{(1)}}\right],$$

$$b_{ij}^{(2)} := -\inf\left\{\frac{\partial f_i}{\partial u_j}(t,x) : t \geqslant 0,\ x \in B\right\}.$$

Assumptions (2) and (4) guarantee that $0 \leqslant b_{ij}^{(2)} < \infty$. Further, define

$$a^{(1)} := \min\{a_i^{(1)} : i = 1, \dots, N\},$$

$$b^{(2)} := \max\{b_{ij}^{(2)} : i,j = 1, \dots, N\}.$$

Lemma 2 *There is $\delta > 0$ such that if $u(t) = (u_1(t), \ldots, u_N(t))$ is a positive solution of (K) then*

$$\liminf_{t\to\infty} \sum_{i=1}^{N} u_i(t) \geqslant \delta.$$

Proof Proof. See for example [5, 6, 7].

□

Definition 2 *System (K) is* permanent *if there exist positive constants $\underline{\nu}$ and $\overline{\nu}$ such that for each positive solution $u(t) = (u_1(t), \ldots, u_N(t))$of (K) there is $T > 0$ with the property $\underline{\nu} \leqslant u_i(t) \leqslant \overline{\nu}$ for each $t \geqslant T$.*

Define the lower and upper averages of a function g which is continuous and bounded above and below on $[0, \infty)$. If $0 < t < s$ we set

$$A[g, t, s] := \frac{1}{t-s} \int_s^t g(\tau) d\tau.$$

The lower and upper averages of g denote by $m[g]$ and $M[g]$ respectively are define by

$$m[g] := \liminf_{t-s\to\infty} A[g, t, s],$$

$$M[g] := \limsup_{t-s\to\infty} A[g, t, s].$$

In [5] we proved that if

$$m[f_i(\cdot, 0, \ldots, 0)] > \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{b_{ij}^{(2)} M[f_j(\cdot, 0, \ldots, 0)]}{b_{jj}^{(1)}}, \quad \text{for} \quad i = 1, \ldots, N.$$

then system (K) is permanent and globally attractive.

3 Main Results

In [6] we introduced average conditions which insure that all but one of the species are driven to extinction.

Theorem 3 *Assume that for all $k > 1$ there exists $i_k < k$ such that for all $j \leqslant k$*

$$\frac{M[f_k(\cdot, 0, \ldots, 0)]}{m[f_{i_k}(\cdot, 0, \ldots, 0)]} < \frac{b_{kj}^{(1)}}{b_{i_k j}^{(2)}}.$$

If $u = (u_1(t), \ldots, u_N(t))$ is a positive solution of (K) then for all $i = 2, \ldots, N$ $u_i(t) \to 0$ as $t \to \infty$

See Theorem 1 in [6].

Denote by $U_1(t)$ a fixed positive solution of

$$U_1'(t) = U_1(t) f_1(t, U_1(t), 0, \ldots, 0)$$

It can be proved that $U_1(t)$ is defined on $[0, \infty)$, bounded above and below by positive constants, and globally attractive.

Theorem 4 *If $u(t) = (u_1(t), \ldots, u_N(t))$ is a positive solution of (K) then $u_1(t) \to U_1(t)$ as $t \to \infty$.*

For the proof see Theorem 2 in [6].

In [7] we showed that for any $r \leqslant N$ the average conditions guarantee that r of the species in system (K) are permanent while remaining $N - r$ are driven to extinction.

Theorem 5 *Let r be a given integer with $1 \leqslant r < N$. Assume for any $k > r$ there is an $i_k < k$ such that for any $j \leqslant k$*

$$\frac{M[f_k(\cdot, 0, \ldots, 0)]}{m[f_{i_k}(\cdot, 0, \ldots, 0)]} < \frac{b_{kj}^{(1)}}{b_{i_k j}^{(2)}}$$

holds. If $u = (u_1(t), \ldots, u_N(t))$ is a positive solution of (K) then for all $i > r$ $u_i(t) \to 0$ exponentially as $t \to \infty$.

For the proof see Theorem 1 in [7].

Theorem 6 *Suppose that all the conditions of Theorem 3 hold. Assume that*

$$m\left[f_i(t, 0, \ldots, 0) - \sum_{\substack{j=1 \\ j \neq i}}^{N} b_{ij}^{(2)} U_j(t)\right] > 0, \quad \textit{for} \quad i = 1, \ldots, r \tag{M}$$

where U_j is a positive solution of the equation

$$U_j'(t) = U_j(t) \left(f_j(t, 0, \ldots, 0) - b_{jj}^{(1)} U_j(t) \right).$$

Then there exist $\overline{\nu} > 0$ and $\underline{\nu} > 0$ such that for each positive solution $u(t) = (u_1(t), \ldots, u_N(t))$ of system (K) there exists $T > 0$ such that

$$\underline{\nu} \leqslant u_i(t) \leqslant \overline{\nu} \quad \textit{for} \quad t \geqslant T \quad \textit{and} \quad i = 1, \ldots, r.$$

For a proof see Theorem 2 in [7].

Denote $I = \{1, \ldots, r\}$.

Theorem 7 *Suppose that all the conditions of Theorem 3 hold. Assume that there exist positive constants* $\rho_1, \ldots, \rho_r > 0$ *and* $\varepsilon > 0$ *such that for* $j = 1, \ldots, r$

$$\rho_j b_{jj}^{(1)} - \sum_{\substack{i=1 \\ i \neq j}}^{r} \rho_i b_{ij}^{(2)} \geqslant \varepsilon.$$

If $u(t) = (u_1(t), \ldots, u_N(t))$ *is any positive solution of (K) and* $\tilde{u}(t) = (\tilde{u}_1(t), \ldots, \tilde{u}_r(t))$ *is any positive solution of subsystem*

$$u_i' = u_i f_i(t, u), \quad i \in I,$$

then

$$\lim_{t \to \infty} |u_j(t) - \tilde{u}_j(t)| = 0 \quad j = 1, \ldots, r.$$

For a proof see Theorem 3 in [7].

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FORWARD AND PULLBACK ATTRACTION ON PULLBACK ATTRACTORS

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Abstract

Pullback attractors are important elements to study the asymptotic behaviour for nonautonomous PDEs because they copy the pullback dynamic of the system inside them. Although pullback and forward dynamic may not be related, there exist some cases when the trajectories converge forward in time to the pullback attractor. In this work we prove how the pullback attractor copy the forward dynamic in these cases.

Key words: *pullback attractor, forward attraction, evolution process.*
AMS subject classifications: *35B40, 35B41, 35L05, 35Q35.*

1 Introduction

One of the central problems in dynamical systems is the study of the asymptotic behaviour of evolution processes associated to the modeling of real world phenomena. When the model under study is an autonomous differential equation, the asymptotic behaviour is rather well established and many references on the subject are available (cf. Temam [16], Hale [10], Ladyzhenskaya [12], Babin-Vishik [1], Robinson [14] for example). However, if the evolution process comes from a nonautonomous differential equation, even though some nice references are already available ((cf. Cheban [7], Chepyzhov-Vishik [8], Kloeden [11], Sell-You[15], Caraballo *et. al.* [2]), much is yet to be done.

In general, a nonautonomous system shows two different dynamics without relation between them: *forward dynamic* (the behaviour when final time goes to infinity) and *pullback dynamic* (the behaviour when the initial time goes to minus infinity). An interesting task concerns the analysis of the case in which both kinds of attraction take place. Our aim is to show how the pullback attractor copies the whole dynamic in this case. To do this, we will use the framework of evolution processes, because we can identify the solution of problems with this kind of families.

Definition 1 *An evolution process in X is a family of maps $\{S(t,s) : t \geqslant s\} \subset \mathcal{C}(X)$ with the following properties*

1) $S(t,t) = I$ for all $t \in \mathbb{R}$,

2) $S(t,s) = S(t,\tau)S(\tau,s)$, for all $t \geqslant \tau \geqslant s$,

3) $\{(t,s) \in \mathbb{R}^2 : t \geqslant s\} \times X \ni (t,s,x) \mapsto S(t,s)x \in X$ is continuous.

Since a fixed set A in X will not, in general, remain fixed by a nonautonomous process, invariance for an evolution process is defined as:

Definition 2 *A family of nonempty sets $\{B(t) : t \in \mathbb{R}\}$ is* invariant under *$\{S(t,s) : s \leqslant t\}$ if $S(t,s)B(s) = B(t)$ for all $t \geqslant s$. We say that $\{B(t) : t \in \mathbb{R}\}$ is* positive invariant *if we only have the inclusion $S(t,s)B(s) \subset B(t)$.*

Now we can define the pullback attractor for an evolution process.

Definition 3 *A family of compact sets $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ is the* pullback attractor for *$\{S(t,s) : s \leqslant t\}$ if it is invariant, attracts all bounded subsets of X 'in the pullback sense', that is,*

$$\lim_{s \to -\infty} \operatorname{dist}(S(t,s)B, \mathcal{A}(t)) = 0, \ \forall t \in \mathbb{R}, \ \forall B \subset X \ \textit{bounded,}$$

and is minimal in the sense that if there exists a family of closed sets $\{C(t) : t \in \mathbb{R}\}$ such that attracts bounded sets of X, then $\mathcal{A}(t) \subset C(t)$, for all $t \in \mathbb{R}$.

We use the Hausdorff semi-distance. Let A, B be subsets of X and $d : X \longrightarrow \mathbb{R}$ the distance in X, then we define the Hausdorff semi-distance as

$$\operatorname{dist} \sup_{a \in A} \inf_{b \in B} d(a,b)$$

In the nonautonomous case we can also define forward attraction as follows: we say that $\{C(t) : t \in \mathbb{R}\}$ attracts the bounded set $B \subset X$ if

$$\lim_{t \to \infty} \operatorname{dist}(S(t+s,s)B, C(t+s)) = 0.$$

The pullback attractor does not necessarily have forward attraction. Consider the following simple examples of nonautonomous equations giving different answers to the relation on pullback and forward attraction. Indeed, consider

$$y_1'(t) = -2ty_1(t) + 2t^2$$

and

$$y_2'(t) = 2ty_2(t) + 2t^2.$$

Both can be solved explicitly with initial value $y_0 \in \mathbb{R}$ at time $s \in \mathbb{R}$ by

$$y_1(t,s) = (y_0 - s)e^{-(t^2-s^2)} + t - e^{-t^2}\int_s^t e^{r^2}\,dr,$$

$$y_2(t,s) = (y_0+s)e^{t^2-s^2} + t + e^{t^2}\int_s^t e^{-r^2}\,dr.$$

In the first case we can observe how the trajectory is more and more close to $\mathcal{A}_1(t) = t - e^{-t^2}\int_0^t e^{r^2}dr$ when t goes to infinity. In the same way, the trajectories of the second equation are attracted in a pullback sense by the family $\mathcal{A}_2(t) = -t + e^{t^2}\int_{-\infty}^t e^{-r^2}dr$, that is, when initial time $s \to -\infty$. However, $\{\mathcal{A}_1(t) : t \in \mathbb{R}\}$ is forward but not pullback attracting and $\{\mathcal{A}_2(t) : t \in \mathbb{R}\}$ is pullback but not forward.

2 Pullback and forward attraction in evolution processes

The uniform forward attraction gives us trivial examples of pullback attractors that have forward attraction, because a pullback uniform attractor is also a forward uniform attractor and vice versa. In this case we need a uniform concept of attraction, that is, we say that $B \subset X$ attracts uniformly under the process $\{S(t,s) : s \leqslant t\}$ if for any $C \subset X$

$$\lim_{t\to\infty}\sup_{s\in\mathbb{R}} \operatorname{dist}(S(t+s,s)C, B) = 0. \tag{1}$$

We do not distinguish between pullback and forward because if we perform a simple change of variables we obtain

$$\lim_{s\to\infty}\sup_{t\in\mathbb{R}} \operatorname{dist}(S(t,t-s)C, B) = 0. \tag{2}$$

The first definition of uniform attractor is given in [8] and is based on the autonomous definition of global attractor, so the authors define it as a not necessarily invariant, in autonomous sense ($S(t,s)A = T(t-s)A = A$), compact subset that is uniformly attracting. However, afterwards in the same section, we can find the concept of kernel sections of the uniform attractor, a particular concept of pullback attractor. In [6] we can find a definition of the uniform attractor where the invariant property holds. In both cases, all the results appear in the skew-product framework. Below we write a general definition and an existence result within the framework of evolution processes.

Definition 4 *Let $\{S(t,s) : s \leqslant t\}$ be an evolution process. A family of bounded closed sets $\{\mathcal{A}_u(t) : t \in \mathbb{R}\}$ is called the uniform attractor if the following properties hold:*

1. *There exists a compact set $\hat{A} \subset X$ such that $\bigcup_{t\in\mathbb{R}} \mathcal{A}_u(t) \subset \hat{A}$.*
2. *It is uniformly attracting under $\{S(t,s) : s \leqslant t\}$.*
3. *It is minimal in the sense of Definition 3.*

Theorem 1 *If there exists a compact uniformly attracting set, then there exists the uniform attractor.*

Although we need a pullback attracting family, the forward attraction comes from the compact set $\hat{A}$. Actually, in this case we have a global attractor that is attracting in the pullback sense too.

In [9], the existence of a uniform exponential attractor for the nonautonomous equation

$$\begin{cases} u_t = a\Delta u - f(u) + g(t) \text{ in } \Omega \\ u(x,t) = 0 \text{ in } \partial\Omega, \end{cases} \tag{3}$$

is proved under some restrictions for functions f and g.

Other examples are nonautonomous perturbations of gradient-like semigroups, where the forward attraction comes from the autonomous nature of the limit problem. This kind of processes possesses a concrete structure as the union of unstable manifolds of some specific sets. Let $\{S_\eta(t,s) : t \geqslant s\}$ be evolution processes, with $\eta \in [0,1]$, such that $S_\eta \stackrel{\eta \to 0}{\longrightarrow} S_0$ in a certain sense, and $\{S_0(t,s) = T(t-s) : t \geqslant s\}$ is a gradient-like semigroup, that is, there exists a finite number of equilibria and all the global trajectories converge forward and backward to them. Let us suppose that there exits a pullback attractor $\{\mathcal{A}_\eta(t) : t \in \mathbb{R}\}$ and a finite number of isolated invariant families $\{\Xi_{1,\eta}, ..., \Xi_{n,\eta}\}$ with traces $\{\Gamma_{1,\eta}, ..., \Gamma_{n,\eta}\}$, where $\Gamma_{i,\eta} = \bigcup_{t\in\mathbb{R}} \Xi_{i,\eta}(t)$. In this general case, and doing a comparison between autonomous and nonautonomous case, those isolated invariant families play the role of equilibrium points. We need to introduce the concept of trace because the dynamic of each $\Xi_{i,\eta}$ is not constant in general (see [5] for more details and definitions). Let also suppose that for $\eta = 0$ we have a gradient-like global attractor. Under some conditions we can write the pullback attractor as $\mathcal{A}_\eta(t) = \bigcup_{i=1}^n W^u(\Xi_{i,\eta})(t)$ for all $\eta \leqslant \eta_0$ for some $\eta_0 > 0$. The following result is Theorem 1.12 of [5] and show how a pullback attractor possesses a forward attraction too.

Theorem 2 *Suppose all the stationary points of* $\{S_0(t,s) = T(t-s) : t \geqslant s\}$ *are hyperbolic. If we also assume that there is* $\gamma > 0$ *and, for each* $1 \leqslant i \leqslant n$, *a neighborhood* $V_{i,\eta}$ *of the trace* $\Gamma_{i,\eta}$ *of* $\Xi_{i,\eta}$ *such that for any* $u_0 \in V_{i,\eta}$, $s \in \mathbb{R}$ *and as long as* $S_\eta(t+s,s)u_0 \in V_{i,\eta}$

$$\sup_{s\in\mathbb{R}} \operatorname{dist}(S_\eta(t+s,s)u_0, W^u(\Xi_{i,\eta})(t)) \leqslant Me^{-\gamma t},$$

then for any bounded set $B \subset X$, *there is a constant* $c(B) > 0$ *such that*

$$\sup_{s\in\mathbb{R}} \operatorname{dist}(S_\eta(t+s,s)u_0, \mathcal{A}_\eta(t+s)) \leqslant c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \tag{4}$$

The following nonautonomous damped wave equation gives us an example of pullback attractor with exponential forward attraction. In this case we have a nonautonomous problem that does not come from an autonomous one (see [3] for more details). Let us consider the following equation

$$\begin{cases} u_{tt} + \beta(t)u_t = \Delta u + f(u) \text{ in } \Omega \\ u(x,t) = 0 \text{ in } \partial\Omega, \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $f \in C^2(\mathbb{R})$ satisfies some growth conditions, and $\beta : \mathbb{R} \to \mathbb{R}$ is bounded and globally Lipschitz. The existence of the pullback attractor in $H_0^1(\Omega) \times L^2(\Omega)$ for this problem has been recently proved in [4], and if we assume that there only exists a finite number of equilibrium points, and all of them are hyperbolic, then the pullback attractor has exponential forward attraction as in (4).

3 Trajectories inside the pullback attractor

One of the most important result in the theory of pullback attractors is related of its finite fractal dimension. As in the autonomous case (see [14]), results in [13] prove that for each trajectory of $\{S(t,s) : s \leqslant t\}$, another one can be found inside the pullback attractor that tracks the original one. The following theorem gives an analogous result based on forward attracting families for evolution processes.

Theorem 3 *Suppose that the process $\{S(t,s) : s \leqslant t\}$ is Lipschitz in X,*

$$\sup_{s\in\mathbb{R}} \|S(t+s,s)u - S(t+s,s)v\|_X \leqslant \kappa(t)\|u-v\|_X, \tag{6}$$

with $\kappa : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ bounded in compact subsets and u, v in any bounded subset $B \in X$. Suppose also that there exists a family of compacts sets $\{A(t) : t \in \mathbb{R}\}$ that forward attracts bounded sets and is positive invariant under $\{S(t,s) : s \leqslant t\}$. Then, for each trajectory $u(t,s) \in X$ of $\{S(t,s) : s \leqslant t\}$ and positive sequences $\{\varepsilon_n\}_{n=0}^\infty$ and $\{T_n\}_{n=0}^\infty$ with $\varepsilon_n \overset{n\to\infty}{\longrightarrow} 0$, $T_n < T_{n+1}$ and $T_n \overset{n\to\infty}{\longrightarrow} \infty$, there exists a sequence $t_n \overset{n\to\infty}{\longrightarrow} \infty$ and $v_n \in A(t_n + s)$ such that

$$\sup_{t\in[0,T_n]} \|u(t+t_n+s,s) - S(t+t_n+s,t_n+s)v_n\|_X \leqslant \varepsilon_n. \tag{7}$$

Moreover, the 'jumps' $\|v_{n+1} - S(T_n+t_n+s,t_n+s)v_n\|_X$ decrease to zero.

Proof. By the forward attraction and the compactness of each set of the family $\{A(t) : t \in \mathbb{R}\}$, there exists a time $t_0 = t_0(\varepsilon_0, T_0)$ and a $v_0 \in A(t_0 + s)$ such that

$$\|S(t_0+s,s)u(s) - v_0\|_X \leqslant \frac{\varepsilon_0}{\max_{t\in[0,T_0]} \kappa(t)}.$$

Hence, using (6) we have

$$\begin{aligned}
&\|S(t+t_0+s,s)u(s) - S(t+t_0+s,t_0+s)v_0\|_X \\
&\qquad = \|S(t+t_0+s,t_0+s)S(t_0+s,s)u(s) - S(t+t_0+s,t_0+s)v_0\|_X \\
&\qquad \leqslant \max_{t\in[0,T_0]} \kappa(t)\|S(t_0+s,s)u(s) - v_0\|_X \\
&\qquad \leqslant \varepsilon_0 \text{ for all } t \in [0,T_0].
\end{aligned}$$

Now, for ε_1 and T_1 we can find a t_1 and a $v_1 \in A(t_1 + s)$ such that $t_0 < t_1$ and

$$\|S(t_1+s,s)u(s) - v_1\|_X \leqslant \frac{\varepsilon_1}{\max_{t\in[0,T_1]} \kappa(t)},$$

therefore,

$$\|S(t+t_1+s,s)u(s)-S(t+t_1+s,t_1+s)v_1\|_X \leqslant \varepsilon_1 \text{ for all } t\in[0,T_1].$$

In the same manner, we can see that for any ε_n and T_n there exist a time $t_{n-1}<t_n$ and a $v_n \in A(t_n+s)$ such that

$$\|S(t+t_n+s,s)u(s)-S(t+t_n+s,t_n+s)v_1\|_X \leqslant \varepsilon_n \text{ for all } t\in[0,T_n].$$

Finally, we have

$$\begin{aligned}\|v_{n+1}-&S(T_n+t_n+s,t_n+s)v_n\|_X\\ &\leqslant \|v_{n+1}-S(T_n+t_n+s,t_n+s)u(t_n+s)\|_X\\ &+\|S(T_n+t_n+s,t_n+s)u(t_n+s)-S(T_n+t_n+s,t_n+s)v_n\|_X\\ &\leqslant \varepsilon_{n+1}+\varepsilon_n,\end{aligned}$$

which completes the proof. □

Remark 1 *As t_n does not depend on the initial time, we can track $u(t,s)$ by trajectories in $\{A(t):t\in\mathbb{R}\}$ of length T_n from t_n+s to $t_{n+1}+s$ within a distance ε_n.*

Due to Theorem 3, the uniform exponential attractor in example (3) tracks the forward trajectories of the system in $H_0^1(\Omega)$. If we have forward attraction in the pullback attractor (as in example (5)), we can understand all the dynamics of the system only by the study of the dynamic inside it, obtaining a complete representation of the dynamic of the system in a finite fractal dimensional set. This shows the importance of studying pullback attractor with forward attraction to understand the whole dynamic of nonautonomous systems.

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Título:	Towards the Numerical Simulation of Ship Generated Waves Using a Cartesian Cut Cell Based Free Surface Solver.
Doctorando:	José Antonio Armesto Álvarez.
Director/es:	Clive G. Mingham, Ling Qian and Derek M. Causon.
Defensa:	21 de Noviembre de 2008, Manchester.
Calificación:	Apto (bajo sistema de calificación Apto/No Apto).

Resumen:

The Cartesian Cut Cell method has been applied to different flow configurations by researchers at the Centre of Mathematical Modelling and Flow Analysis. This method has been implemented to define flow domains around obstacles using a Godunov-type high order upwind scheme to solve Shallow Water Equations and Navier-Stokes (Euler) equations in two phase flows.

A new idea to study Navier-Stokes (Euler) equations in just one phase flows where the domain is accurately described using the Cartesian Cut Cell Method around the moving free surface is presented. The solution technique involves three stages for every time step: the definition of the domain, the solution of the flow equations and the movement of the free surface. The Cartesian Cut Cell Method only requires to recompute cells affected by the movement of the free surface providing quickly the new domain. The flow equations are solved using the Artificial Compressibility Method and a Godunov-type high order upwind scheme involving the solution of Riemann problems. The Heigh Function method is applied to study the evolution on time of the free surface. This method involves the solution of the kinematic equations, where a fourth order Runge-Kutta method is employed. Boundary conditions at the free surface are discussed.

The technique proposed is very quick and allows the use of big time steps. In comparison with the two phase version, the proposed techniques used one thousand times bigger time steps and require around 25 times less computational effort. On the other hand, the results show dependence on the artificial compressibility parameter introduced as part of the solution of the flow equations. Extensions to the presented study are proposed including the use of different flow solvers.

An algorithm to solve free surface flows in a single phase system is presented. The Cartesian Cut Cell Method is used to define the grid in a domain involving free surface and/or the presence of an obstacle. The algorithm approximate the solutions of the incompressible Navier-Stokes equations based on the Artificial

Compressibility Method and uses the cell-centred finite volume approach. A Godunov-type high order upwind scheme is applied to compute fluxes at cell interfaces, involving polynomial reconstruction and the solution of a Riemann problem. The HLL Riemann solver and Roe's Riemann solver are implemented as part of the Godunov-type upwind scheme. An implicit scheme is used for the time discretization in problems without free surface while an explicit forth order Runge-Kutta method is used in free surface problems. An introduction to problem where the free surface and the obstacle cut each other is presented.

Nota del encargado de la sección: los anuncios presentes en la sección son el resultado de avisos recibidos por medios diversos y de una búsqueda en *internet*. Puede ser de interés para los socios consultar las siguientes páginas *web* (además de la de nuestra sociedad `http://www.sema.org.es/`):

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`http://www.austms.org.au/Events`
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`http://www.bcamath.org/public_activities/ctrl_activities.php`
- *Centre de Recerca Matemàtica* (CMR), calendario de eventos:
`http://www.crm.es/Conferences/IndexConferencesEng.htm`
- *European Mathematical Society* (EMS), calendario de eventos:
`http://www.euro-math-soc.eu/conferences.html`
- *Ingenio Mathematica* (I-Math):
`http://www.i-math.org/`
- *Ingenio Mathematica* (I-Math), nodo CESGA:
`http://mathematica.nodo.cesga.es/`
- *London Mathematical Society* (LMS), calendario de eventos:
`http://www.lms.ac.uk/newsletter/calendar.html`
- *Societá Italiana di Matematica Applicata e Industriale* (SIMAI), calendario de eventos:
`http://www.simai.eu/convegni-ed-eventi/Convegni/`
- *Society for Industrial and Applied Mathematics* (SIAM), calendario de eventos:
`http://www.siam.org/meetings/calendar.php`
- *Real Sociedad Matemática Española* (RSME), calendario de eventos:
`http://www.rsme.es/content/blogsection/9/59/`
- *Unione Matematica Italiana* (UMI), calendario de eventos:
`http://umi.dm.unibo.it/agenda_congressi--6.html`

Tipo de evento:	Curso de Verano
Nombre:	Marte y sus Enigmas
Lugar:	El Escorial, Madrid, ESPAÑA
Fecha:	12–16 July 2010
Organiza:	Luis Vázquez, (UCM), Héctor Guerrero (INTA), Instituto de Matemática Interdisciplinar
Información:	
E-mail:	
WWW:	`http://www.ucm.es/info/cv/subweb/prog/programas/72101.html`

Tipo de evento:	Workshop
Nombre:	Numerical Solutions of Partial Differential Equations: Novel Discretization Techniques
Lugar:	Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota, USA
Fecha:	01–05 November 2010
Organiza:	Institute for Mathematics and its Applications (IMA)
Información:	
E-mail:	
WWW:	`http://www.ima.umn.edu/2010-2011/W11.1-5.10/`

Tipo de evento:	Escuela
Nombre:	School of Applied Mathematics and Innovation SAMI 2010: Non–smooth systems and applications
Lugar:	Universidad Sergio Arboleda, Sede Rodrigo Noguera Laborde. Santa Marta, COLOMBIA
Fecha:	08–12 November 2010
Organiza:	Instituto de Matemáticas y sus Aplicaciones (IMA)
Información:	
E-mail:	`mayer@ima.usergioarboleda.edu.co`
WWW:	`http://ima.usergioarboleda.edu.co/SAMI/SAMI2010.htm`

Tipo de evento:	Workshop
Nombre:	SIAM/MSRI Workshop on Hybrid Methodologies for Symbolic–Numeric Computation
Lugar:	Mathematical Sciences Research Institute (MSRI) Berkeley, California, USA
Fecha:	17–19 November 2010
Organiza:	Society for Industrial and Applied Mathematics (SIAM), Mathematics Sciences Research Institute (MSRI)
Información:	
E-mail:	`575@msri.org`
WWW:	`http://www.scg.uwaterloo.ca/siam-msri-hybrid`

Tipo de evento:	Workshop
Nombre:	Numerical Solutions of Partial Differential Equations: Fast Solution Techniques
Lugar:	Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota, USA
Fecha:	29 November – 03 December 2010
Organiza:	Institute for Mathematics and its Applications (IMA)
Información:	
E-mail:	`575@msri.org`
WWW:	`http://www.ima.umn.edu/2010-2011/W11.29-12-3.10/`

Tipo de evento:	Congreso
Nombre:	International Conference on Advances in Optimization and Related Topics (ADORT–2010)
Lugar:	Centre de Recerca Matemàtica (CRM), Bellaterra, Barcelona, ESPAÑA
Fecha:	29 November – 03 December 2010
Organiza:	Aris Daniilidis (Universitat Autónoma de Barcelona) Albert Ferrer (Universitat Politécnica de Catalunya) Juan Enrique Martínez–Legaz (Universitat Autónoma de Barcelona)
Información:	
E-mail:	`adort2010@crm.cat`
WWW:	`http://www.crm.cat/adort2010/`

Tipo de evento:	Workshop
Nombre:	WAVES AND MULTISCALE PROCESSES IN THE TROPICS
Lugar:	American Institute of Mathematics, Palo Alto, California, USA
Fecha:	06–10 December 2010
Organiza:	Joseph Biello, Boualem Khouider, and George Kiladis
Información:	
E-mail:	workshops@aimath.org
WWW:	http://aimath.org/ARCC/workshops/multiscale.html

Tipo de evento:	Congreso
Nombre:	2ND AFRICAN CONFERENCE ON COMPUTATIONAL MECHANICS
Lugar:	Cape Town, SOUTH AFRICA
Fecha:	05–08 January 2011
Organiza:	A. G. Malan, South Africa; P. Nithiarasu, United Kingdom; B. D. Reddy, South Africa
Información:	
E-mail:	amalan@csir.co.za (A. G. Malan); P.Nithiarsu@swansea.ac.uk (P. Nithiarasu); daya.reddy@uct.ac.za (B. D. Reddy)
WWW:	http://www.africomp.com/

Tipo de evento:	Workshop
Nombre:	High Performance Computing and Emerging Architectures
Lugar:	American Institute of Mathematics, Palo Alto, California, USA
Fecha:	10–14 January 2011
Organiza:	Lorena A. Barba (Mechanical Engineering, Boston University), Eric Darve (Mechanics and Computation Group and Flow Physics and Computational Engineering Group, Stanford University), David Keyes (Applied Physics & Applied Mathematics, Columbia University / Kaust)
Información:	
E-mail:	
WWW:	http://www.ima.umn.edu/2010-2011/W1.10-14.11/

Tipo de evento:	Congreso
Nombre:	Congreso de la Real Sociedad Matemática Española. Centenario de la RSME
Lugar:	Ávila, ESPAÑA
Fecha:	01–05 Febrero 2011
Organiza:	José María Muñoz Porras (USAL, Presidente), Esteban Gómez González (USAL), Daniel Hernández Ruipérez (USAL), Ana Cristina López Martín (USAL), Luis Manuel Navas Vicente (USAL), Francisco José Plaza Martín (USAL), Fernando Sancho de Salas (USAL), Carlos Tejero Prieto (USAL)
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E-mail:	
WWW:	http://campus.usal.es/~rsme2011/

Tipo de evento:	Workshop
Nombre:	Fourth School and Workshop on Mathematical Methods in Quantum Mechanics
Lugar:	Casa della Gioventù, University of Padova, Bressanone, ITALIA
Fecha:	14–19 February 2011
Organiza:	R. Adami (Milano), L. Barletti (Firenze), F. Cardin (Padova), G. Dell'Antonio (Roma "La Sapienza"), R. Figari (Napoli), S. Graffi (Bologna), G. Panati (Roma "La Sapienza"), M. Pulvirenti (Roma "La Sapienza"), A. Sacchetti (Modena), A. Teta (L'Aquila)
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E-mail:	`info.mmqm@unimore.it`
WWW:	`http://www.mmqm.unimore.it/`

Tipo de evento:	Conference
Nombre:	SIAM Conference on Computational Science and Engineering (CSE11)
Lugar:	Grand Sierra Resort and Casino, Reno, Nevada, USA
Fecha:	28 February – 04 March 2011
Organiza:	Society for Industrial and Applied Mathematics (SIAM)
Información:	
E-mail:	
WWW:	`http://www.siam.org/meetings/cse11/`

Tipo de evento:	Workshop
Nombre:	Computing in Image Processing, Computer Graphics, Virtual Surgery, and Sports
Lugar:	Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota, USA
Fecha:	07–11 March 2011
Organiza:	Institute for Mathematics and its Applications (IMA)
Información:	
E-mail:	
WWW:	`http://www.ima.umn.edu/2010-2011/W3.7-11.11/`

Tipo de evento:	Conference
Nombre:	SIAM Conference on Mathematical & Computational Issues in the Geosciences (GS11)
Lugar:	Hilton Long Beach & Executive Meeting Center, Long Beach, California, USA
Fecha:	21–24 March 2011
Organiza:	Society for Industrial and Applied Mathematics (SIAM)
Información:	
E-mail:	
WWW:	`http://www.siam.org/meetings/gs11/`

Tipo de evento:	Congress
Nombre:	Computational Challenges in Partial Differential Equations
Lugar:	Faraday Building, Swansea University, UK
Fecha:	14–18 April 2011
Organiza:	The Isaac Newton Institute for Mathematical Sciences (INI), Cambridge, and the Wales Institute of Mathematical and Computational Sciences (WIMCS)
Información:	
E-mail:	`k.morgan@swansea.ac.uk`
WWW:	`http://www.wimcs.ac.uk/INI_Meeting.html`

Tipo de evento:	Conference
Nombre:	Eleventh SIAM International Conference on Data Mining
Lugar:	Phoenix, Arizona, USA
Fecha:	28–30 April 2011
Organiza:	Society for Industrial and Applied Mathematics (SIAM)
Información:	
E-mail:	
WWW:	`http://www.siam.org/meetings/sdm11/`

Tipo de evento:	Conference
Nombre:	SIAM Conference on Optimization (OP11)
Lugar:	Darmstadt, ALEMANIA
Fecha:	16–19 May 2011
Organiza:	Society for Industrial and Applied Mathematics (SIAM)
Información:	
E-mail:	
WWW:	`http://www.siam.org/meetings/op11/`

Tipo de evento:	Congress
Nombre:	7th International Congress on Industrial and Applied Mathematics (ICIAM 2011)
Lugar:	Vancouver, BC, CANADA
Fecha:	18–22 July 2011
Organiza:	Canadian Applied and Industrial Mathematics Society / Société canadienne de mathématiques appliquées et industrielles (CAIMS / SCMAI), Society for Industrial and Applied Mathematics (SIAM), Mathematics of Information Technology and Complex Systems (MITACS)
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