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Estimados socios,

Aquí tenéis un nuevo número del Boletín el en que incluimos dos interesantes artículos de J. Garcia-Azorero, J. J. Manfredi, I. Peral y J.D. Rossi, y de C. Bourdarias, M. Gisclon y S. Junca, además de la segunda parte de las Actas del Workshop Iberoamericano de Matemáticas Aplicadas celebrado en Chillán (Chile), cuya primera parte publicamos en el volumen 41. Junto a esto presentamos el trabajo galardonado con el Premio S$\vec{e}$MA a la Divulgación de la Matemática Aplicada 2008, que recayó en X. Mora y una revisión de Michel Bernadou de las Escuelas Hispano-Francesas de Simulación Numérica en Física e Ingeniería celebradas hasta la fecha.

Os recordamos que la Sociedad tiene un importante encuentro en el marco de la XIII Escuela Hispano-Francesa sobre Simulación Numérica en Física e Ingeniería que se celebrará en el próximo mes de septiembre en Valladolid.

Recibid un cordial saludo,

Grupo Editor
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ARTÍCULOS

NEUMANN BOUNDARY CONDITIONS FOR THE INFINITY LAPLACIAN AND THE MONGE-KANTOROVICH MASS TRANSPORT PROBLEM

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Abstract

In this note we review some recent results concerning the natural Neumann boundary condition for the ∞-Laplacian and its relation with the Monge-Kantorovich mass transport problem.

1. We study the limit as $p \to \infty$ of solutions of $-\Delta_p u_p = 0$ in a domain Ω with $|Du_p|^{p-2}\partial u_p/\partial \nu = g$ on $\partial\Omega$. We obtain a natural minimization problem that is verified by a limit point of $\{u_p\}$ and a limit problem that is satisfied in the viscosity sense. It turns out that the limit variational problem is related to the Monge-Kantorovich mass transport problems when the measures are supported on $\partial\Omega$.

2. Next, we study the limit of Monge-Kantorovich mass transport problems when the involved measures are supported in a small strip near the boundary of a bounded smooth domain, Ω. Given an absolutely continuous measure (with respect to the surface measure) supported on the boundary $\partial\Omega$ with zero mean value, by performing a suitable extension of the measures to a strip of width ε near the boundary of the domain Ω we consider the mass transfer problem for the extensions. Then we study the limit as ε goes to zero of the Kantorovich potentials for the extensions and obtain that it coincides with a solution of the original mass transfer problem.

3. Also we present a Steklov like eigenvalue problem that appears as the limit of the usual Steklov eigenvalue problem for the $p-$Laplacian as $p \to \infty$.

Key words: *Quasilinear elliptic equations, Neumann boundary conditions, Monge-Kantorovich mass transport problem.*

AMS subject classifications: *35J65 35J50 35J55*

1 Introduction. The Monge-Kantorovich mass transportation problem and the $\infty-$Laplacian

In this note we review some recent results obtained by the authors in [8], [13], [14] and [15]. We study the Monge-Kantorovich mass transport problem when the involved measures are supported on the boundary of the domain. This problem is related to the natural Neumann boundary conditions that appear when one considers the ∞-Laplacian in a smooth bounded domain as limit of the Neumann problem for the p-Laplacian as $p \to \infty$.

To formalize the mass transportation problem, let g be a measure with zero total mass and let Ω be a domain with $\mathrm{supp}(g) \subset \overline{\Omega}$. We want to determine the most efficient way of transport the measures g_+ to g_-, that is, we want to find a function $T : \mathrm{supp}(g_+) \to \mathrm{supp}(g_-)$ in such a way that T minimizes the total transport cost

$$L(T) = \int_\Omega |x - T(x)| dg_+(x).$$

We refer to [24] and [10] for references and details.

On the other hand, let $\Delta_p u = \mathrm{div}\left(|Du|^{p-2}Du\right)$ be the p-Laplacian. The ∞-Laplacian is the limit operator $\Delta_\infty = \lim_{p\to\infty} \Delta_p$ given by

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i}$$

in the viscosity sense (to be more precise, in the sense that a uniform limit of $p-$harmonic functions (solutions to $\Delta_p u = 0$) is $\infty-$harmonic (a solution to $\Delta_\infty u = 0$)). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function f; see [2], [3], and [17]. A fundamental result of Jensen [17] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense.

The close relation between the Monge-Kantorovich problem and the limit as $p \to \infty$ for solutions to $\Delta_p u = f$ was first noticed by Evans and Gangbo in [11]. They considered mass transfer optimization problems between absolutely continuous measures (with respect to the Lebesgue measure) that appear as limits of p-Laplacian problems. A very general approach is discussed in [7], where a problem related to but different from ours is discussed (see Remark 4.3 in [7]).

Here we study the Neumann problem for the ∞-Laplacian obtained as the limit as $p \to \infty$ of the problems

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. The boundary data g is a continuous function that necessarily verifies the compatibility condition $\int_{\partial\Omega} g = 0$, otherwise there is no solution to (1). Imposing the normalization $\int_\Omega u = 0$ there exists a unique solution to problem (1) that we denote by u_p.

We will find a variational problem that is verified by a limit point of $\{u_p\}$ and a limit partial differential equation that is satisfied in the viscosity sense. Next, we will study the limit of Monge-Kantorovich mass transfer problems when the involved measures are supported in a small strip near the boundary. Given an absolutely continuos measure (with respect to the surface measure) supported on the boundary $\partial\Omega$ with zero mean value, by performing a suitable extension of the measures to a strip of width ε near the boundary of the domain Ω we consider the mass transfer problem for the extensions. Then we study the limit as ε goes to zero of the Kantorovich potentials for the extensions and obtain that it coincides with a solution of the original mass transfer problem. Moreover, recent results from game theory allow to give a probabilistic interpretation of the infinity Laplacian (see Section 5 for details). Here, we use these results and show that the PDE that is solved by the continuous value of the game is actually a mixed boundary value problem for the infinity Laplacian. In addition, this game theory interpretation provides a proof of the uniqueness of viscosity solutions to this mixed problem. Also, at the end of this paper, we will indicate a Steklov like eigenvalue problem that appears as the limit of the usual Steklov eigenvalue problem for the $p-$Laplacian as $p \to \infty$.

When considering the Neumann problem, boundary conditions that involve the outer normal derivative, $\partial u/\partial \nu$ have been addressed from the point of view of viscosity solutions for fully nonlinear equations in [4] and [16]. In these references it is proved that there exist viscosity solutions and comparison principles between them when appropriate hypothesis are satisfied. In particular a suitable strict monotonicity is needed and such property does not hold in our case of interest.

The rest of the paper is organized as follows: in Section 2 we deal with a variational setting, in Section 3 we perform a viscosity analysis of the limit as $p \to \infty$ in (1), in Section 4 we approximate these problems (both variationally and in the viscosity sense) by problems with measures supported in small strips near the boundary, in Section 5 we present some results using a game theory approach and finally in Section 6 we present a related eigenvalue problem.

2 A variational approach

A solution to (1) can be obtained by a variational principle. In fact, up to a Lagrange multiplier, $\lambda_p \to 1$ as $p \to \infty$, we can write

$$\int_{\partial\Omega} u_p\, g = \max\left\{ \int_{\partial\Omega} wg \colon w \in W^{1,p}(\Omega),\ \int_\Omega w = 0\,,\ \int_\Omega |Dw|^p \le 1 \right\}. \qquad (2)$$

Our first result states that there exist accumulation points of the family $\{u_p\}_{p>1}$ as $p \to \infty$ which are maximizers of a variational problem that is the natural limit of variational problems (2). Observe that for $q > N$ the set $\{u_p\}_{p>q}$ is bounded in $C^{1-p/q}(\overline{\Omega})$.

Theorem 1 *Let v_∞ be a uniform limit of a subsequence $\{u_{p_i}\}$, $p_i \to \infty$, then v_∞ is a solution to the maximization problem*

$$\int_{\partial\Omega} v_\infty g = \max\left\{ \int_{\partial\Omega} wg \colon w \in W^{1,\infty}(\Omega),\ \int_\Omega w = 0,\ \|Dw\|_\infty \le 1 \right\}. \tag{3}$$

An equivalent dual statement is the minimization problem

$$\|Dv_\infty\|_\infty = \min\left\{ \|Dw\|_\infty \colon w \in W^{1,\infty}(\Omega),\ \int_\Omega w = 0,\ \int_{\partial\Omega} wg \ge 1 \right\}. \tag{4}$$

The maximization problem (3) is also obtained by applying the Kantorovich optimality principle to a mass transfer problem for the measures $\mu^+ = g^+\mathcal{H}^{N-1}\llcorner\,\partial\Omega$ and $\mu^- = g^-\mathcal{H}^{N-1}\llcorner\,\partial\Omega$ that are concentrated on $\partial\Omega$. The mass transfer compatibility condition $\mu^+(\partial\Omega) = \mu^-(\partial\Omega)$ holds since g has zero average on $\partial\Omega$. The maximizers of (3) are called maximal Kantorovich potentials [1].

To prove Theorem 1 we follow [13]. We review some previous estimates.

Suppose that we have a sequence $\{u_p\}$ of solutions to (1). Since we are interested in large values of p we may assume that $p > N$ and hence $u_p \in C^\alpha(\overline{\Omega})$. Multiplying the equation by u_p and integrating we obtain,

$$\int_\Omega |Du_p|^p = \int_{\partial\Omega} u_p\, g \le \left(\int_{\partial\Omega} |u_p|^p \right)^{1/p} \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} \tag{5}$$

where p' is the exponent conjugate to p, that is $1/p' + 1/p = 1$. Recall the following trace inequality, see for example [9],

$$\int_{\partial\Omega} |\phi|^p d\sigma \le Cp \left(\int_\Omega |\phi|^p + |D\phi|^p dx \right),$$

where C is a constant that does not depend on p. Going back to (5), we get,

$$\int_\Omega |Du_p|^p \le \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} C^{1/p} p^{1/p} \left(\int_\Omega |u_p|^p + |Du_p|^p dx \right)^{1/p}.$$

On the other hand, for large p we have

$$|u_p(x) - u_p(y)| \le C_p |x-y|^{1-\frac{N}{p}} \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

Since we are assuming that $\int_\Omega u_p = 0$, we may choose a point y such that $u_p(y) = 0$, and hence

$$|u_p(x)| \le C(p,\Omega) \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

The arguments in [9], pag. 266-267, show that the constant $C(p,\Omega)$ can be chosen uniformly in p. Hence, we obtain

$$\int_\Omega |Du_p|^p \le \left(\int_{\partial\Omega} |g|^{p'}\right)^{1/p'} C^{1/p} p^{1/p} (C_2^p+1)^{1/p} \left(\int_\Omega |Du_p|^p dx\right)^{1/p}.$$

Taking into account that $p' = p/(p-1)$, for large values of p we get

$$\left(\int_\Omega |Du_p|^p\right)^{1/p} \le \alpha_p \left(\int_{\partial\Omega} |g|^{p'}\right)^{1/p}$$

where $\alpha_p \to 1$ as $p \to \infty$. Next, fix m, and take $p > m$. We have,

$$\left(\int_\Omega |Du_p|^m\right)^{1/m} \le |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_\Omega |Du_p|^p\right)^{1/p} \le |\Omega|^{\frac{1}{m}-\frac{1}{p}} \alpha_p \left(\int_{\partial\Omega} |g|^{p'}\right)^{1/p},$$

where $|\Omega|^{\frac{1}{m}-\frac{1}{p}} \to |\Omega|^{\frac{1}{m}}$ as $p \to \infty$. Hence, there exists a weak limit in $W^{1,m}(\Omega)$ that we will denote by v_∞. This weak limit has to verify

$$\left(\int_\Omega |Dv_\infty|^m\right)^{1/m} \le |\Omega|^{\frac{1}{m}}.$$

As the above inequality holds for every m, we get that $v_\infty \in W^{1,\infty}(\Omega)$ and moreover, taking the limit $m \to \infty$,

$$|Dv_\infty| \le 1, \qquad \text{a.e. } x \in \Omega.$$

Lemma 1 *The subsequence u_{p_i} converges to v_∞ uniformly in $\overline{\Omega}$.*

Proof. From our previous estimates we know that

$$\left(\int_\Omega |Du_p|^p dx\right)^{1/p} \le C,$$

uniformly in p. Therefore we conclude that u_p is bounded (independently of p) and has a uniform modulus of continuity. Hence u_p converges uniformly to v_∞. □

Proof Proof of Theorem 1. Multiplying by u_p, passing to the limit, and using Lemma 1, we obtain,

$$\lim_{p\to\infty} \int_\Omega |Du_p|^p = \lim_{p\to\infty} \int_{\partial\Omega} u_p g = \int_{\partial\Omega} v_\infty g.$$

If we multiply (1) by a test function w, we have, for large enough p,

$$\begin{aligned}\int_{\partial\Omega} wg &\le \left(\int_\Omega |Du_p|^p\right)^{(p-1)/p} \left(\int_\Omega |Dw|^p\right)^{1/p} \\ &\le \left(\int_{\partial\Omega} v_\infty g d\sigma + \delta\right)^{(p-1)/p} \left(\int_\Omega |Dw|^p\right)^{1/p}.\end{aligned}$$

As the previous inequality holds for every $\delta > 0$, passing to the limit as $p \to \infty$ we conclude,

$$\int_{\partial\Omega} wg \le \left(\int_{\partial\Omega} v_\infty g \right) \|Dw\|_\infty .$$

Hence, the function v_∞ verifies,

$$\int_{\partial\Omega} v_\infty g = \max \left\{ \int_{\partial\Omega} wg \, : \, w \in W^{1,\infty}(\Omega), \, \int_\Omega w = 0, \, \|Dw\|_\infty \le 1 \right\},$$

or equivalently,

$$\|Dv_\infty\|_\infty = \min \left\{ \|Dw\|_\infty : \, w \in W^{1,\infty}(\Omega), \, \int_\Omega w = 0, \, \int_{\partial\Omega} wg \, \le 1 \right\}.$$

This ends the proof. □

On the other hand, taking as a test function in the maximization problem v_∞ itself we obtain the following corollary.

Corollary 1 *If $g \not\equiv 0$, then $\|Dv_\infty\|_{L^\infty(\Omega)} = 1$.*

3 Viscosity setting

In this section we discuss the equation that v_∞, a uniform limit of u_p as $p \to \infty$, satisfies in the viscosity sense.

Following [4] let us recall the definition of viscosity solution taking into account general boundary conditions. Assume

$$F : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{N \times N} \to \mathbb{R}$$

a continuous function. The associated equation

$$F(x, Du, D^2u) = 0$$

is called (degenerate) elliptic if

$$F(x, \xi, X) \le F(x, \xi, Y) \quad \text{ if } X \ge Y.$$

Definition 1 *Consider the boundary value problem*

$$\begin{cases} F(x, \nabla u, D^2u) = 0 & \text{in } \Omega, \\ B(x, u, \nabla u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

1. *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\max\{B(x_0, \phi(x_0), \nabla\phi(x_0)), \; F(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \ge 0$$

holds and if $x_0 \in \Omega$ then we require

$$F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \ge 0.$$

2. *An upper semi-continuous function u is a subsolution if for every $\psi \in C^2(\overline{\Omega})$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\min\{B(x_0, \psi(x_0), \nabla\psi(x_0)),\ F(x_0, \nabla\psi(x_0), D^2\psi(x_0))\} \leq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, \nabla\psi(x_0), D^2\psi(x_0)) \leq 0.$$

3. *Finally, u is a viscosity solution if it is a super and a subsolution.*

The main result in this section is the following theorem.

Theorem 2 *A limit v_∞ is a solution of*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ B(x, u, Du) = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

in the viscosity sense. Here

$$B(x,u,Du) \equiv \begin{cases} \min\left\{|Du| - 1\ ,\ \frac{\partial u}{\partial \nu}\right\} & \text{if } g(x) > 0, \\ \max\{1 - |Du|\ ,\ \frac{\partial u}{\partial \nu}\} & \text{if } g(x) < 0, \\ H(|Du|)\frac{\partial u}{\partial \nu} & \text{if } g(x) = 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{if } x \in \{g(x) = 0\}^o, \end{cases}$$

where $\{g(x) = 0\}^o$ is the interior of the zero level set of g and $H(a)$ is given by

$$H(a) = \begin{cases} 1 & \text{if } a \geq 1, \\ 0 & \text{if } 0 \leq a < 1. \end{cases}$$

Proof Proof of Theorem 2. Again we follow [13]. First, let us check that $-\Delta_\infty v_\infty = 0$ in the viscosity sense in Ω. Let us recall the standard proof. Let ϕ be a smooth test function such that $v_\infty - \phi$ has a strict maximum at $x_0 \in \Omega$. Since u_{p_i} converges uniformly to v_∞ we get that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \Omega$ with $x_i \to x_0$. Next we use the fact that u_{p_i} is a viscosity solution (see [13] for a proof of this fact) of

$$-\Delta_{p_i} u_{p_i} = 0$$

and we obtain

$$-(p_i - 2)|D\phi|^{p_i - 4}\Delta_\infty\phi(x_i) - |D\phi|^{p_i - 2}\Delta\phi(x_i) \leq 0. \tag{8}$$

If $D\phi(x_0) = 0$ we get $-\Delta_\infty\phi(x_0) \leq 0$. If this is not the case, we have that $D\phi(x_i) \neq 0$ for large i and then

$$-\Delta_\infty\phi(x_i) \leq \frac{1}{p_i - 2}|D\phi|^2\Delta\phi(x_i) \to 0, \quad \text{as } i \to \infty.$$

We conclude that

$$-\Delta_\infty \phi(x_0) \leq 0.$$

That is v_∞ is a viscosity subsolution of $-\Delta_\infty u = 0$.

A similar argument shows that v_∞ is also a supersolution and therefore a solution of $-\Delta_\infty v_\infty = 0$ in Ω.

Let us check the boundary condition. There are six cases to be considered. Here we deal only with one and refer to [13] for the rest of the cases.

Assume that $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \geq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \geq g(x_i).$$

Since $g(x_0) > 0$, we have $D\phi(x_0) \neq 0$, and we obtain

$$|D\phi|(x_0) \geq 1.$$

Moreover, we also have

$$\frac{\partial \phi}{\partial \nu}(x_0) \geq 0.$$

Hence, if $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$, we have

$$\max\left\{ \min\{-1 + |D\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\} \, , -\Delta_\infty \phi(x_0) \right\} \geq 0. \tag{9}$$

This ends the proof. □

Remark 1 *The function v_∞ is a viscosity solution of $\Delta_\infty v_\infty = 0$ in Ω and therefore it is an absolutely minimizing function, [3]. It is a minimizer of the Lipschitz constant of u among functions that coincide with v_∞ on $\partial\Omega'$ in every subdomain Ω' of Ω. Therefore we can rewrite the maximization problem* (3) *as a maximization problem on $\partial\Omega$: $v_\infty|_{\partial\Omega}$ is a function that has Lipschitz constant less or equal than one on $\partial\Omega$ and maximizes $\int_{\partial\Omega} ug$.*

Concerning the limit PDE, note that there is no uniqueness of viscosity solutions of (7), see [13]. Nevertheless we can say something about uniqueness under some favorable geometric assumptions on g and Ω. The proof of uniqueness is based on some tools from [11]. To state our uniqueness result let us describe the required geometrical hypothesis on the boundary data. Let $\partial\Omega_+ = \text{supp} g^+$ and $\partial\Omega_- = \text{supp} g^-$. For a given v_∞ a maximizer in (3) following [11] we define the transport set as

$$T(v_\infty) = \left\{ \begin{array}{rl} z \in \overline{\Omega} : \exists x \in \partial\Omega_+, y \in \partial\Omega_-, & v_\infty(z) = v_\infty(x) - |x - z| \\ & \text{and} \quad v_\infty(z) = v_\infty(y) + |y - z| \end{array} \right\}.$$

Observe that this set T is closed. We have the following property (see [11])

Proposition 1 *Suppose that Ω is a convex domain. Let v_∞ be a maximizer of (3) with $\Delta_\infty v_\infty = 0$, then $|Dv_\infty(x)| = 1$, for a.e. $x \in T(v_\infty)$.*

Define a transport ray by $R_x = \{z \,|\, |v_\infty(x) - v_\infty(z)| = |x - z|\}$. Notice that two transport rays cannot intersect in Ω unless they are identical. Indeed, assume $z \in T$ then there exist $x, y \in \overline{\Omega}$ such that $v_\infty(x) - v_\infty(z) = |x - z|$ and $v_\infty(z) - v_\infty(y) = |z - y|$, then $|x - y| \leq |x - z| + |z - y| = v_\infty(x) - v_\infty(y)$. If x, y and z are not colinear we contradict the Lipschitz condition verified by v_∞.

Our first geometric hypothesis for uniqueness is then

$$\partial\Omega \subset T(v_\infty).$$

Note that with similar ideas but using the uniqueness of viscosity solutions to a mixed problem for the infinity Laplacian, this hypothesis can be relaxed (see the last remark of Section 3).

We have:

Theorem 3 *Assume that we have a convex domain Ω and a boundary datum g on $\partial\Omega$ such that every maximizer v_∞ with $\Delta_\infty v_\infty = 0$ verifies $\partial\Omega \subset T(v_\infty)$, then there exists a unique infinite harmonic solution, u_∞ to (3). Hence, the limit $\lim_{p\to\infty} u_p = u_\infty$, uniformly in Ω exists.*

Remark 2 *Observe that if $\{g = 0\}$ has empty interior on the boundary then the uniqueness of the limit holds since for every v_∞ we get $\partial\Omega \subset T(v_\infty)$.*

Examples. To illustrate our results we present some examples. In an interval $\Omega = (-L, L)$ with $g(L) = -g(-L) > 0$ the limit of the solutions of (1), u_p, turns out to be $u_\infty(x) = x$. It is easy to check that this function is indeed the unique solution of the maximization problem (3) and of the problem (7).

This example can be easily generalized to the case where Ω is an annulus, $\Omega = \{r_1 < |x| < r_2\}$, and the function g is a positive constant g_1 on $|x| = r_1$ and a negative constant g_2 on $|x| = r_2$ with the constraint $\int_{\partial\Omega} g = \int_{|x|=r_1} g + \int_{|x|=r_2} g = 0$. The solutions u_p of (1) in the annulus converge uniformly as $p \to \infty$ to a cone $u_\infty(x) = C - |x|$. However one can modify the function g on $|x| = r_2$ in such a way it does not change its sign and that the cone does not maximize (3). Hence, there is no uniqueness for (7) even for non-vanishing boundary data.

An example of a domain and boundary data such that uniqueness of the limit holds is a disk in $\mathbb{R}^2$, $D = \{|(x, y)| < 1\}$ with $g(x, y) > 0$ for $x > 0$ and $g(x, y) < 0$ for $x < 0$ with $\int_{\partial D} g = 0$.

4 Approximations by measures supported in small strips near the boundary

In this section we will show that these variational problems can be achieved as a singular limit of mass transport problems where the measures are supported in small strips near the boundary. In this sense we get a natural Neumann problem

for the $p-$Laplacian while in the paper [11] it appears a Dirichlet condition in a large ball.

Precisely, let us consider the subset of Ω,

$$\omega_\delta = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) < \delta\}.$$

Note that this set has measure $|\omega_\delta| \sim \delta \mathcal{H}^{N-1}(\partial\Omega)$ for small values of δ. Then for sufficiently small $s > 0$ we can define the *parallel* interior boundary $\Gamma_s = \{z - s\nu(z),\ z \in \partial\Omega\}$ where $\nu(z)$ denotes the outwards normal unit at $z \in \partial\Omega$. Note that $\Gamma_0 = \partial\Omega$. Then we can also look at the set ω_δ as the neighborhood of Γ_0 defined by

$$\omega_\delta = \{y = z - s\nu(z),\ z \in \partial\Omega,\ s \in (0, \delta)\} = \bigcup_{0<s<\delta} \Gamma_s$$

for sufficiently small δ, say $0 < \delta < \delta_0$. We also denote $\Omega_s = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > s\}$ and for s small we have that $\partial\Omega_s = \Gamma_s$.

Let us consider the transport problem for a suitable extension of g. To define this extension, as we have mentioned, let us denote by $d\sigma$ and $d\sigma_s$ the surface measures on the sets $\partial\Omega$ and Γ_s respectively. Given a function ϕ defined on $\bar{\Omega}$, and given $y \in \Gamma_s$ (with s small) , there exists $z \in \partial\Omega$ such that $y = z - s\nu(z)$. Hence, we can change variables:

$$\int_{\Gamma_s} \phi(y) d\sigma_s = \int_{\partial\Omega} \phi(z - s\nu(z)) G(s, z)\, d\sigma$$

where $G(s, z)$ depends on Ω (more precisely, it depends on the surface measures $d\sigma$ and $d\sigma_s$), and by the regularity of $\partial\Omega$, $G(s, z) \to 1$ as $s \to 0$ uniformly for $z \in \partial\Omega$.

Using these ideas, we define the following extension of g in Ω. Consider $\eta : [0, \infty) \to [0, \infty)$ a $\mathcal{C}^\infty$ such that $\eta(s) = 1$ if $0 \le s \le \frac{1}{2}$, $\eta(s) = 0$ if $s > 1$, $0 \le \eta(s) \le 1$ and $\int_0^\infty \eta(s)\, ds = A$. Defining $\eta_\varepsilon(s) = \frac{1}{A\varepsilon}\eta\left(\frac{s}{\varepsilon}\right)$, we get $\int_0^\infty \eta_\varepsilon(s)\, ds = 1$. For $\delta < \varepsilon$ consider Γ_s and

$$g_\varepsilon(y) = \eta_\varepsilon(s) \frac{g(z)}{G(s, z)}, \qquad y = z - s\nu(z).$$

We have $g_\varepsilon \equiv 0$ in $\Omega - \omega_\varepsilon$ and $g_\varepsilon \in \mathcal{C}(\Omega)$. Moreover,

$$\begin{aligned} \int_\Omega g_\varepsilon(x)\, dx &= \int_0^\varepsilon \int_{\Gamma_s} g_\varepsilon(y)\, d\sigma_s\, ds \\ &= \int_0^\varepsilon \int_{\partial\Omega} g_\varepsilon(z - s\nu(z)) G(s, z)\, d\sigma\, ds \\ &= \int_0^\varepsilon \eta_\varepsilon(s) \int_{\partial\Omega} g(z)\, d\sigma\, ds = 0. \end{aligned}$$

Associated to this extension we could consider the following two variational problems. First, the maximization problem in $W^{1,p}(\Omega)$,

$$\max\left\{\int_{\omega_\varepsilon} w g_\varepsilon : w \in W^{1,p}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^p(\Omega)} \le 1\right\}, \tag{10}$$

and the maximization problem in $W^{1,\infty}(\Omega)$,

$$\max\left\{\int_{\omega_\varepsilon} wg_\varepsilon : w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1\right\}. \tag{11}$$

We call $u_{p,\varepsilon}$ a solution to (10) and $u_{\infty,\varepsilon}$ a solution to (11).

Our first result says that we can take the limits as $\varepsilon \to 0$ and $p \to \infty$ in these variational problems. With the above notations we have the following commutative diagram

$$\begin{array}{ccc} u_{\infty,\varepsilon} & \to & u_{\infty,0} \\ p\to\infty \quad \uparrow & & \uparrow \\ u_{p,\varepsilon} & \underset{\varepsilon\to 0}{\to} & u_{p,0} \end{array} \tag{12}$$

This diagram can be understood in two senses, either taking into account the variational properties satisfied by the functions, or considering the corresponding PDEs that the functions satisfy.

From the variational viewpoint, first we can state the following result:

Theorem 4 *Diagram* (12) *is commutative in the following sense:*

1. *Maximizers of* (10), $u_{p,\varepsilon}$, *converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{p,0}$ *a maximizer of* (2) *as* $\varepsilon \to 0$.
2. *Maximizers of* (10), $u_{p,\varepsilon}$, *converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,\varepsilon}$ *a maximizer of* (11) *as* $p \to \infty$.
3. *Maximizers of* (11), $u_{\infty,\varepsilon}$, *converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,0}$ *a maximizer of* (3) *as* $\varepsilon \to 0$.
4. *Maximizers of* (2), $u_{p,0}$, *converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,0}$ *a maximizer of* (3) *as* $p \to \infty$.

Proof Proof of Theorem 4. The proof of the uniform convergence (along subsequences) of $u_{p,0}$ to $u_{\infty,0}$ is contained in [13].

Let us prove that $u_{p,\varepsilon}$ converges to $u_{p,0}$ as $\varepsilon \to 0$. We have

$$\|Du_{p,\varepsilon}\|_{L^p(\Omega)} \le 1.$$

Therefore we can extract a subsequence (that we still call $u_{p,\varepsilon}$) such that

$$u_{p,\varepsilon} \rightharpoonup v, \qquad \text{as } \varepsilon \to 0,$$

weakly in $W^{1,p}(\Omega)$ and, since $p > N$,

$$u_{p,\varepsilon} \to v, \qquad \text{as } \varepsilon \to 0,$$

uniformly in Ω (in fact, convergence holds in C^β). This limit v verifies the normalization constraint

$$\int_\Omega v = 0$$

and moreover

$$\|Dv\|_{L^p(\Omega)} \leq 1.$$

On the other hand, thanks to the uniform convergence and to the definition of the extension g_ε we obtain,

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_{\omega_\varepsilon} g_\varepsilon \, u_{p,\varepsilon} &= \lim_{\varepsilon \to 0} \int_0^\varepsilon \int_{\Gamma_s} g_\varepsilon(y) u_{p,\varepsilon}(y) \, d\sigma_s \, ds \\
&= \lim_{\varepsilon \to 0} \int_0^\varepsilon \int_{\partial\Omega} g_\varepsilon(z - s\nu(z)) u_{p,\varepsilon}(z - s\nu(z)) G(s,z) \, d\sigma \, ds \\
&= \lim_{\varepsilon \to 0} \int_0^\varepsilon \eta_\varepsilon(s) \int_{\partial\Omega} g(z) u_{p,\varepsilon}(z - s\nu(z)) \, d\sigma \, ds \\
&= \int_{\partial\Omega} g v \, d\sigma
\end{aligned}$$

and hence

$$\int_\Omega |Dv|^p - \int_{\partial\Omega} g v \, d\sigma \leq \liminf_{\varepsilon \to 0} \left(\int_\Omega |Du_{p,\varepsilon}|^p - \int_{\omega_\varepsilon} g_\varepsilon \, u_{p,\varepsilon} \right). \tag{13}$$

On the other hand for every $w \in C^1(\overline{\Omega})$ we have

$$\int_\Omega |Dw|^p - \int_{\partial\Omega} g w \, d\sigma = \lim_{\varepsilon \to 0} \int_\Omega |Dw|^p - \int_{\omega_\varepsilon} g_\varepsilon \, w.$$

Hence, the extremal characterization of $u_{p,\varepsilon}$ implies

$$\inf_{u \in W^{1,p}(\Omega), \int_\Omega u = 0} \left\{ \int_\Omega |Du|^p - \int_{\partial\Omega} g u d\sigma \right\} \geq \liminf_{\varepsilon \to 0} \int_\Omega |Du_{p,\varepsilon}|^p - \int_{\omega_\varepsilon} g_\varepsilon \, u_{p,\varepsilon} \, .$$

And by (13) we obtain

$$\inf_{u \in W^{1,p}(\Omega), \ \int_\Omega u = 0} \left\{ \int_\Omega |Du|^p - \int_{\partial\Omega} g u \, d\sigma \right\} = \int_\Omega |Dv|^p - \int_{\partial\Omega} g v \, d\sigma,$$

and therefore all possible limits $v = u_{p,0}$ satisfy the extremal property (2).

Now, let us prove that $u_{\infty,\varepsilon}$ converges to $u_{\infty,0}$, a maximizer of (3). Recall that $u_{\infty,\varepsilon}$ is a solution to the problem

$$M_\varepsilon = \max \left\{ \int_{\omega_\varepsilon} w g_\varepsilon : w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

That is,

$$M_\varepsilon = \int_{\omega_\varepsilon} u_{\infty,\varepsilon} \, g_\varepsilon \, .$$

Therefore $u_{\infty,\varepsilon}$ is bounded in $W^{1,\infty}(\Omega)$ and then there exists a subsequence (that we still denote by $u_{\infty,\varepsilon}$) such that,

$$\begin{aligned} &u_{\infty,\varepsilon} \overset{*}{\rightharpoonup} v \text{ weakly-* in } W^{1,\infty}(\Omega) \text{ and} \\ &u_{\infty,\varepsilon} \to v \text{ uniformly in } \overline{\Omega}, \end{aligned} \tag{14}$$

as $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon \to 0} \int_{\omega_\varepsilon} u_{\infty,\varepsilon}\, g_\varepsilon = \int_{\partial\Omega} v g \, d\sigma.$$

On the other hand, for every $z \in C^1(\overline{\Omega})$ it holds that

$$\lim_{\varepsilon \to 0} \int_{\omega_\varepsilon} g_\varepsilon\, z = \int_{\partial\Omega} g z \, d\sigma.$$

Hence, if we call

$$M = \max\left\{ \int_{\partial\Omega} w g \, d\sigma \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}, \tag{15}$$

we obtain, from (14),

$$M \le \liminf_{\varepsilon \to 0} M_\varepsilon = \int_{\partial\Omega} v g \, d\sigma.$$

Therefore $v = u_{\infty,0}$ is a maximizer of (15), as we wanted to prove.

Finally, let us prove that $u_{p,\varepsilon} \to u_{\infty,\varepsilon}$. Recall that

$$\int_{\omega_\varepsilon} u_{p,\varepsilon}\, g_\varepsilon = \max\left\{ \int_{\omega_\varepsilon} w g_\varepsilon : w \in W^{1,p}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^p(\Omega)} \le 1 \right\}.$$

Therefore, for any $q < p$

$$\|Du_{p,\varepsilon}\|_{L^q(\Omega)} \le \left(\|Du_{p,\varepsilon}\|_{L^p(\Omega)} |\Omega|^{\frac{p-q}{p}} \right)^{1/q} \le |\Omega|^{\frac{p-q}{pq}}.$$

Hence, we can extract a subsequence (still denoted by $u_{p,\varepsilon}$) such that,

$$u_{p,\varepsilon} \to u, \qquad \text{uniformly in } \overline{\Omega},$$

as $p \to \infty$ with

$$\|Du\|_{L^\infty(\Omega)} \le 1.$$

Then

$$\int_{\omega_\varepsilon} u_{p,\varepsilon}\, g_\varepsilon \to \int_{\omega_\varepsilon} u g_\varepsilon, \qquad \text{as } p \to \infty.$$

This limit u verifies that

$$\int_{\omega_\varepsilon} u g_\varepsilon \le \max\left\{ \int_{\omega_\varepsilon} w g_\varepsilon : w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}.$$

Let us prove that we have an equality here. If not, there exists a function v such that $v \in W^{1,\infty}(\Omega)$, $\int_\Omega v = 0$, $\|Dv\|_{L^\infty(\Omega)} \le 1$ with

$$\int_{\omega_\varepsilon} u g_\varepsilon < \int_{\omega_\varepsilon} v g_\varepsilon \, .$$

If we normalize, taking $\varphi = v/|\Omega|^{1/p}$, we obtain a function in $W^{1,p}(\Omega)$ with $\int_\Omega \varphi = 0$, $\|D\varphi\|_{L^p(\Omega)} \le 1$ and such that

$$\lim_{p\to\infty} \int_{\omega_\varepsilon} u_{p,\varepsilon}\, g_\varepsilon = \int_{\omega_\varepsilon} u g_\varepsilon < \int_{\omega_\varepsilon} v g_\varepsilon = \lim_{p\to\infty} |\Omega|^{1/p} \int_{\omega_\varepsilon} \varphi g_\varepsilon \, .$$

This contradiction proves that

$$\int_{\omega_\varepsilon} u g_\varepsilon = \max\left\{ \int_{\omega_\varepsilon} w g_\varepsilon : w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}.$$

This ends the proof. □

Now, we turn our attention to the PDE verified by the limits in the viscosity sense (see Section 3 for the precise definition) or in the weak sense.

Up to a Lagrange multiplier λ_p the functions $u_{p,0}$ are viscosity (and weak) solutions to the problem,

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = \lambda_p\, g & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Let us to point out that it is easily seen that $\lambda_p \to 1$ as $p \to \infty$. Hence, to simplify the notation, we will drop this Lagrange multiplier in the sequel.

In the previous section, see [13] and also [14], the limit as $p \to \infty$ of the family $u_{p,0}$ is studied in the viscosity setting. It is proved that the problem that is satisfied by a uniform limit $u_{\infty,0}$ in the viscosity sense is (7) that we recall below,

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega, \\ B(x,u,Du) = 0, & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where

$$B(x,u,Du) \equiv \begin{cases} \min\left\{ |Du| - 1 \, , \, \frac{\partial u}{\partial \nu} \right\} & \text{if } g > 0, \\ \max\{1 - |Du| \, , \, \frac{\partial u}{\partial \nu}\} & \text{if } g < 0, \\ H(|Du|) \frac{\partial u}{\partial \nu} & \text{if } g = 0, \end{cases}$$

and $H(a)$ is given by

$$H(a) = \begin{cases} 1 & \text{if } a \ge 1, \\ 0 & \text{if } 0 \le a < 1. \end{cases}$$

Moreover, $u_{\infty,0}$ satisfies in the sense of viscosity the estimates:

$$|Du_{\infty,0}| \le 1 \quad \text{and} \quad -|Du_{\infty,0}| \ge -1,$$

see [6].

On the other hand, when we deal with the problems in the strips, again up to a Lagrange multiplier that converges to one, the functions $u_{p,\varepsilon}$ are weak (and hence viscosity) solutions to the problem,

$$\begin{cases} -\Delta_p u = g_\varepsilon & \text{in } \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Passing to the limit as $p \to \infty$ in these problems we get that the function $u_{\infty,\varepsilon}$ satisfy the following properties in the viscosity sense (see again [6]):

$$\begin{cases} |Du| \le 1 & \text{in } \Omega, \\ -|Du| \ge -1 & \text{in } \Omega, \end{cases} \tag{19}$$

and, in the different regions determined by g_ε:

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega \setminus \omega_\epsilon, \\ \min\{|Du| - 1, -\Delta_\infty u\} = 0 & \text{in } \{g_\epsilon > 0\}, \\ \max\{1 - |Du|, -\Delta_\infty u\} = 0 & \text{in } \{g_\epsilon < 0\}, \\ -\Delta_\infty u \ge 0 & \text{in } \Omega \cap \partial\{g_\epsilon > 0\} \cap (\partial\{g_\epsilon < 0\})^c, \\ -\Delta_\infty u \le 0 & \text{in } \Omega \cap \partial\{g_\epsilon < 0\} \cap (\partial\{g_\epsilon > 0\})^c. \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Notice that the equations in $\{g_\epsilon > 0\}$ and $\{g_\epsilon < 0\}$ can be simplified by the estimate (19), however to understand the boundary condition in viscosity sense it is necessary to consider such equations in its full generality.

We split our following results in two theorems.

First, we have,

Theorem 5

1. *The limit $u_{p,0}$ of a uniformly converging sequence $u_{p,\varepsilon}$ of weak solutions to* (18) *as $\varepsilon \to 0$ is a weak solution to* (16) *(and hence a viscosity solution).*

2. *The limit $u_{\infty,0}$ of a uniformly converging sequence $u_{p,0}$ of viscosity solutions to* (16) *as $p \to \infty$ is a viscosity solution to* (17).

Let us to point out that when $\varepsilon \to 0$, g_ε concentrates on the boundary, and therefore the sequence $\{g_\varepsilon\}$ is not uniformly bounded. This makes difficult to give a sense to pass to the limit in the viscosity framework when $\varepsilon \to 0$. Hence in this case we consider the variational characterization of the sequence $\{u_{p,\varepsilon}\}$ (that is equivalent to the fact of being a weak solution). To the best of our knowledge, it is not known that the notions of viscosity and weak solutions

coincide for solutions to (18), cf. [19] where such equivalence is only proved for Dirichlet boundary conditions.

Now, we deal with the rest of the commutative diagram. To pass to the limit in the sequence $u_{\infty,\varepsilon}$ we need the variational characterization, and we also need a uniqueness result for the limit problem, which has been proved in [13]. This uniqueness result says that:

If Ω is convex and $\{g = 0\}^o = \emptyset$, then there is a unique function which satisfies the extremal property (3).

Let us to point out that the hypothesis $\{g = 0\}^o = \emptyset$ implies also the uniqueness of the extremals to (11). Therefore, under this hypothesis there exists a unique $u_{\infty,\varepsilon}$ reached as a limit of the solutions $u_{p,\varepsilon}$ as $p \to \infty$.

Now, we can state our second theorem, see [15] for the proof.

Theorem 6

1. *The limit $u_{\infty,\varepsilon}$ of a uniformly converging sequence $u_{p,\varepsilon}$ of viscosity solutions to* (18) *as $p \to \infty$ is a viscosity solution to* (19)-(20).

2. *Assume that Ω is convex and $\{g = 0\}^o = \emptyset$. Consider the viscosity solutions $u_{\infty,\varepsilon}$ to* (19)-(20)*, obtained as a uniform limit as $p \to \infty$ of the solutions $u_{p,\varepsilon}$. Then, the sequence $\{u_{\infty,\varepsilon}\}$ converges uniformly to a viscosity solution to* (17)*, $u_{\infty,0}$.*

5 Connections with game theory. Tug-of-War games

In this section we deal with an approach to these type of problems based on game theory.

A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome). Recently, these type of games have been used in connection with some PDE problems, see [5], [20], [22], [23].

For the reader's convenience, let us first describe briefly the game introduced in [23] by Y. Peres, O. Schramm, S. Sheffield and D. Wilson. Consider a bounded domain $\Omega \subset \mathbb{R}^n$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$. Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\varepsilon(x_0)} \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\varepsilon(x_{k-1})} \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function*. In more general models, it is considered also a *running payoff* $f(x)$ defined on Ω, which represents the reward (respectively, the cost) at each intermediate state x, and gives rise to nonhomogeneous problems. We will

assume throughout this paper that $f \equiv 0$. This procedure gives a sequence of game states $x_0, x_1, x_2, \ldots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now we want to give a definition of the *value of the game*. To this end we introduce some notation and the normal or strategic form of the game (see [22] and [21]). The initial state $x_0 \in \overline{\Omega} \setminus \Gamma_D$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in \overline{B_\varepsilon(0)}$; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\varepsilon(0)} \times \overline{B_\varepsilon(0)}$ which is announced to the other player. Then, the new state $x_1 \in \overline{B_\varepsilon(x_0)}$ (namely, the current state plus the action) is selected according to the distribution $p(\cdot|x_0, a_0)$ in $\overline{\Omega}$. At stage k, knowing the history $h_k = (x_0, a_0, x_1, a_1, \ldots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), each player i chooses an action a_k^i. If the game terminated at time $j < k$, we set $x_m = x_j$ and $a_m = 0$ for $j \le m \le k$. The current state x_k and the profile $a_k = \{a_k^1, a_k^2\}$ determine the distribution $p(\cdot|x_k, a_k)$ of the new state x_{k+1}.

Denote $H_k = (\overline{\Omega} \setminus \Gamma_D) \times \left(\overline{B_\varepsilon(0)} \times \overline{B_\varepsilon(0)} \times \overline{\Omega}\right)^k$, the set of *histories up to stage* k, and by $H = \bigcup_{k \ge 1} H_k$ the set of all histories. Notice that H_k, as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \ldots, a_{k-1}, x_k, \ldots)$ endowed with the product topology. Then, the final payoff for Player I, i.e. F, induces a Borel-measurable function on H_∞. A *pure strategy* S_i for Player i, is a mapping from histories to actions, namely, a mapping from H to $\overline{B_\varepsilon(0)}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\varepsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\varepsilon(0)}$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}^{x_0}_{S_I, S_{II}}$ on the space of plays H_∞. We denote by $\mathbb{E}^{x_0}_{S_I, S_{II}}$ the corresponding expectation. Then, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

The ε-value of the game for Player I is given by

$$u_I^\varepsilon(x_0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)],$$

while the ε-value of the game for Player II is defined as

$$u_{II}^\varepsilon(x_0) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)].$$

In some sense, $u_I^\varepsilon(x_0), u_{II}^\varepsilon(x_0)$ are the least possible outcomes that each player expects to get when the ε-game starts at x_0. As in [23], we penalize severely the games that never end. If the game does not stop then we define $u_I^\varepsilon(x_0) = -\infty$ and $u_{II}^\varepsilon(x_0) = +\infty$.

If $u_I^\varepsilon = u_{II}^\varepsilon := u_\varepsilon$, we say that *the game has a value*. In [23] it is shown that, under very general hypotheses, that are fulfilled in the present setting, the ε-Tug-of-War game has a value.

All these $\varepsilon-$values are Lipschitz functions which converge uniformly when $\varepsilon \to 0$. The uniform limit as $\varepsilon \to 0$ of the game values u_ε is called *the continuous value* of the game that we will denote by u. Indeed, see [23], it turns out that u is a viscosity solution to the problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \Gamma_D, \end{cases} \tag{21}$$

where $\Delta_\infty u = |\nabla u|^{-2} \sum_{ij} u_{x_i} u_{x_i x_j} u_{x_j}$ is the $1-$homogeneous infinity Laplacian.

When $\Gamma_D \equiv \partial\Omega$, it is known that problem (21) has a unique viscosity solution, (as proved in [17], and in a more general framework, in [23]). Moreover, it is the unique AMLE (absolutely minimal Lipschitz extension) of $F : \Gamma_D \to \mathbb{R}$ in the sense that $Lip_U(u) = Lip_{\partial U \cap \Omega}(u)$ for every open set $U \subset \overline{\Omega} \setminus \Gamma_D$. AMLE extensions were introduced by Aronsson, see the survey [3] for more references and applications of this subject.

When $\Gamma_D \neq \partial\Omega$ the PDE problem (21) is incomplete, since there is a missing boundary condition on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Our main concern is to find the boundary condition that completes the problem. We have that it is in fact the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n}(x) = 0.$$

On the other hand, we give an alternative proof of the property $-\Delta_\infty u(x) = 0$ by using direct viscosity arguments. We have the following result:

Theorem 7 *Let $u(x)$ be the continuous value of the Tug-of-War game introduced in [23]. Then,*

i) $u(x)$ is a viscosity solution to the mixed boundary value problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases} \tag{22}$$

ii) Reciprocally, assume that Ω verifies for every $z \in \overline{\Omega}$ and every $x^* \in \Gamma_N$ $z \neq x^*$ that

$$\left\langle \frac{x^* - z}{|x^* - z|}; n(x^*) \right\rangle > 0.$$

Then, if $u(x)$ is a viscosity solution to (22)*, it coincides with the unique continuous value of the game.*

The hypothesis imposed on Ω in part ii) holds for the case which Γ_N is strictly convex. The first part of the theorem comes as a consequence of the Dynamic Programming Principle read in the viscosity sense. To prove the second part we use that the continuous value of the game enjoys comparison with quadratic functions, and this property uniquely determine the value of the game.

We have found a PDE problem, (22), which allows to find both the continuous value of the game and the AMLE of the Dirichlet data F (which is given only on a subset of the boundary) to $\overline{\Omega}$. To summarize, we point out that a complete equivalence holds, in the following sense:

Theorem 8 *It holds*

$$u \text{ is AMLE of } F \Leftrightarrow u \text{ is the value of the game} \Leftrightarrow u \text{ solves } (22).$$

The first equivalence was proved in [23] and the second one is just Theorem 7.

Another consequence of Theorem 7 is the following:

Corollary 2 *There exists a unique viscosity solution to* (22)*.*

The existence of a solution is a consequence of the existence of a continuous value for the game together with part i) in the previous theorem, while the uniqueness follows by uniqueness of the value of the game and part ii).

Note that to obtain uniqueness we have to invoke the uniqueness of the game value. It should be interesting to obtain a direct proof (using only PDE methods) of existence and uniqueness for (22) but we have not been able to find the appropriate perturbations near Γ_N to obtain uniqueness (existence follows easily by taking the limit as $p \to \infty$ in the mixed boundary value problem problem for the $p-$laplacian).

Remark 3 *Corollary 2 allows to improve the convergence result given in [13] for solutions to the Neumann problem for the $p-$laplacian as $p \to \infty$. The uniqueness of the limit holds under weaker assumptions on the data (for example, Ω strictly convex).*

6 Eigenvalue problems

We end this note by briefly describe a similar limit in an eigenvalue problem.

Eigenvalues of $-\Delta_p u = \lambda |u|^{p-2}u$ with Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, have been extensively studied since [12]. The limit as $p \to \infty$ was studied in [18].

Our last aim here is to state a result concerning the limit as $p \to \infty$ for the Steklov eigenvalue problem

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Theorem 9 *For the first eigenvalue of* (23) *we have,*

$$\lim_{p\to\infty} \lambda_{1,p}^{1/p} = \lambda_{1,\infty} = 0,$$

with eigenfunction given by $u_{1,\infty} = 1$.
For the second eigenvalue, it holds

$$\lim_{p\to\infty} \lambda_{2,p}^{1/p} = \lambda_{2,\infty} = \frac{2}{\operatorname{diam}(\Omega)}.$$

Moreover, given $u_{2,p}$ *eigenfunctions of* (23) *of* $\lambda_{2,p}$ *normalized by* $\|u_{2,p}\|_{L^\infty(\partial\Omega)} = 1$, *there exits a sequence* $p_i \to \infty$ *such that* $u_{2,p_i} \to u_{2,\infty}$, *in* $C^\alpha(\overline{\Omega})$. *The limit* $u_{2,\infty}$ *is a solution of*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ \Lambda(x,u,\nabla u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{24}$$

in the viscosity sense, where

$$\Lambda(x,u,\nabla u) \equiv \begin{cases} \min\left\{|\nabla u| - \lambda_{2,\infty}|u| \ , \ \frac{\partial u}{\partial \nu}\right\} & \text{if } u > 0, \\ \max\{\lambda_{2,\infty}|u| - |\nabla u| \ , \ \frac{\partial u}{\partial \nu}\} & \text{if } u < 0, \\ \frac{\partial u}{\partial \nu} & \text{if } u = 0. \end{cases}$$

For the k-th eigenvalue we have that if $\lambda_{k,p}$ *is the k-th variational eigenvalue of* (23) *with eigenfunction* $u_{k,p}$ *normalized by* $\|u_{k,p}\|_{L^\infty(\partial\Omega)} = 1$, *then for every sequence* $p_i \to \infty$ *there exists a subsequence such that*

$$\lim_{p_i\to\infty} \lambda_{k,p}^{1/p} = \lambda_{*,\infty}$$

and $u_{k,p_i} \to u_{*,\infty}$ *in* $C^\alpha(\overline{\Omega})$, *where* $u_{*,\infty}$ *and* $\lambda_{*,\infty}$ *is a solution of* (24).

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HYPERBOLIC MODELS IN GAS-SOLID CHROMATOGRAPHY

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Abstract

We present different models arising in chemical engineering and essentially related to isothermal gas chromatography. These models describe a fixed bed adsorption process of separation of a gaseous mixture: each compound can exist either in a mobile phase or a solid and static one, with a finite or infinite mass-exchange kinetics. Many authors, in the fields of chemical engineering and mathematics, have investigated these models under various assumptions, from a theoretical or numerical point of view. We explain first the relations between some of these approaches. Next, we present some results related to these models, some of them being new, particularly in the case of a monovariant system with one or two active compounds for the Cauchy problem. Lastly, we mention some open problems.

Key words: *gas chromatography, nonlinear chromatography, mass transfer kinetics, adsorption, systems of conservation laws*

1 Introduction

Chromatography is the collective term for a family of laboratory techniques for the separation of mixtures. It involves passing a mixture dissolved in a "mobile phase" (liquid or gaseous) through a stationary phase, which separates the analyte to be measured from other molecules in the mixture and allows it to be isolated (source: Wikipedia).
The principal methods are

Frontal Chromatography: a procedure in which the sample (liquid or gas) is fed continuously into the chromatographic bed. In frontal chromatography no additional mobile phase is used.

Displacement Chromatography: a procedure in which the mobile phase contains a compound (the Displacer) more strongly retained than the compounds of the sample under examination. The sample is fed into the system as a finite slug.

Elution Chromatography: a procedure in which the mobile phase is continuously passed through or along the chromatographic bed and the sample is fed into the system as a finite slug.

Chromatography may be preparative or analytical. Preparative chromatography seeks to separate the compounds of a mixture for further use (and is thus a form of purification). It is a process that has lately become of considerable interest in the pharmaceutical industry: only chromatography is sufficiently flexible and powerfull to satisfy the practical requirements encountered in most difficult separations of pharmaceutical intermediates. Analytical chromatography normally operates with smaller amounts of material and seeks to measure the relative proportions of analytes in a mixture. The two are not mutually exclusive.

Another method of separating chemical substances is distillation, based on differences in their volatilities in a boiling liquid mixture. It is a process used for instance in petroleum industry. A common feature both to chromatography and distillation is that the separation follows from the interaction between two phases in motion one with respect to the other. A distillation column is heated at the bottom, thus separating the mixture in a gaseous phase moving upwards, and a liquid one moving downwards by gravity. In standard chromatography the mixture, in gaseous or liquid form, is injected in a column filled with some porous medium. The chemical compounds are partially retained by the pores, thus generating stationary phase in the column (fixed bed adsorption). In the modelling of such process, two types of phenomena are to be considered. On the one hand, the propagation of the mobile phase is ruled by the laws of fluid dynamics, gas dynamics, porous media,... On the other hand, the repartition of matter between the two phases relies on thermodynamics and the notion of diphasic equilibrium is involved.

There are many reference works in the field of Chromatography. G. Guiochon and B. Lin [17], for instance, describe the different mathematical models of chromatography, examine the assumptions on which they are based, consider their properties and discuss their solutions. In [18], one can find the fundamentals of thermodynamics, mass-transfer kinetics and flow through porous media that are relevant to chromatography. The authors present the models used in chromatography, the applications, describe the different processes used and the methods of optimization of the experimental conditions.

In this article we mainly consider the case of isothermal gas-solid chromatography, a procedure in which the temperature of the column is kept constant during the process. The mixture analyzed is vaporized at the entrance of a column that contains a solid substance (the adsorber) called the stationary phase and then is transported across it by a carrier gas. The carrier gas, or vector gas (usually nitrogen, sometimes hydrogen or helium), is the mobile

phase (see Fig. 1). In most cases it has to be inerted vis a vis the solutes and the stationary phase.

The paper is organized as follows. First, we examine and discuss some models arising in gas-solid chromatography (or closely related processes). Next we give some mathematical results which seem to us significant in that field. We emphasize in particular the case of the so-called Pressure Swing Adsorpion process (PSA) in the case of two compounds, for which we give some new results by the authors. Lastly we mention some open problems.
We specify that we do not seek to be exhaustive, neither from the point of view of modelling, nor of that of the mathematical analysis. Our goal is to concentrate on some problems which appear more particularly interesting to us in terms of potential developments, pointing out various open problems.

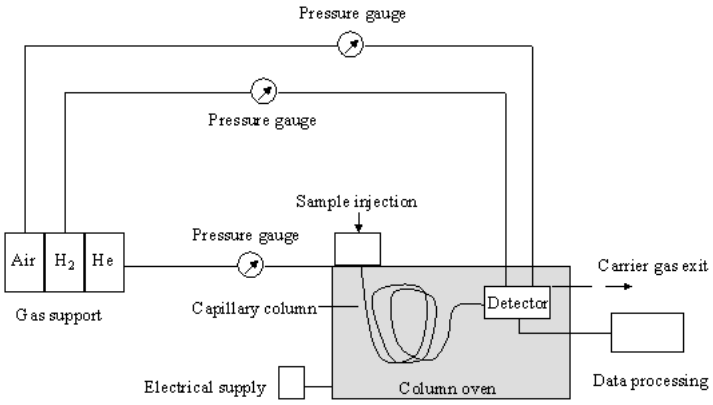

Figure 1: Construction scheme of a gas chromatograph. Source: home page of Zentrales Analytisches Labor, Brandenburgische Technische Universität Cottbus.

2 Some models in nonlinear chromatography

General references in this field are, for instance, [29, 17, 18]. One may consult [27] for a complete nomenclature for Chromatography.
In this section we choose to start with the model described by P. Rouchon and *al.* ([30]). The assumptions of the model are completely stated and some of them are used as a basis for a later discussion: finite exchange kinetics or not (Subsections 2.2 and 2.4), pressure law and sorption effect (Subsections 2.3 and

2.4). The mathematical results given in Section 3 are related to the models presented here. Some of them are new.

2.1 The model of Rouchon-Schonauer-Valentin-Guiochon

We recall the model described by P. Rouchon and *al.* ([30]) which accounts for the migration and transformation of the large concentration band of a single pure gaseous compound along a chromatography column. The main assumptions of this model are the following:

1. The column is supposed to be radially homogeneous and so is the input profile. Therefore the problem is monodimensional. The only variables are the abscissa x along the column and the time t.
2. Gases follow ideal gas laws for compressibility and mixing.
3. Darcy's law is valid in the range of flow velocity u investigated. The column permeability is constant independent of the abscissa.
4. The local pressure p is constant during an experiment, depends on the abscissa not on the time even during the passage of a large concentration band.
5. The carrier gas is not sorbed by the stationary phase.
6. Temperature T is constant during an experiment, independent of the position or the time.
7. Mass and heat energy exchanges between the mobile and the stationary phases are infinitely fast. The two phases are constantly at thermal and composition equilibrium.
8. Axial diffusion proceeds at negligible speed.

Notice that the assumption of isothermality 6 is easily justified provided that adequate time is allowed for exchange of energy with the surroundings and also for systems with little adsorption. Assumption 7 may be relaxed: it will be investigated in the next subsection. As pointed out in [30], combination of Assumptions 7 and 8 results in an infinite efficiency of the column. For a thorough discussion of the hypothesis, we refer to the book by Guiochon *et al.* [18] where a large amount of bibliography can also be found.

Let N_M^i and N_S^i, $1 \leq i \leq d$, be the number of moles of compound i per unit length of column at equilibrium, where the subscripts S and M stand for stationary and mobile phase respectively. We assume that the index d stands for the carrier gas, if present, thus $N_S^d = 0$. The equations for the conservation of mass are:

$$\partial_t(N_M^i + N_S^i) + \partial_x(uN_M^i) = 0,\ 1 \leq i \leq d. \tag{1}$$

The quantities N_M^i and N_S^i are not independent, they are related by the so-called equilibrium isotherm k^i: $RT\,N_S^i = k^i(N_M^1, \cdots, N_M^d)$ where T is the

temperature, R a positive constant. In particular we have $k^d = 0$ because the carrier gas is not sorbed.

Notice that the precise form of the isotherms is usually unknown but can be experimentally obtained. Simple examples of such a function are the linear isotherm $k^i = K_i N_M^i$, with $K_i \geq 0$, the Langmuir isotherm $k^i = \dfrac{Q_i K_i N_M^i}{1 + \sum_{j=1}^{d} K_j N_M^j}$, with $K_i \geq 0$, $Q_i > 0$ (see for instance [31]) and the BET isotherm defined by

$$k^1 = \frac{QKN_M^1}{(1 + KN_M^1 - (N_M^1/N_M^s))(1 - (N_M^1/N_M^s))}, \; Q > 0, K > 0, N_M^s > 0, k^2 = 0$$

for one adsorbable compound in an inert gas.

The unknowns are therefore the local mobile phase velocity u and the values of N_M^i for each compound i.

Introducing the local pressure p and the mole fraction $X_i = \dfrac{N_M^i}{\sum_{k=1}^{d} N_M^k}$ of each compound i, Rouchon and *al.* [30] write the equations (1) under the form:

$$\partial_x(upX_i) + \partial_t(pX_i + k^i(pX_1, ..., pX_d)) = 0, \; 1 \leq i \leq d. \tag{2}$$

Because $\sum_{i=1}^{d} X_i = 1$ and $k^d = 0$, the equation for the carrier gas may be replaced by the sum of all equations (1). This gives the total mass balance equation of the column:

$$\partial_x(u\,p) + \partial_t(p + \sum_{i=1}^{d-1} k^i(pX_1, ..., pX_d)) = 0. \tag{3}$$

The law of ideal gas writes:

$$pX_i\varepsilon^1 = N_M^i RT$$

where ε^1, the porosity, and the temperature T are assumed here to be constant. Setting

$$F_i = \frac{up}{RT}X_i = \frac{uN_M^i}{\varepsilon^1}, \; 1 \leq i \leq d, \; F_0 = \frac{up}{RT}, \; \varepsilon^2 = \sum_{i=1}^{d} N_S^i,$$

James ([19]) write these equations under the form:

$$\partial_x F_i + \partial_t(\frac{p}{RT}\frac{F_i}{F_0} + \frac{\varepsilon^2}{\varepsilon^1}\, h_i(F_1, \cdots, F_d, F_0)) = 0, \; 1 \leq i \leq d,$$

$$\partial_x F_0 + \partial_t(\frac{p}{RT} + \frac{\varepsilon^2}{\varepsilon^1}\sum_{i=1}^{d-1} h_i(F_1, \cdots, F_d, F_0)) = 0,$$

thus keeping track of the temperature: this may be useful in case of experiments with prescribed temperature as a function of the time.

The functions $h_i(F_1, \cdots, F_d, , F_0) = \dfrac{\varepsilon^1 k_i}{\varepsilon^2 RT} = \dfrac{\varepsilon^1}{\varepsilon^2} N_S^i$ are the isotherms, written in the F-variables.

In the sequel we will make use of the concentrations c_i (moles/m^3) of the i^{th} compound in the mobile phase as main unknowns (joined to u). The corresponding concentrations in the stationnary phase are denoted q_i. At equilibrium they are given by $q_i = q_i^*(c_1, ..., c_d))$ where q_i^*, $1 \leq i \leq d$ are the equilibrium isotherms corresponding to this new set of variables. Setting $\rho = \sum_{i=1}^{d} c_i$, equations (2) and (3) read

$$\partial_x(u\, c_i) + \partial_t(c_i + q_i^*(c_1, ..., c_d)) = 0,\ 1 \leq i \leq d, \tag{4}$$

$$\partial_x(u\, \rho) + \partial_t(\rho + \sum_{i=1}^{d} q_i^*(c_1, ..., c_d)) = 0. \tag{5}$$

This is our reference model in the sequel.

2.2 A model with finite exchange kinetics

A question is that, given a certain amount of mixture, there exists a privileged repartition of the matter between the two phases (the so called stable equilibrium state): the equilibrium state can be reached in a short time with respect to the relative velocities of the phases (infinite mass-transfer kinetics: Assumption 7 above) or not. When the time needed to reach the equilibrium is not negligible with respect to the characteristic times induced by the velocity, we must give up Assumption 7 and take the deviation from equilibrium into account: the actual concentration q_i in the solid phase differs from $q_i^*(c_1, \cdots, c_d)$. This phenomenon is known as a finite exchange kinetics. Finite exchange kinetics can be modelled by the following system of equations (with a suitable pressure law) for a column of length L:

$$\partial_t c_i + \partial_x(u\, c_i) = A_i\,(q_i - q_i^*(c_1, \cdots, c_d)), \tag{6}$$

$$\partial_t q_i = -A_i\,(q_i - q_i^*(c_1, \cdots, c_d)), \quad t \geq 0, \quad x \in (0, 1). \tag{7}$$

The right hand sides of the equations quantify the attraction of the system to the equilibrium state: it is a pulling back force proportional to the deviation from equilibrium. A compound with concentration c_i is said to be inert if $A_i = 0$ and $q_i^* = 0$.

When the coefficients A_i in (6)-(7) tend to infinity (instantaneous equilibrium),

say $A_i = 1/\varepsilon$ with $\varepsilon \to 0$ for instance, we get formally $q_i - q_i^* = -\dfrac{1}{A_i}\,\partial_t q_i \to 0$ and Eqs. (6)-(7) reduce to (4).
In Section 2.4 we present some theoretical results for a particular pressure law arising in the so-called "Pressure Swing Adsorption" process, dealing with a non constant velocity u.

In [21], James studied a system of semi linear transport equations, closely related to (6)-(7), modelling diphasic propagation arising in chemical engineering in which two phases are in motion with distinct **constant** speeds $u > 0$ and $v \leq 0$, covering the cases of liquid-solid chromatography and distillation. The model is the following:

$$\partial_t c_\varepsilon + \partial_x(u c_\varepsilon) = \frac{1}{\varepsilon}(q_\varepsilon - h(c_\varepsilon)), \tag{8}$$

$$\partial_t q_\varepsilon + \partial_x(v q_\varepsilon) = -\frac{1}{\varepsilon}(q_\varepsilon - h(c_\varepsilon)), \tag{9}$$

where $\varepsilon > 0$. The unknowns are the concentrations vectors c_ε, $q_\varepsilon \in \mathbb{R}^d$.
The right hand side, which has the same interpretation as in Eqs. (6)-(7), is written in an academic form, of course not standard in the chemical engineering literature. Note that this set of equations can be used at two levels: on the one hand, specific phenomena due to slow exchange kinetics are related to large values of ε, on the other hand, we can let ε go to zero as above.

2.3 Velocity and pressure law

In gas chromatography, velocity variations accompany changes in gas composition, especially in the case of high concentration solute: it is known as the sorption effect. To neglect this effect or not leads of course to models with very different mathematical properties. The sorption effect is of major importance in gas chromatography but often close to being insignificant in liquid-solid chromatography or distillation, which is the context of the model (8)-(9), for instance, where the velocities are kept constant.

2.3.1 Neglecting the sorption effect

According to Assumption 3 we write $u = -C\,\partial_x p$ where C is a constant depending on the porosity. Next, Assumption 4 gives $\partial_t p = 0$. Neglecting the sorption effect in first approximation, we assume that the total flow rate is constant i.e. $u\,p = \text{cste}$. Thus $p\,\partial_x p$ is a constant and we obtain immediately

$$p(x) = \sqrt{P_{in}^2 - \frac{x}{L}(P_{in}^2 - P_{out}^2)}.$$

In this expression P_{in} is the inlet pressure ratio, P_{out} is the outlet pressure ratio and L is the column length. For very small variations of pressure between inlet and outlet, we can therefore assume a constant velocity.

As pointed out by James ([21]), the assumption of constant total flow rate is not really physically relevant but the derivation of the pressure law may be viewed as an independent reasoning and others models are of course admissible.

2.3.2 Taking the sorption effect into account

Assume that the pressure is given by $p = k\rho^\gamma,\ k > 0,\ \gamma > 0$ (pressure law for a polytropic ideal gas) and that the speed follows Darcy's law: we have then

$$u = -K\,\partial_x \rho^\gamma,\ \text{with } K > 0. \tag{10}$$

To our knowledge the problem (6)-(7)-(10) has never been investigated from a mathematical point of view. As a first approach we propose an existence result for two simplified models: see Theorem 5 and Remark 2 in Subsection 3.2.

Notice that setting $K = \dfrac{1}{\varepsilon}$ in (10) we get formally, when ε tends to zero, $\partial_x \rho^\gamma = 0$, that is $\rho = \rho(t)$. This means, in the isothermal case, that the total pressure is only time dependant (which dramatically differs from Assumption 4): the velocity $u(t,x)$ of the mixture has to be found in order to achieve a given pressure. This is exactly the context of the model developed in the next section.

2.4 Pressure Swing Adsorption (PSA)

2.4.1 Introduction

"Pressure Swing Adsorption (PSA) is a technology that is used to separate some species from a gas under pressure according to these species' molecular characteristics and affinity for an adsorbent material. It operates at near-ambient temperatures and so differs from cryogenic distillation techniques of gas separation. Special adsorptive materials (e.g., zeolites) are used as a molecular sieve, preferentially adsorbing the undesired gases at high pressure. The process then swings to low pressure to desorb the adsorbent material. Using two adsorbent vessels allows near-continuous production of the target gas. It also permits so-called pressure equalization, where the gas leaving the vessel being depressurized is used to partially pressurize the second vessel. This results in significant energy savings, and is common industrial practice." (Wikipedia)
PSA is used extensively in the production and purification of oxygen, nitrogen and hydrogen for industrial uses. PSA can be used to separate a single gas from a mixture of gases. A typical PSA system involves a cyclic process where a number of connected vessels containing adsorbent material undergo successive pressurization and depressurization steps in order to produce a continuous stream of purified product gas (see Fig. 2).

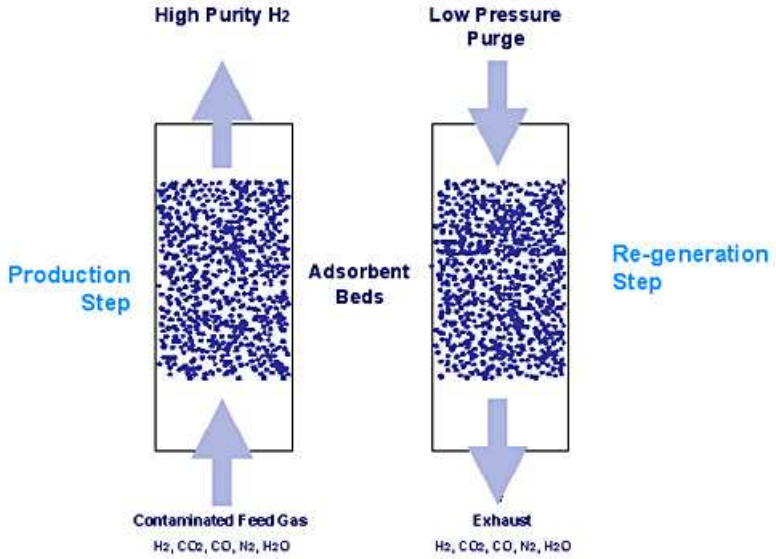

Figure 2: PSA process: production and regeneration steps. Source: questairinc.com

2.4.2 Modelling

Our purpose is to model a step of the cyclic process, restricted to isothermal behavior. As in general fixed bed chromatography, each of the d species ($d \geq 2$) simultaneously exists under two phases, a gaseous and movable one with concentration $c_i(t,x)$ or a solid (adsorbed) other with concentration $q_i(t,x)$, $1 \leq i \leq d$. The sorption effect is taken into account through a constraint on the pressure (see the end of 2.3.2). Following Ruthwen (see [31]) we can describe the evolution of u, c_i, q_i according to the system (6)-(7) with suitable initial and boundary data:

$$c_i(0,x) = c_i^0(x), \quad q_i(0,x) = q_i^0(x) \text{ in }]0,1[, \tag{11}$$

$$c_i(t,0) = c_i^{in}(t), \quad u(t,0) = u_0(t) > 0, \tag{12}$$

$$c_i(t,1) = c_i^{out}(t) \text{ if } u(t,1) < 0. \tag{13}$$

In (6)-(7) the velocity $u(t,x)$ of the mixture has to be found in order to achieve a given pressure (or density in this isothermal model):

$$\sum_{i=1}^{d} c_i = \rho(t), \tag{14}$$

where ρ represents the *given* total density of the mixture. The experimental device is realized so that it is a given function depending only upon time, which

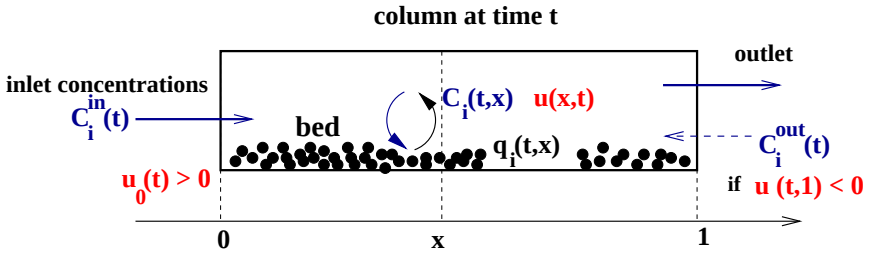

Figure 3: Unknown and boundary data in a step of the PSA process.

differs drastically from Assumption 4 (see Subsection 2.1). Adding (6) for $i = 1$ to d, we get, thanks to (14):

$$\partial_x u = -\frac{\partial_t \rho}{\rho} + \frac{1}{\rho} \sum_{i=1}^{d} A_i \, (q_i - q_i^*)(c_1, \cdots, c_d), \tag{15}$$

which gives an integral dependency of u upon the concentrations.

3 Some mathematical results

In this section we investigate some of the preceding models from a theoretical or numerical point of view. We will focus in particular on the PSA system with finite or infinite exchange kinetics and give some recent results by the authors.

3.1 On a model with constant velocity

We present here some results due to James [21] for System (8)-(9). Some other results in a closely related case are given in Subsection 3.5.1 via the kinetic approach. In so far as the set of problems are connected (the case $v = 0$ corresponds to chromatography), it may be interesting to give an account of the main results in this direction.
First, as pointed out by James, the system admits a natural set of "diphasic entropies" under the form $\eta(c) = \eta_1(c) + \eta_2(h(c))$ where η_1, $\eta_2 : \mathbb{R}^d \to \mathbb{R}$ are two convex functions satisfying $\nabla_c \eta_1(c) = \nabla_c \eta_2(h(c))$ for all $c \in \mathbb{R}^d$.
Next, a quite natural set of boundary conditions is the following, as far as chromatography and distillation are concerned:

$$c_\varepsilon(0,t) = a(t) \in L^\infty(]0,+\infty[) \text{ (injection)}, \tag{16}$$

$$uc_\varepsilon(1,t) + vq_\varepsilon(1,t) = b(t) \in L^\infty(]0,+\infty[) \text{ (reflux) if } v < 0. \tag{17}$$

If $v = 0$ (case of chromatography), the reflux boundary condition simply disappears.

From the point of view of distillation, the boundary conditions are natural: the first one is a Dirichlet-like "injection" at one end of a column and only acts on the incoming variable ($u > 0$); the second one looks like a Neumann

condition on the other hand and imposes $v < 0$ (it is a simplified model of "reflux" in a distillation column).

It turns out that System (8)-(9) with (16)-(17) is well posed. Applying the fixed point theorem, there is an existence and uniqueness result for (8)-(9) in $L^\infty(]0,T[;L^1(]0,1[)^{2d})$:

Theorem 1
For a given $T > 0$, assume that a and b are in $L^\infty(]0,T[)$, $c^0 \in L^\infty \cap L^1(]0,1[)$ and that the function h is of class $\mathcal{C}^1$. Then there exists an unique solution to (8)-(9) which lies in $L^\infty(]0,T[;L^1(]0,1[))$.

When ε tends to zero we get formally a set of equations which express the conservation of matter:

$$\partial_t(c + h(c)) + \partial_x(u\,c + v\,h(c)) = 0. \tag{18}$$

However a difficulty arises if one lets ε go to 0 because the boundary conditions are not at equilibrium for $\varepsilon > 0$ so that boundary layers may appear.

James imposes the condition

$$f(c) = u\,c + v\,h(c) \leq \min_{t>0} b(t).$$

This condition implies some restrictions on the initial and boundary data, which lead to uniform L^∞ estimates for the solution to (8)-(9) for a broader class of fluxes.

In the scalar case and using compensated compactness, James proves that the solution of this system converges, as $\varepsilon \to 0$, to a solution of the preceding equation satisfying a set of entropy inequalities:

Theorem 2
Let be $T > 0$, a, $b \in L^\infty(]0,T[)$, $a \geq 0$, $b \leq 0$, $c^0 \in L^1(]0,1[) \cap L^\infty(]0,1[)$, $c^0 \geq 0$ and

$$c^* = \sup\{c \geq 0, \exists c' \leq c, f(c') \leq \min b(t)\} \geq \max[||\ a\ ||_\infty, ||\ c^0\ ||_\infty].$$

Let $c_\varepsilon, q_\varepsilon$ be a solution of (8)-(9) with initial data at equilibrium:

$$c_\varepsilon(x,0) = c^0(x) \in L^1(]0,1[) \cap L^\infty(]0,1[), \quad q_\varepsilon(x,0) = h(c^0(x)).$$

Then there exists a subsequence of solutions which converges a.e. and strongly in $]0,1[\times]0,+\infty[$ to $c \in L^\infty(]0,T[;L^1(]0,1[)$ satisfying for any $\phi \in \mathcal{D}'_+([0,1] \times \mathbb{R}_+)$, $k \in \mathbb{R}$:

$$\int_0^\infty \int_0^1 [(|c-k| + |h(c) - h(k)|)\,\partial_t\phi + (u|c-k| + v|h(c) - h(k)|)\,\partial_x\phi]\ dx\,dt$$

$$\leq \int_0^\infty (u\ |\ a(t) - k\ |\ \phi(0,t) + |\ b(t) - f(k)\ |\ \phi(1,t))\ dt -$$

$$\int_0^1 \left(|\ c^0(x) - k\ | + |\ h(c^0(x)) - h(k)\ |\right)\ \phi(x,0)\,dx.$$

This result is meaningful for $c^* > 0$, which occurs only if $f(c)$ becomes non positive for some c: notice that this excludes the case of the chromatography ($v = 0$).

In [23], James and *al.* study numerically the same model to take in account the finite exchange kinetics. The resulting hyperbolic system with a non linear relaxation term is then formally treated with a Chapman-Enskog type expansion. A first order correction to the classical quasilinear hyperbolic model is derived which consists in a nonlinear diffusion term. Numerical schemes for both models, relaxed and parabolic, are then tested and compared for different initial and boundary values.

Lastly, in [22], the authors describe and validate a numerical solution of the inverse problem of nonlinear chromatography using the model given by Eq. (18). The method allows the determination of best numerical estimates of the coefficients of an isotherm model from the individual elution profiles of the two compounds of a binary mixture. In two cases, when the isotherm model is satisfactory, the authors observed a very good agreement between the equilibrium isotherm equations obtained by this new method and those determined by the classical combination of elution by characteristic points and binary frontal analysis. Practically, this method would significantly reduce the amounts of products required to determine a set of competitive binary isotherms.

3.2 On a model with Darcy's velocity

As a first approach to a study of the complete model (6)-(7)-(10) we propose an existence result for the following simplified one:

$$\partial_t c_i + \partial_x(u\, c_i) = 0, \quad \text{in } (0,T)\times\mathbb{R}, \tag{19}$$

$$u = -\partial_x \rho^\gamma, \quad \gamma > 0,\ \rho = \sum_{i=1}^{d} c_i, \tag{20}$$

$$c_i(0,x) = c_i^0(x), \quad x \in \mathbb{R}. \tag{21}$$

Notice that we work on $\mathbb{R}$ in order to focus on the main difficulties, but we assume that the initial data c_i^0 are compactly supported. Furthermore we assume $A_i = 0$ to avoid some problems related to the nonlinearities q_i^*. Adding (19) for $i = 1$ to d we get, thanks to (10), $\partial_t \rho - \partial_x(\rho\, \partial_x \rho^\gamma) = 0$, that we write under the form:

$$\partial_t \rho - \frac{\gamma}{\gamma+1} \Delta \rho^{\gamma+1} = 0$$

which is a porum medium equation. There exists a large amount of works on this subject. The first results are due to Oleinik, Kalashnikov and Chzhou [28]. For regularity results on ρ consult for instance Aronson [1, 2, 3], Aronson-Benilan [4], Caffarelli-Friedmann [5], Dibenedetto [14]. Concerning the domain of dependency $\mathcal{P}[\rho] = \{\rho > 0\}$ in one dimension we know, under suitable

assumptions, that if the initial data ρ^0 is supported in a bounded interval, then ρ is compactly supported for all time and $\mathcal{P}[\rho]$ is enclosed between two monotonic and lipschitz continuous curves $x = \xi_i(t)$ (see [24, 34] for instance). More precisely we will use explicitly the following results:

consider the problem

$$\partial_t \rho = \Delta \rho^m \quad \text{in} \quad S = (0,T] \times \mathbb{R}, \quad \rho(0,x) = \rho^0(x),\ x \in \mathbb{R} \tag{22}$$

where $m > 1$. Following [1] we say that a function $\rho(t,x)$ is a weak solution of (22) if

- ρ^0 is non negative, continuous, bounded in $\overline{S}$,
- $\partial_x \rho^m \in L^\infty$,
- ρ satisfies:

$$\forall \phi \in \mathcal{D}([0,T) \times \mathbb{R}) \quad \int_S (\partial_x \rho^m\, \partial_x \phi - \rho\, \partial_t \phi)\, dx\, dt = \int_{\mathbb{R}} \rho^0(x)\, \phi(0,x)\, dx.$$

Proposition 3 *If $(\rho^0)^m$ is lipschitz continuous, the problem (22) has a unique weak solution ρ. Moreover ρ is a classical solution on $\mathcal{P}[\rho] = \{\rho > 0\}$.*

Remark 1 *The L^∞ bound for $\partial_x \rho^m$ depends only upon the lipschitz constant of $(\rho^0)^m$.*

Proposition 4 *Assume that $(\rho^0)^m$ is lipschitz continuous and let ρ be the weak solution of (22). For all $\tau > 0$ there exists a positive constant $C = C(m, \tau, \|\rho^0\|_\infty)$ such that*

$$\forall (t,x),\ (t,x') \in [\tau,T] \times \mathbb{R}, \quad \left|\rho^{m-1}(t,x) - \rho^{m-1}(t,x')\right| \le C\ |x - x'|\,.$$

If $(\rho^0)^{m-1}$ is lipschitz continuous the same conclusion holds for all (t,x), $(t,x') \in [0,T] \times \mathbb{R}$ and C depends on the lipschitz constant of $(\rho^0)^{m-1}$ instead of τ.

We are looking a *priori* for a weak solution with $c_i \in L^\infty$ and we will get, depending on the assumption on ρ^0, $u \in L^\infty((\tau,T) \times \mathbb{R})$ for all $\tau > 0$ or $u \in L^\infty((0,T) \times \mathbb{R})$ which does not allow to pass to the limit in a sequence of smooth approximate solutions in the weak formulation of (19). However, if ρ is smooth, we have $u\, c_i = -\dfrac{\gamma}{\gamma+1}\, \dfrac{c_i}{\rho}\, \partial_x \rho^{\gamma+1}$ with $\dfrac{c_i}{\rho} \in L^\infty$ and we will get $\partial_x \rho^{\gamma+1} \in L^\infty$ and a BV estimate for $\dfrac{c_i}{\rho}$. Thus we choose to formulate the problem as follows:

$$\partial_t c_i - \frac{\gamma}{\gamma+1}\, \partial_x \Big(\frac{c_i}{\rho}\, \partial_x \rho^{\gamma+1}\Big) \;=\; 0 \ \text{in}\ (0,T) \times \mathbb{R},\ 1 \le i \le d, \tag{23}$$

$$\text{with} \quad \gamma > 0,\ \rho = \sum_{i=1}^{d} c_i. \tag{24}$$

For mere technical reasons, the initial data $c_i^0(x)$ are written under the form $c_i^0(x) = \rho^0(x)\, a_i(x)$ where the functions a_i are defined on $\mathbb{R}$, non negative, and satisfy $\sum_{i=1}^{d} a_i = 1$. We make the following assumptions:

$$\forall i \in \{1, \cdots, d\} \quad a_i \text{ has bounded variation on } \mathbb{R}, \tag{25}$$

$$\rho^0 \text{ is lipschitz continuous, non negative and compactly supported,} \tag{26}$$

$$\exists C > 0, \quad \forall \eta > 0 \quad \int_{\mathbb{R}} \left| \partial_x \left(\frac{1}{\rho^0 + \eta} \right) \right| \leq C/\eta. \tag{27}$$

The technical assumption (27) is related to the the behavior of ρ^0 near the point where this function vanishes. It is satisfied for instance if ρ^0 is lipschitz continuous, supported in some interval $[a, b]$, positive in $]a, b[$, monotone in the neighborhood of a and b.
We state now our main result:

Theorem 5 *Under the assumptions (25)-(26)-(27), Problem (23)-(24) has at least a solution* $(c_1, ..., c_d)$ *with* $c_i \in L^\infty$, $c_i \geq 0$, $\rho = \sum_{i=1}^{d} c_i$ *continuous on* $\mathbb{R}^+ \times \mathbb{R}$ *and such that:*

$$\forall T > 0,\ \partial_x \rho^{\gamma+1} \in L^\infty((0,T) \times \mathbb{R}),\ \frac{c_i}{\rho} \in L^\infty((0,T); BV(\mathbb{R})),$$

$$\forall T > 0, \forall \tau > 0, u = -\partial_x \rho^\gamma \in L^\infty((\tau, T) \times \mathbb{R}).$$

Moreover, if $(\rho^0)^\gamma$ *is lipschitz continuous then* $u \in L^\infty((0,T) \times \mathbb{R})$.

Outlines of the proof- The estimates on u are straightforward consequences of the general properties of the solution of the porous medium equation: consult for instance Aronson [1].
Assume the initial data in $\mathcal{C}^2(\mathbb{R})$ (thanks to a regularization step that we skip), let be $\eta > 0$, $c_{i,\eta}^0 = c_i^0 + \eta/d$, $1 \leq i \leq d$ and $\rho_\eta^0 = \rho^0 + \eta = \sum_{1 \leq i \leq d} c_{i,\eta}^0$. Let ρ_η be the unique solution of the following problem $(\mathcal{P}_\eta)$:

$$\partial_t \rho_\eta - \frac{\gamma}{\gamma+1} \Delta \rho_\eta^{\gamma+1} = 0, \tag{28}$$

$$\rho_\eta(0, x) = \rho_\eta^0. \tag{29}$$

Notice that ρ_η satisfies:

$$\partial_t \rho_\eta + \partial_x (u_\eta\, \rho_\eta) = 0, \tag{30}$$

with $u_\eta = -\partial_x(\rho_\eta^\gamma)$ which is Lipschitz continuous on $\mathbb{R}$, uniformly with respect to $t \in [0, T]$. Let $c_{i,\eta}$, $1 \leq i \leq d$, be the solution of the advection equation

$$\partial_t c_{i,\eta} + \partial_x(u_\eta\, c_{i,\eta}) = 0 \tag{31}$$
$$c_{i,\eta}(0,x) = c^0_{i,\eta}, \tag{32}$$

then we have clearly $\rho_\eta = \sum_{1\le i\le d} c_{i,\eta}$.
Notice that by comparison principle we have

$$\eta \le \rho_\eta \le \| \rho^0 \|_\infty + \eta \quad \text{in} \quad [0,T]\times\mathbb{R}$$

and also

$$0 < \eta_1 \le \eta_2 \Longrightarrow \rho_{\eta_1} \le \rho_{\eta_2} \quad \text{in} \quad [0,T]\times\mathbb{R}.$$

Thanks to standard arguments we can then assume that (up to a subsequence):

$$\rho_\eta \to \rho \text{ in } L^p_{loc} \quad \forall p \ge 1, \qquad u_\eta \rightharpoonup u = -\partial_x \rho^\gamma \text{ in } \mathcal{D}', \qquad c_{i,\eta} \rightharpoonup c_i \text{ in } L^\infty \, weak*$$
(33)

but it is not enough to pass to the limit in the weak formulation of $(\mathcal{P}_\eta)$. More precisely, according to (23), we have to show that for all $\phi \in \mathcal{D}'([0,T]\times\mathbb{R})$:

$$\int_0^T \int_{\mathbb{R}} u_\eta\, c_{i,\eta}\, \partial_x\phi \xrightarrow[\eta\to 0]{} \int_0^T \int_{\mathbb{R}} -\frac{\gamma}{\gamma+1}\,\frac{c_i}{\rho}\,\partial_x \rho^{\gamma+1}\,\partial_x\phi. \tag{34}$$

Let be $\alpha > 0$ fixed. The left hand side of (34) is written as:

$$\int_0^T \int_{\mathbb{R}} u_\eta\, c_{i,\eta}\, \partial_x\phi = I_\eta + J_\eta$$

with $I_\eta = \int_0^T \int_{\{\rho<\alpha\}} u_\eta\, c_{i,\eta}\, \partial_x\phi$ and $J_\eta = \int_0^T \int_{\{\rho\ge\alpha\}} u_\eta\, c_{i,\eta}\, \partial_x\phi$.

The integral I_η: we have $u_\eta\, c_{i,\eta} = -(\partial_x \rho_\eta^{\gamma+1})\, c_{i,\eta} = -\dfrac{\gamma}{\gamma+\frac12}\,\rho_\eta^{\frac12}\,\partial_x\left(\rho_\eta\right)^{\gamma+\frac12}\,\dfrac{c_{i,\eta}}{\rho_\eta}$

and thus

$$\left| u_\eta\, c_{i,\eta} \right| \le \rho_\eta^{\frac12} \left| \partial_x \left(\rho_\eta\right)^{\gamma+\frac12} \right|. \tag{35}$$

Using Assumption (26) and Eq. (28) multiplied by ρ_η^γ, we get the following lemma:

Lemma 6

$$\exists C>0,\ \forall \eta>0,\ \forall (a,b)\in\mathbb{R}^2,\ a\ge b,\ \forall T>0\ \int_0^T\int_a^b \left|\partial_x\left(\rho_\eta\right)^{\gamma+\frac12}\right|^2 dx\,dt \le C.$$

Notice that we use here that there exists a uniform L^∞ bound for $\partial_x\left(\rho_\eta\right)^{\gamma+1}$ as a solution of (28): see the remark following Prop. 3. Using (35), this last lemma and the Hölder inequality, we finally get

$$\exists C>0, \quad \limsup I_\eta \le C\sqrt{\alpha} \tag{36}$$

The integral J_η: we write $u_\eta\, c_{i,\eta} = -\dfrac{\gamma}{\gamma+1}\,\partial_x\left(\rho_\eta\right)^{\gamma+1}\,\dfrac{c_{i,\eta}}{\rho_\eta}$. Thanks to the uniform L^∞ bound for $\partial_x\left(\rho_\eta\right)^{\gamma+1}$ and (33) we can assume that

$$\partial_x\left(\rho_\eta\right)^{\gamma+1} \rightharpoonup \partial_x\rho^{\gamma+1}\, weak*$$

and it remains to study the sequence $\left(\dfrac{c_{i,\eta}}{\rho_\eta}\right)_{\eta>0}$. From (30)-(31) we deduce that

$$\partial_t\left(\frac{c_{i,\eta}}{\rho_\eta}\right) + u_\eta\,\partial_x\left(\frac{c_{i,\eta}}{\rho_\eta}\right) = 0 \tag{37}$$

and we get classically (recall that $c_{i,\eta}/\rho_\eta$ is compactly supported):

$$\partial_t \int_{\mathbb{R}} \left|\partial_x\left(\frac{c_{i,\eta}}{\rho_\eta}\right)\right| dx = 0.$$

Next, we have:

$$\partial_x\left(\frac{c_i^0+\eta/d}{\rho^0+\eta}\right) = (\partial_x a_i)\left(\frac{\rho^0}{\rho^0+\eta}\right)^2 + \frac{\eta\rho^0}{(\rho^0+\eta)^2}\,\partial_x a_i + (a_i - \frac{1}{d})\,\frac{\eta}{(\rho^0+\eta)^2}\,\partial_x\rho^0$$

and thus, thanks to Assumptions (25)-(26)-(27), we get the uniform bound:

$$\int_0^T \int_{\mathbb{R}} \left|\partial_x\left(\frac{c_{i,\eta}}{\rho_\eta}\right)\right| dx \le C \tag{38}$$

for some positive constant C. Recall that u_η is uniformly bounded in $L^\infty((\tau,T)\times\mathbb{R})$ for all $\tau>0$. Thus, from (37) and (38) we deduce

$$\int_{\{\rho\ge\alpha\}\cap\{\tau\le t\le T\}} \left|\partial_t\left(\frac{c_{i,\eta}}{\rho_\eta}\right)\right| \le C(\tau,\alpha). \tag{39}$$

From (38) and (39) we deduce that the sequence $\left(\dfrac{c_{i,\eta}}{\rho_\eta}\right)_{\eta>0}$ is uniformly bounded with respect to η in $BV(\{\rho\ge\alpha\}\cap\{\tau\le t\le T\})$, and the following lemma holds:

Lemma 7 *The sequence* $\left(\dfrac{c_{i,\eta}}{\rho_\eta}\right)_{\eta>0}$ *is relatively compact in* $L^p_{loc}(\{\rho\ge\alpha\}\times\{t\ge\beta\})$ *for all* $p\ge 1$, $\alpha>0$ *and* $\beta\in]0,T[$.

Then, using lemma 7 and a diagonal extraction process we deduce that there exists a sequence $(\alpha_n,\tau_n)\in\mathbb{R}_+\times\mathbb{R}_+$ such that α_n, $\tau_n\to 0$ and

$$\limsup J_\eta = -\frac{\gamma}{\gamma+1}\int_{\{\rho\ge\alpha_n\}\cap\{t>\tau_n\}} \frac{c_i}{\rho}\,\partial_x\rho^{\gamma+1}\,\partial_x\phi + R_n \tag{40}$$

with $|R_n|\le C\,\tau_n$.
Thanks to (36) and (40) we get easily (34), and thus Theorem 5 holds. □

Notice that the method used for the proof of Lemma 7 is no longer valid in more than one dimension.

Remark 2 *Dealing with a single compound and working in* $\mathbb{R}^n$*, the problem (6)-(7) with* u *given by (10) writes :*

$$\begin{aligned} \partial_t \rho - \frac{\gamma}{\gamma+1}\,\Delta \rho^{\gamma+1} &= A\,(q - q^*(\rho)), \\ \partial_t q &= -A\,(q - q^*(\rho)), \end{aligned}$$

with initial data ρ_0, q_0*. For this problem, Bourdarias ([7]) obtained an existence result with continuous bounded solutions. The solution is easily obtained via a monotonicity argument. The method cannot be extended to the case of many compounds, which is an open problem.*

3.3 PSA with finite exchange kinetics

In [6], both BV and L^∞ theory are developed for the isothermal model (6)-(7)-(14) and the main results are summarized below. Let us point out that one of the mathematical interests of the model is its analogies and differences compared to various other classical equations of physics or chemistry. First, this model shares a similar structure with conservation laws under the form

$$\partial_t \rho + \partial_x(\rho\, u(\rho)) = 0, \quad \partial_x u(\rho) = F(\rho)$$

where $u(\rho)$ has an integral dependency upon ρ , while in scalar conservation laws u depends upon ρ. It is underlined that, due to this particular dependency, oscillations can propagate thus differing from Burger's example (see Lions-Perthame-Tadmor[25]), but we will not insist on this aspect.

3.3.1 Theoretical results in BV or L^∞ framework

As a first result, we give an existence result for a solution with concentrations having bounded variation and a lipschitz continuous velocity.

Theorem 8 *Assume that the initial and boundary data are at equilibrium, that the initial data have bounded variation in* $(0,1)$ *and that the boundary data, the isotherms and the total density are lipschitz continuous. Then the problem (6)-(7)-(14) has an unique solution* $((c_i),(q_i),u)$ *such that* $c_i \geq 0, q_i \geq 0$ *and*

$$\forall T>0,\ c_i, q_i \in L^\infty((0,T)\times(0,1)) \cap L^\infty(0,T;BV(0,1)),$$

$$\partial_x u \in L^\infty((0,T)\times(0,1)) \cap L^1(0,T;BV(0,1)).$$

Outlines of the proof- the proof is performed in two steps. First, for a fixed velocity u uniformly Lipschitz continuous in $(0,1)$ we get a solution $((c_i),(q_i))$ of (6)-(7) via a fixed point procedure in $L^1((0,T)\times(0,1))^{2d}$. This solution satisfies some L^∞ estimates. Moreover, the fundamental BV estimate holds:

$$\forall T>0,\ \exists C>0,\ \forall t\leq T,\ \forall i, \quad TV(c_i(t,\cdot))+TV(q_i(t,\cdot)) \leq C\,(1+\int_0^t\int_0^1 |\partial^2_{xx} u|), \tag{41}$$

where the constant C depends on T, the L^∞ norms of c_i, q_i, u, $\partial_x u$ and the boundary data, and on the total variation of the initial data. Next, thanks to (41) and the L^∞ estimates we can built a fixed point procedure to define the velocity u. □

In the L^∞ framework, which is the more natural for this problem, there is an existence result but uniqueness is still an open problem:

Theorem 9
Assume that the initial and boundary data are at equilibrium, that initial data are in L^∞, that the boundary data have lipschitz regularity and $\partial^2_{tt}\rho \in L^\infty(0,T)$. Then the problem (6)-(7)-(14) has at least a solution $c_i, q_i \in L^\infty_{t,x}$, $u \in L^\infty_t(W^{1,\infty}_x)$ for all $T > 0$.

Outlines of the proof- the initial data are approximated in L^1 by a sequence of data in $BV(0,1)$, bounded in L^∞. Thanks to a compactness property in L^1 and some regularization lemma due to DiPerna-Lions ([15]) it can be shown that the corresponding solutions (given by Theorem 8) converge in L^1 towards a solution of the problem (see [6] for details). □

Remark 3 *It is possible to deal without the assumption of equilibrium for the initial and boundary data. This is illustrated in Fig. 4 where the initial data are zero for the solid phases, and in Fig. 5 where the boundary data c_i^{in} are not at equilibrium: in this last case, a boundary layer occurs. It may be shown, using a classical analysis, that the limits of the solutions $c_i(t,x)$ when x goes to zero are $\widetilde{c_i^{in}}(t) = \dfrac{\rho\, c_i^{in}(t)}{\sum_i c_i^{in}(t)}$ and this is observed at the numerical level, via the numerical scheme presented below.*

3.3.2 Numerical approximation

A numerical analysis of this problem was performed in [8]. For the time discretization, using an operator splitting, an intermediate spatial regularization step is introduced in order to obtain some BV estimates for the approximate solutions (with BV initial data). With respect to the fully discrete finite volume scheme, it appears that it is possible to get such estimates without any regularization step because of the dissipative effect of the upwind scheme used to treat the transport part of the equations. The scheme is built as follows: we use a uniform spatial mesh $m_j =]x_{j-1/2}, x_{j+1/2}[$, $j = 1, \cdots, N$, with center x_j and size Δx and a time step Δt. The initial data are assumed to be piecewise constant over cells m_j. Assume that the discrete unknown $(c_i)_j^n$ and $(q_i)_j^n$ corresponding to the meshes m_j at a given time $t_n = n\,\Delta t$ are already computed, then:

First step: adsorption. We solve on a time step $[t_n = n, t_{n+1}]$ the system of ODEs

$$\partial_t c_i = A_i\,(q_i - q_i^*(c_1, \cdots, c_d)), \tag{42}$$
$$\partial_t q_i = -A_i\,(q_i - q_i^*(c_1, \cdots, c_d)), \quad t_n \le t \le t_{n+1}, \tag{43}$$

for $x = x_j$, $j = 1, \cdots, N$, with $(c_i)_j^n$ and $(q_i)_j^n$ as initial data. We get the updated values $(q_i)_j^{n+1}$, intermediate values $(c_i)_j^{n+1/2}$ and we set $\rho_j^{n+1} = \sum_j (c_i)_j^{n+1/2}$. We assume that we proceed with an arbitrary accuracy, in such a way that $(c_i)_j^{n+1/2}$, $(q_i)_j^{n+1/2} \ge 0$ and $0 < \alpha \le \rho_j^{n+1} \le \beta$ where α and β depend only upon the data of the problem (following the properties of the exact solution).

Second step: transport. The transport part is discretized with a classical upwind finite volume scheme as

$$(c_i)_j^{n+1} = (c_i)_j^{n+1/2} - \lambda \left(u_{j+1/2}^n (c_i)_{j+1/2}^{n+1/2} - u_{j-1/2}^n (c_i)_{j-1/2}^{n+1/2} \right) \tag{44}$$

with $\lambda = \Delta t / \Delta x$ and

$$(c_i)_{j+1/2}^{n+1/2} = \begin{cases} (c_i)_j^{n+1/2} & \text{if} \quad u_{j+1/2}^n \ge 0, \\ u_{j+1/2}^n & \text{if} \quad u_{j+1/2}^n < 0, \end{cases}$$

where $u_{j+1/2}^n$ has to be defined in such a way that the constraint (14) holds. It is easy to show that this is realized through the formula

$$u_{j+1/2}^n = \frac{\rho_j^{n+1} - \rho(t_n) + \lambda\, u_{j-1/2}^n \sum_j (c_i)_{j-1/2}^{n+1/2}}{\lambda \sum_j (c_i)_{j+1/2}^{n+1/2}}$$

which allows to compute all the discrete velocities used in (44), starting from the given velocity $u_{-1/2}^n = u_0(t_n)$. The positivity of the updated concentrations $(c_i)_j^{n+1}$ is ensured thanks to the CFL condition $\Delta t \le \Delta x / \parallel u \parallel_\infty$.

This scheme is convergent and an $O(\Delta t)^{1/2}$ convergent rate is obtained. Notice that it may be extended to second order, adapting Van-Leer's method. As a conclusion we present two numerical experiments illustrating Remark 3. The results have been obtained with Langmuir isotherms. We do not precise the values of all the parameters involved, because we just emphasize the qualitative aspect.

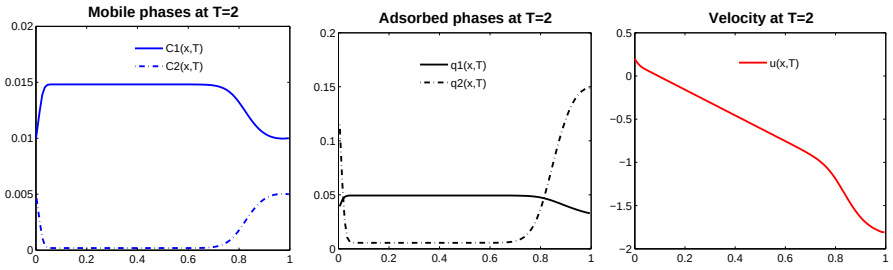

Figure 4: A case of outlet reflux due to a rapid adsorption of one compound. The initial values are not at equilibrium because $c_1^0 = 0.01$, $c_2^0 = 0.005$ whereas $q_1^0 = q_2^0 = 0$ (the bed is supposed to be purged) and the mass-transfer kinetics is greater for the second compound ($A_2 >> A_1$). This case show the relevance of the outlet boundary condition (13): it may actually occur that the velocity changes its sign.

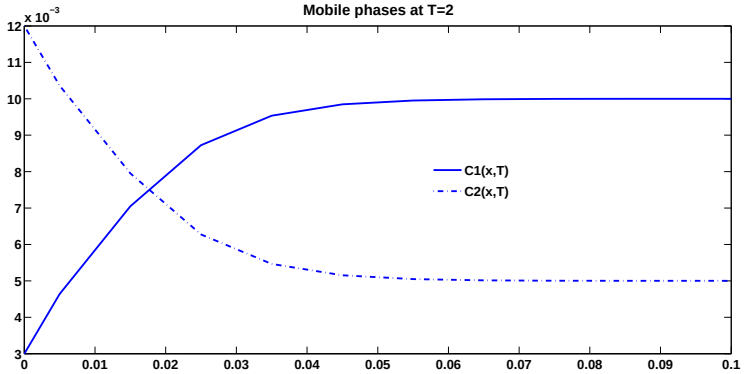

Figure 5: Boundary layer phenomenon: the boundary data are not at equilibrium (here $c_1^{in} = 0.001$, $c_2^{in} = 0.004$, $\rho = 0.015$) and the computed solutions take near $x = 0$ the theoretical values given in Rem. 3, that is here $\widetilde{c_1^{in}} = 0.003$ and $\widetilde{c_2^{in}} = 0.012$.

3.4 Some results on the PSA system with infinite exchange kinetics

When the coefficients A_i in (6)-(7) tend to infinity (instantaneous equilibrium), say $A_i = 1/\varepsilon$ with $\varepsilon \to 0$ for instance, we get formally, as in the model studied by James (see Subsection 3.1)

$$q_i - q_i^* = -\frac{1}{A_i}\,\partial_t q_i \to 0$$

and Equations (6)-(7) reduce to

$$\partial_t(c_i + q_i^*(c_1, \cdots, c_d)) + \partial_x(u\,c_i) = 0, \quad 1 \le i \le d, \tag{45}$$

with the constraint (14), thus dealing with a non constant speed. This equation is similar to (18) but here $v = 0$ and u is not constant.

Recently ([9]) the authors focused on the problem (45)-(14) with two compounds and a constant total density $\rho \equiv 1$, assuming an isobaric behavior (which is not really restrictive from a theoretical point of view). The system is:

$$\begin{aligned} \partial_t(c_1 + q_1^*(c_1, c_2)) + \partial_x(u\, c_1) &= 0, & (46)\\ \partial_t(c_2 + q_2^*(c_1, c_2)) + \partial_x(u\, c_2) &= 0, & (47)\\ c_1 + c_2 = \rho &= 1. & (48) \end{aligned}$$

Douglas and *al.* ([16]) developed a general criterion for determining the transitions type for this monovariant system and gave several examples (see also [33]). A first attempt, in the simpler case where one of the compounds is inert, led the authors to an existence and uniqueness result (see [9]) summarized in the next subsection. Rouchon and *al.* ([30]) discussed this model and performed a numerical simulation in the case of one inert compound. The general case, presented in Subsection 3.4.2, is now better understood ([10]) from a mathematical point of view.

3.4.1 Case of an inert carrier gas

In this subsection, we assume that one of the compounds is inert with concentration c_2 and that the spatial domain is $\mathbb{R}_+$ for the sake of simplicity. We set $c = c_2$ and $h(c) = -q_1^*(c_1, c_2) = -q_1^*(1 - c, c)$, thus, using (15), the system (46)-(47)-(48) may be written under the form:

$$\left\{ \begin{array}{rcl} \partial_t c + \partial_x(uc) &=& 0,\\ \partial_t h(c) - \partial_x u &=& 0, \end{array} \right. \tag{49}$$

supplemented by initial and boundary data:

$$\left\{ \begin{array}{rcll} c(0,x) &=& c_0(x) \in [0,1], & x > 0,\\ c(t,0) &=& c_b(t) \in [0,1], & t > 0,\\ u(t,0) &=& u_b(t), & t > 0. \end{array} \right. \tag{50}$$

We assume in (50) an influx boundary condition, i.e. $\forall t > 0,\ u_b(t) > 0$ and, in this first simplified approach

$$H'(c) \geq 0,\ h'(c) > 0, \tag{51}$$

where $H(c) = 1 + ch'(c)$. The reference for the results described in this subsection is [9].

First we obtain an existence result for smooth solutions using a classical characteristic method. The interest of this result is mainly to suggest us an entropy condition for shockwaves:

$$(EC) \qquad \text{“c increases through a shock”}.$$

For smooth solutions, the active gas desorbs and u increases to evacuate gases. Next, we obtain a global existence theorem for a weak solution of System (49)-(50):

Theorem 10 (Global large weak solution)
Let be $X > 0$, $T > 0$. Assume (51) and that $c_0 \in BV(0,X)$, $c_b \in BV(0,T)$, $u_b \in L^\infty(0,T)$, satisfying $0 \leq c_0, c_b \leq 1$ and $\inf_{0<t<T} u_b(t) > 0$. Then the system (49)-(50) admits a weak solution given by an adapted Godunov scheme. Furthermore, c and u satisfy:

$$c \in L^\infty((0,T)\times(0,X)) \cap L^\infty((0,T); BV(0,X)),$$

$$c \in Lip(0,T\,;\, L^1(0,X)),\ c \in BV((0,T)\times(0,X)),$$

$$u \in L^\infty((0,T)\times(0,X)) \cap L^\infty((0,T); BV(0,X)),$$

with bounds on c and u and these functions satisfy initial boundary conditions (50) strongly.

The proof relies on a precise study of the solutions to the Riemann problem satisfying the entropy condition (EC). We use a Godunov scheme to construct an approximate weak solution of Problem (49)-(50) and we give some L^∞ and BV bounds. Lastly, we show that a sequence of approximate solutions converges in some sense towards a global weak entropy solution.

The uniqueness problem for weak entropy solutions was solved in some class of piecewise smooth functions. Nevertheless, this case is relevant in most practical cases and involve global solutions with shock waves and contact discontinuities. Notice that the assumptions in (51) are not really physically relevant because they hold for particular isotherm only. This restriction is avoided in the general approach below (but we no longer have an uniqueness result).

As an illustration for this particular problem we consider the case of an increasing smooth initial concentration of inert gas: a shock develops and then propagates (Fig. 6). Dealing with a finite exchange kinetics, the same case is investigated using the numerical method described in Section 3.3.2 with some increasing values of the coefficient A governing the kinetics for the active gas. The numerical experiments suggest that the model (6)-(7)-(14) could converge in some sense towards the model (45)-(14), but this is still an open problem.

3.4.2 General case

In [10], the authors deal with the case of two compounds which may be active or not, with physically relevant assumptions. Following Rouchon and *al.* ([30]) the problem is analyzed as an hyperbolic system with respect to the (x,t) variables, that is with x as the evolution variable. It appears to be really the key for the mathematical analysis of the problem.

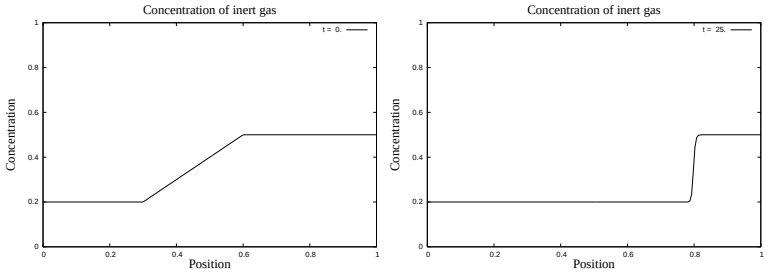

Figure 6: Development of a shock (right side). The initial concentration (left side) is continuous and nondecreasing.

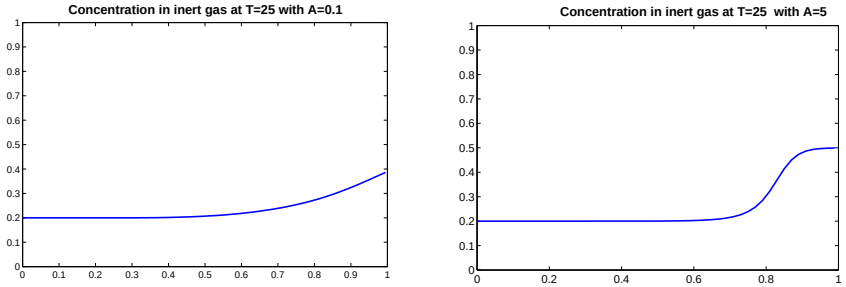

Figure 7: Evolution of an increasing smooth initial concentration of inert gas with a finite exchange kinetics and two different "drawback" coefficients A for the active gas: $A = 0.1$ (left) and $A = 5$ (right). The initial and boundary data are the same as in Fig. 6

.

We set $c = c_1$ (as in [16], instead of c_2 in [9]), $m = u\,c$ (flow rate of the first species) and

$$\begin{aligned} q_i(c) &= q_i^*(c, 1-c), \quad i = 1, 2, \\ h(c) &= q_1(c) + q_2(c), \\ I(c) &= c + q_1(c), \end{aligned}$$

(notice the change of sign in h with respect to [9]). The problem (46)-(47)-(48) is then written under the form

$$\left\{ \begin{array}{rcl} \partial_t I(c) + \partial_x(u\,c) &=& 0. \\ \partial_t h(c) + \partial_x u &=& 0, \end{array} \right. \tag{52}$$

or equivalently

$$\partial_x U + \partial_t \Phi(U) = 0 \text{ with } U = \left(\begin{array}{c} u \\ m \end{array} \right) \text{ and } \Phi(U) = \left(\begin{array}{c} h(m/u) \\ \\ I(m/u) \end{array} \right) \tag{53}$$

supplemented by initial and boundary data.
We assume that $q_1' \geq 0$ and $q_2' \leq 0$ (general property of all isotherms, see [16]).

This system (53) is hyperbolic, with eigenvalues 0 and $\lambda = \dfrac{H(c)}{u}$. Introducing the function $f = q_1\, c_2 - q_2\, c_1$ defined by Douglas and *al.* in [16], written here under the form $f(c) = q_1(c) - c\, h(c)$, it appears that λ is genuinely nonlinear in each domain where $f" \neq 0$.
Next, the system admits the two Riemann invariants: c and $w = \ln u + g(c)$, where g satisfies $g'(c) = \dfrac{-h'(c)}{H(c)}$ and the smooth entropy families

$$S(c,u) = \phi(w) + u\,\psi(c) \tag{54}$$

where ϕ and ψ are any smooth real functions.
Let be $G = e^g$: if the sign of G'' changes then the system does not admit any convex smooth entropy but for each convex or degenerate convex smooth function ψ (i.e. $\psi'' \geq 0$) the corresponding entropy $S = u\,\psi(c)$ is degenerate convex. Introducing a suitable notion of entropy solution, the authors in [10] show the existence of at least such a solution.

The mathematical definition of weak entropy solution is the following: let be $T > 0$, $X > 0$, $u \in L^\infty((0,T)\times(0,X),\mathbb{R}^+)$, $0 \leq c(t,x) \leq 1$ for almost $(t,x) \in (0,T)\times(0,X)$. Then (c,u) is a weak entropy solution if for all convex (or degenerate convex) ψ

$$\frac{\partial}{\partial x}(u\psi(c)) + \frac{\partial}{\partial t}Q(c) \leq 0,$$

in the distribution sense, where $Q' = H\psi' + h'\psi$.

In addition, if $G'' \geq 0$ on $[0,1]$, (c,u) has to satisfy $\dfrac{\partial}{\partial x}(uG(c)) \leq 0$.

The main result, relying on the resolution of the Riemann problem and on the Godunov scheme, is the following:

Theorem 11
Let be $T > 0$, $X > 0$, $c_0 \in BV(0,X)$, $c_b \in BV(0,T)$, satisfying $0 \leq c_0$, $c_b \leq 1$ and $\inf_{0<t<T} u_b(t) > 0$. Then the system admits a weak entropy solution.

A natural framework for this problem, as in the case of the PSA system with finite exchange kinetics, would be the L^∞ one. Some tracks are presently explored but it is still an open problem.

3.5 Kinetic approaches

The theory of kinetic formulation of hyperbolic systems, introduced by P.-L. Lions, B. Perthame and E. Tadmor [25], consists in representing a whole family of entropy inequalities by a single equation, using a supplementary variable, for the fundamental solution of the wave equation for entropies. It often leads to numerical schemes enjoying interesting properties and easy to implement.
We give first a result due to James, Peng and Perthame ([20]) concerning the

system of electrophoresis. Next, using the results stated in 3.4.2, we present a similar approach for the PSA system, currently developed by the authors.

3.5.1 Kinetic formulation for Electrophoresis

Electrophoresis is a process of separation for compounds in a aqueous solution with an electric field. The corresponding model is close to the chromatography system with Langmuir isotherm ([13, 32]). F. James, Y.-J. Peng and B. Perthame ([20]) have complemented on the system of electrophoresis the theory of kinetic formulation of hyperbolic systems. They gave several applications of this kinetic formulation: a maximum principle and a stability result in L^∞ using compensated compactness. Even though this system is rather related to liquid-solid chromatography (the sorption effect is neglected) we mention these results because the method is powerful and may be extended in other contexts.

The system of electrophoresis is

$$\partial_t c_i + \partial_x \frac{\alpha_i c_i}{D} = 0,\ t \geq 0,\ 1 \leq i \leq d, \tag{55}$$

where $c_1, \cdots, c_d$ are the positive unknowns, $D = 1 + c_1 + ... + c_d$ and the α_i are given numbers satisfying $0 < \alpha_1 < ... < \alpha_d$.
This system admits d eigenvalues $0 < \lambda_1(c) \leq ... \leq \lambda_d(c)$ and $w_i = \lambda_i D$ is a $i-$Riemann invariant.

The first step consists in building $d+1$ families of nonlinear entropies: it is summarized in the following proposition.

Proposition 12
For $0 \leq i \leq d$, the function

$$\chi_i(\xi; w_1, \cdots, w_d) = \prod_{j=1}^{d} \left|1 - \frac{\xi}{w_j}\right| \mathbb{1}_{]w_i, w_{i+1}[}(\xi)$$

is an entropy of System (55), convex in c, with the entropy flux

$$\phi_i(\xi; w_1, \cdots, w_d) = \frac{\xi}{D}\, \chi_i(\xi; w_1, \cdots, w_d).$$

An entropy solution of (55) is a solution satisfying in the distribution sense the inequality

$$\partial_t E + \partial_x F \leq 0$$

for any convex entropy of the form

$$E(c) = \int_{\mathbb{R}_+} g(\xi)\, \chi_i(\xi; w_1, \cdots, w_d)\, d\xi$$

for some i, $1 \leq i \leq d$, $g \in L^1(\mathbb{R}_+)$. Then the authors in [20] derive the following kinetic formulation of (55): there are $d+1$ non-positive measures $m_i(x, \xi, t)$ such that

$$\partial_t \chi_i(\xi, c(x,t)) + \partial_x \left(\frac{\xi}{D}\, \chi_i(\xi, c(x,t)) \right) = m_i \quad \text{in } \mathcal{D}'(\mathbb{R}_x \times \mathbb{R}_{+\xi} \times \mathbb{R}_{+t}).$$

Two applications are given in [20]: the maximum and minimum principle on the Riemann invariants and a compactness result. They are recalled in the following two theorems.

Theorem 13
Let c be an entropy solution satisfying $c \in L^\infty(0,T;L^1_{loc}(\mathbb{R}))$, $c_i \geq 0,\ 1 \leq i \leq d$. Then

$$\alpha_{i-1} \leq \inf_{y\in\mathbb{R}} w_i(y,0) \leq w_i(x,t) \leq \sup_{y\in\mathbb{R}} w_i(y,0) \leq \alpha_i$$

and for some constant $L > 0$

$$L \inf_{y\in\mathbb{R}} c_i^0(y) \leq c_i(x,t) \leq \frac{1}{L} \sup_{y\in\mathbb{R}} c_i^0(y),\ 1 \leq i \leq d.$$

Theorem 14
Consider an uniformly bounded family $(c_{i,n}(x,t))_{n\geq 0}$ of entropy solutions to the electrophoresis who satisfy that the initial data are uniformly bounded and

$$c_{i,n}(x,t) \rightharpoonup c_i(x,t) \in L^\infty((0,T)\times\mathbb{R}),\ 1 \leq i \leq d$$

weak for all $T \in (0,\infty)$ as n tends to $+\infty$. Then $(c_{i,n})_n$ converges pointwise to c_i an entropy solution.*

3.5.2 A kinetic approach for the PSA system

A kinetic formulation built on the entropies (54) given in Section 3.4.2 as well as several applications were investigated and will be presented in a forthcoming paper ([11]). More precisely, using the entropy family $u\,\psi(c)$ where $\psi'' \geq 0$ we first state:

Theorem 15
If (u,c) is a weak entropy solution of System (52), then there exists a nonnegative measure $m(t,x,\xi)$ such that:

$$\partial_x(u\chi(c,.)) + (H(\xi) - a(\xi))\,\partial_t\chi(c,.) + \partial_t\left(h(c)\chi(c,.)\right) = \partial_\xi m.$$

where

$$\chi(c,\xi) = \begin{cases} 1 & \text{if } 0 < \xi < c \\ 0 & \text{else} \end{cases} \qquad \text{and} \quad a(\xi) = 1 + f'(\xi) = H(\xi) - h(\xi).$$

Conversely, if there exists a positive function u such that $\ln u \in L^\infty$, a function $f \in L^1_\xi$ such that $0 \leq f \leq 1$ and a nonnegative measure m such that

$$\partial_x(u\,f(t,x,\xi)) + a(\xi)\,\partial_t f(t,x,\xi) + \partial_t\left(h(c)\,f(t,x,\xi)\right) = \partial_\xi m$$

then (u,c) is an entropy solution of (52) with $c(t,x) = \displaystyle\int_0^1 f(t,x,\xi)\,d\xi$.

Notice that this formulation is not purely kinetic, as in the case of the isentropic gas dynamics system (see [26]).

With this kinetic formulation, the authors in [11] derive a kinetic numerical scheme. This first order explicit scheme satisfies in-cell entropy inequalities, the maximum principle on the concentrations, some BV bounds and is much more simple to implement than Godunov's scheme.

4 Open problems

There are a lot of open problems related to the system (46)-(47)-(48). First, the rigorous derivation of this system from (6)-(7)-(14) considered as a relaxation problem, as in James ([21]), with a small parameter $\varepsilon = 1/A_i$ is *a priori* quite more difficult than for the system (8)-(9). In particular, all the BV estimates stated in [7] blow up. Notice however that numerical experiments, as in Section 3.4.1, show a good behavior in physically relevant context.
The study of oscillating solutions is currently performed by the authors ([12]).
A rigorous and general link between the problem with the velocity given by Darcy's law and the constraint (14) has still to be carried out.
There exists some variants of the model taking the axial diffusion or the temperature into account. In particular, the temperature plays a role which may be non negligible. Up to our knowledge, the corresponding problems have never been mathematically studied.

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WORKSHOP IBEROAMERICANO DE MATEMÁTICAS APLICADAS CHILLÁN (CHILE), 6–8 DE AGOSTO DE 2007

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Estimado lector:

En este volumen se incluyen el resto de los textos de las conferencias presentadas en el primer "Workshop Iberoamericano de Matemáticas Aplicadas", realizado en la ciudad de Chillán (Chile), del 6 al 8 de Agosto de 2007, cuya primera parte ya apareció en el volumen 41 de este boletín.

Los trabajos seleccionados en este volumen son los siguientes:

1. C. Conca, R. Orive, M. Vanninathan, Primer y segundo corrector en homogeneización por ondas de Bloch.
2. E. Fernández-Cara, Several questions concerning the control of parabolic systems.
3. E. Ortega-Torres, M. Poblete-Cantellano, M. Rojas-Medar, Fractional time-derivative of some evolution partial differential equations.
4. M. Sepúlveda, O. Vera, Numerical methods for a coupled nonlinear Schrödinger system.

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PRIMER Y SEGUNDO CORRECTOR EN HOMOGENEIZACIÓN POR ONDAS DE BLOCH

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Resumen

En este trabajo se expone una continuación del trabajo iniciado en el artículo [5], de C. Conca y M. Vanninathan, resumiendo resultados contenidos en [4]. Se estudia la homogeneización de operadores elípticos con coeficientes periódicos oscilantes en $\mathbb{R}^N$, en particular, presenta un análisis matemático de los términos correctores asociados a la solución homogeneizada. Para ello, se usan ondas de Bloch y, gracias a este método, se recuperan los diferentes términos del anstaz clásico de la expansión asintótica a dos escalas.

Palabras clave: *homogenization, Bloch waves, correctors*
Clasificación por materias AMS: *35B27, 35A25, 42C30*

1 Presentación del problema

Sea el operador elíptico de coeficientes periódicos

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial y_k}\left(a_{kl}(y)\frac{\partial}{\partial y_l}\right), \qquad \text{donde} \tag{1}$$

$$\begin{cases} a_{kl} \in \mathrm{L}^\infty_{\#}(Y) \text{ con } Y =]0, 2\pi[^N, \text{ i.e., } a_{kl} \text{ es Y-periódica y medible en } \mathbb{R}^N, \\ \exists \alpha > 0 \quad \text{tal que} \quad a_{kl}\eta_k\eta_l \geq \alpha|\eta|^2 \quad \forall \eta \in \mathbb{R}^N \quad \text{(elipticidad)}, \\ a_{kl} = a_{lk} \quad \forall l, k = 1, ..., N \quad \text{(simetría)}. \end{cases} \tag{2}$$

A partir de (1), definimos para cada $\varepsilon > 0$ el siguiente operador elíptico

$$A^\varepsilon \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(a^\varepsilon_{k\ell}(x)\frac{\partial}{\partial x_\ell}\right) \quad \text{con } a^\varepsilon_{k\ell}(x) = a_{k\ell}\left(\frac{x}{\varepsilon}\right), \text{ y } x \in \mathbb{R}^N. \tag{3}$$

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Pedimos como hipótesis que exista una sucesión u^ε de $H^1(\mathbb{R}^N)$ tal que para $f \in L^2(\mathbb{R}^N)$ satisfaga

$$\begin{array}{rrcll} & A^\varepsilon u^\varepsilon & = & f & \text{en } \mathbb{R}^N, \\ \text{Además, cuando } \varepsilon \to 0 & u^\varepsilon & \rightharpoonup & u^* & \text{debilmente en } H^1(\mathbb{R}^N), \\ & u^\varepsilon & \to & u^* & \text{fuerte en } L^2(\mathbb{R}^N). \end{array} \tag{4}$$

Es clásico en homogeneización, que u^* se denomine solución homogeneizada; se sabe que satisface

$$A^* u^* = f \quad \text{en} \quad \mathbb{R}^N, \quad \text{donde} \quad A^* \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(q_{k\ell}\frac{\partial}{\partial x_\ell}\right) \tag{5}$$

es el operador homogeneizado, con coeficientes $q_{k\ell}$ constantes y simétricos (véase [1]).

Nuestro objetivo es recuperar el *primer y segundo corrector* de la solución homogeneizada u^*, de una forma distinta a la clásica de [1]. Se llama primer corrector a una función $\theta_1^\varepsilon \in H^1(\mathbb{R}^N)$, que sea facilmente construible, no dependa de u^ε y satisfaga:

$$\|u^\varepsilon - u^* - \varepsilon\theta_1^\varepsilon\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{cuando} \quad \varepsilon \to 0. \tag{6}$$

Por definición, un segundo corrector $\theta_2^\varepsilon \in H^1(\mathbb{R}^N)$ posee la propiedad

$$\|u^\varepsilon - u^* - \varepsilon\theta_1^\varepsilon - \varepsilon^2\theta_2^\varepsilon\|_{H^1(\mathbb{R}^N)} \leq c\varepsilon. \tag{7}$$

El artículo ha sido redactado en el orden siguiente. Primero, se repasa el método de Bloch y se obtiene el desarrollo de Taylor del primer autovalor y autofunción de Bloch. En la sección "Coeficientes de Bloch", se estudia la convergencia de la primera autofunción a la transformada de Fourier, y se observa que solamente el primer autovalor de Bloch es relevante en la homogeneización del operador A^ε. En la última sección, enunciamos los resultados de correctores y se bosquejan sus demostraciones; para otros detalles el lector podrá consultar [4].

2 Autovalores y autofunciones de Bloch

El método de Bloch, véase [3] ó [5], introduce una familia de problemas espectrales, parametrizados por $\eta \in \mathbb{R}^N$: Encontrar $\lambda_m = \lambda_m(\eta)$ y $\phi_m = \phi_m(y;\eta)$ tales que

$$A(\eta)\phi_m(\cdot,\eta) = \lambda_m(\eta)\phi_m(\cdot,\eta) \quad \text{en} \quad \mathbb{R}^N, \quad \phi(\cdot,\eta) \quad Y\text{-periódica},$$

donde $A(\eta)\phi \stackrel{\text{def}}{=} A(e^{iy\cdot\eta}\phi)$. Gracias a la η-periocidad de ϕ_m, podemos confinar η a la celda $Y' =]-\frac{1}{2},\frac{1}{2}[^N$. Además, sabemos de [5] que

i) $0 \leq \lambda_1(\eta) \leq ... \leq \lambda_m(\eta) \leq ... \to +\infty$, $\lambda_2(\eta) \geq \lambda_N^{(2)} > 0$, $\forall \eta \in Y'$,

ii) $\{\phi_m(\cdot;\eta)\}_m$ es base ortonormal de $L^2_\#(Y)$.

Sean $g, h \in L^2(\mathbb{R}^N)$. El m-ésimo coeficiente de Bloch de g se define por

$$(B_m g)(\eta) = \int_{\mathbb{R}^N} g(y) e^{-iy\cdot\eta} \bar{\phi}_m(y;\eta) dy \quad \forall m \geq 1,\ \eta \in Y'. \tag{8}$$

$$g(y) = \int_{Y'} \sum_{m=1}^{\infty} (B_m g)(\eta) e^{iy\cdot\eta} \phi_m(y;\eta) d\eta, \tag{9}$$

$$\int_{\mathbb{R}^N} |g(y)|^2 dy = \int_{Y'} \sum_{m=1}^{\infty} |(B_m g)(\eta)|^2 d\eta \quad \text{(Parseval)}, \tag{10}$$

$$\int_{\mathbb{R}^N} g(y)\bar{h}(y) dy = \int_{Y'} \sum_{m=1}^{\infty} (B_m g)(\eta)\overline{(B_m h)}(\eta) d\eta \quad \text{(Plancherel)}. \tag{11}$$

Así, la familia $\{\phi_m(y;\eta) : m \geq 1, \eta \in Y'\}$ forma una base de $L^2(\mathbb{R}^N)$, en un sentido generalizado, y $L^2(\mathbb{R}^N)$ se identifica con $L^2(Y', l^2(N))$ por la identidad de Parseval.

En [5] se prueban los siguientes resultados de regularidad: Para cada $m \geq 1$, $\lambda_m(\eta)$ es una función Lipschitziana de η y existe una bola B_δ, de centro $\eta = 0$ y radio $\delta > 0$, tal que, en ella, el primer autovalor de Bloch, $\lambda_1(\eta)$, es simple y define una función analítica de η.

La aplicación $\eta \to \lambda_1(\eta)$, definida de Y' en $\mathbb{R}^N$, posee un mínimo estricto global en $\eta = 0$ y, existe una constante $c > 0$ tal que $\lambda_1(\eta) \geq c|\eta|^2$, para todo $\eta \in Y'$. Más aún, es posible escoger la primera autofunción de modo que sea analítica en B_δ y satisfaga: $Im(\int_Y \phi_1(y;\eta)dy) = 0$ para todo $\eta \in B_\delta$ y $\phi_1(y;0) = (2\pi)^{-N/2}$, que denotamos $p^{(0)}$.

Consideremos las funciones auxiliares $\chi^k, \chi^{k\ell}, \chi^{k\ell m}, \chi^{k\ell mn} \in H^1_{\#}(Y)$, soluciones de:

$$A\chi^k = \frac{\partial a_{k\ell}}{\partial y_\ell} \quad \text{en} \quad \mathbb{R}^N, \quad \mathfrak{M}_Y(\chi^k) \stackrel{\text{def}}{=} \frac{1}{|Y|}\int_Y \chi^k(y)dy = 0. \tag{12}$$

$$A\chi^{k\ell} = (a_{k\ell} - q_{k\ell}) - \frac{1}{2}\left(C_k\chi^\ell + C_\ell\chi^k\right) \quad \text{en} \quad \mathbb{R}^N, \quad \mathfrak{M}_Y(\chi^{k\ell}) = 0. \tag{13}$$

$$\left\{\begin{array}{l} A\chi^{k\ell m} = \frac{1}{3}\Big[(a_{k\ell} - q_{k\ell})\chi^m + (a_{\ell m} - q_{\ell m})\chi^k + (a_{nk} - q_{nk})\chi^\ell - \\ \qquad\qquad C_k\chi^{\ell m} - C_\ell\chi^{mk} - C_m\chi^{k\ell}\Big] \quad \text{en} \quad \mathbb{R}^N, \quad \mathfrak{M}_Y(\chi^{k\ell m}) = 0. \end{array}\right. \tag{14}$$

$$\left\{\begin{array}{l} A\chi^{k\ell mn} = \frac{1}{4!}D^4_{k\ell mn}\lambda_1(0) - \frac{1}{4}\left(C_n\chi^{k\ell m} + C_k\chi^{\ell mn} + C_\ell\chi^{mnk} + C_m\chi^{nk\ell}\right) + \\ \frac{1}{3!}\left[(a_{k\ell} - q_{k\ell})\chi^{mn} + (a_{\ell m} - q_{\ell m})\chi^{kn} + (a_{km} - q_{km})\chi^{\ell n} + (a_{\ell n} - q_{\ell n})\chi^{km}\right. \\ \qquad \left. +(a_{mn} - q_{mn})\chi^{k\ell} + (a_{kn} - q_{kn})\chi^{\ell m}\right] \quad \text{en } \mathbb{R}^N, \quad \mathfrak{M}_Y(\chi^{k\ell mn}) = 0, \end{array}\right. \tag{15}$$

donde $C_k \stackrel{\text{def}}{=} -a_{k\ell}(y)\frac{\partial(.)}{\partial y_\ell} - \frac{\partial}{\partial y_\ell}\left(a_{k\ell}(y)(\cdot)\right)$. Los problemas (12), (13), (14) y (15) admiten solución única. En efecto, para $f \in L^2_{\#}(Y)$, el problema

$$A\varphi = f \quad \text{en} \quad \mathbb{R}^N, \quad \varphi \in H^1_{\#}(Y), \quad \mathfrak{M}(\varphi) = 0,$$

por la *alternativa de Fredholm* (ver [2]), tiene solución única, si y sólo si $\int_Y f(y)dy = 0$.

Proposición 1 *Las derivadas de λ_1 en $\eta = 0$ de orden menor o igual que 4 son*

$$D^\beta \lambda_1(0) = 0 \quad \forall \beta \in \mathbb{Z}^N_+ : \ |\beta| = \beta_1 + ... + \beta_N \ \textit{impar.}$$

$$\frac{1}{2}D^2_{k\ell}\lambda_1(0) = \frac{1}{|Y|}\int_Y a_{k\ell}(y)dy - \frac{1}{2|Y|}\int_Y (C_k\chi^\ell(y) + C_\ell\chi^k(y))dy = q_{k\ell}$$

$$\frac{1}{4!}D^4_{k\ell mn}\lambda_1(0) = \frac{1}{4}\frac{1}{|Y|}\int_Y \left\{C_n\chi^{k\ell m} + C_k\chi^{\ell mn} + C_\ell\chi^{mnk} + C_m\chi^{nk\ell}\right\}dy -$$

$$-\frac{1}{3!}\frac{1}{|Y|}\int_Y \left\{a_{k\ell}\chi^{mn} + a_{\ell m}\chi^{nk} + a_{mn}\chi^{k\ell} + a_{nk}\chi^{\ell m} + a_{km}\chi^{\ell n} + a_{\ell n}\chi^{km}\right\}dy,$$

para todo $k, \ell, m, n = 1, ..., N$. Además, $D^a\lambda_1(0)$ con $a \in \mathbb{N}$ par, es real.

Proposición 2 *Las derivadas de ϕ_1 en $\eta = 0$ de orden menor o igual que 4 son*

$$\begin{aligned}
D_k\phi_1(y;0) &= ip^{(0)}\chi^k(y), \\
\frac{1}{2!}D^2_{k\ell}\phi_1(y;0) &= -p^{(0)}\chi^{k\ell}(y) + \beta^{(2)}_{k\ell}p^{(0)}, \\
\frac{1}{3!}D^3_{k\ell m}\phi_1(y;0) &= -ip^{(0)}\chi^{k\ell m}(y) + \frac{i}{3}\left(\beta^{(2)}_{k\ell}\chi^m(y) + \beta^{(2)}_{\ell m}\chi^k(y) + \beta^{(2)}_{mk}\chi^\ell(y)\right)p^{(0)}, \\
\frac{1}{4!}D^4_{k\ell mn}\phi_1(y;0) &= p^{(0)}\chi^{k\ell mn}(y) - \frac{1}{3!}\left(\beta^{(2)}_{k\ell}\chi^{mn}(y) + \beta^{(2)}_{\ell m}\chi^{nk}(y) + \beta^{(2)}_{mn}\chi^{k\ell} + \right. \\
&\quad \left. + \beta^{(2)}_{nk}\chi^{\ell m}(y) + \beta^{(2)}_{km}\chi^{n\ell}(y) + \beta^{(2)}_{\ell n}\chi^{km}(y)\right)p^{(0)} + \beta^{(4)}_{k\ell mn}p^{(0)},
\end{aligned}$$

donde $\beta^{(2)}_{k\ell}$ y $\beta^{(4)}_{k\ell mn}$ son constantes. Además, $D^a\phi_1(y,0)$ con $a \in \mathbb{N}$ par, es real, y con $a \in \mathbb{N}$ impar, es imaginaria.

Para obtener estos resultados, consideremos la ecuación del primer autovalor

$$(A(\eta) - \lambda_1(\eta))\phi_1(\cdot;\eta) = 0. \tag{16}$$

Derivando con respecto a η_k y evaluando en $\eta = 0$ se obtienen las distintas ecuaciones que han de verificar las derivadas de ϕ_1. Tomamos producto escalar de (16) con $\overline{\phi_1}(\cdot;\eta)$, y evaluando en $\eta = 0$ se tienen las derivadas de λ_1. Para conocer el valor de las constantes β utilizamos que $\|\phi_1(\cdot;\eta)\|_{L^2(Y)} = 1$ para todo η.

3 Análisis de los coeficientes de Bloch

Asociado al operador (3), consideremos las autofunciones $\phi_m^\varepsilon(x;\xi) = \phi_m(y;\eta)$ y los autovalores $\lambda_m^\varepsilon(\xi) = \varepsilon^{-2}\lambda_m(\eta)$, cuyas variables verifican: $y = x/\varepsilon$, $\eta = \varepsilon\xi$. Como en (8), definimos los coeficientes $B_m^\varepsilon u^\varepsilon$ asociados a ϕ_m^ε. Para estos coeficientes serán también válidas las igualdades (9), (10), (11). Notemos que $u^\varepsilon(x) = P_1^\varepsilon(x) + v^\varepsilon(x)$, $x \in \mathbb{R}^N$,donde

$$P_1^\varepsilon(x) = \int\limits_{\varepsilon^{-1}Y'} B_1^\varepsilon u^\varepsilon(\xi) e^{ix\cdot\xi}\phi_1^\varepsilon(x;\xi)d\xi$$

$$v^\varepsilon(x) = \int\limits_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} B_m^\varepsilon u^\varepsilon(\xi) e^{ix\cdot\xi}\phi_m^\varepsilon(x;\xi)d\xi.$$

Proposición 3 *Tenemos que*

(i) $\|v^\varepsilon\|_{L^2(\mathbb{R}^N)} \le c\varepsilon\|f\|_{H^{-1}(\mathbb{R}^N)}$ *para* $f \in H^{-1}(\mathbb{R}^N)$.

(ii) $\|v^\varepsilon\|_{L^2(\mathbb{R}^N)} \le c\varepsilon^2\|f\|_{L^2(\mathbb{R}^N)}$ *para* $f \in L^2(\mathbb{R}^N)$.

(iii) $\|\nabla v^\varepsilon\|_{L^2(\mathbb{R}^N)^N} \le c\varepsilon\|f\|_{L^2(\mathbb{R}^N)}$ *para* $f \in L^2(\mathbb{R}^N)$.

(iv) $\|\nabla v^\varepsilon\|_{L^2(\mathbb{R}^N)^N} \le c\varepsilon^2\|\nabla f\|_{L^2(\mathbb{R}^N)}$ *para* $f \in H^1(\mathbb{R}^N)$.

Sabemos que $\lambda_m^\varepsilon(\xi)B_m^\varepsilon u(\xi) = B_m^\varepsilon f(\xi)$ para todo $m \ge 2$, y que

$$\|v^\varepsilon\|_{L^2}^2 = \int\limits_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} |B_m^\varepsilon u^\varepsilon(\xi)|^2 d\xi, \quad |\nabla v^\varepsilon|_{L^2}^2 \le c \int\limits_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} \lambda_m^\varepsilon(\xi)|B_m^\varepsilon u^\varepsilon(\xi)|^2 d\xi,$$

$$\|f\|_{H^{-1}}^2 = \int\limits_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \frac{|B_m^\varepsilon f(\xi)|^2}{1+\lambda_m^\varepsilon(\xi)} d\xi.$$

Como $\sup\{[\lambda_m^\varepsilon(\xi)]^{-1} \mid m \ge 2,\ \xi \in \varepsilon^{-1}Y'\} \le (\lambda_2^{(N)})^{-1}\varepsilon^2$, esto prueba la Proposición 3. Así los coeficientes de Bloch mayores que 1 son irrelevantes para nuestro estudio.

En [5], se demuestra que dada una sucesión en $L^2(\mathbb{R}^N)$, digamos g^ε, se tiene que
(1) Si $g^\varepsilon \rightharpoonup g$ debilmente en $L^2(\mathbb{R}^N)$, entonces $\chi_{\varepsilon^{-1}Y'}B_1^\varepsilon g^\varepsilon \rightharpoonup \widehat{g}$ debilmente en $L^2_{loc}(\mathbb{R}^N)$ cuando $\operatorname{supp}(g^\varepsilon) \subset K$, compacto.
(2) Si $g^\varepsilon \to g$ en $L^2(\mathbb{R}^N)$, entonces $\chi_{\varepsilon^{-1}Y'}B_1^\varepsilon g \to \widehat{g}$ en $L^2_{loc}(\mathbb{R}^N)$,
donde χ es la función característica de $\varepsilon^{-1}Y'$. Otro resultado en este mismo sentido es:

Proposición 4

(i) Para $g \in L^2(\mathbb{R}^N)$*, con soporte compacto, se tiene que* $\chi_{\varepsilon^{-1}Y'}(\xi)B_1^\varepsilon g(\xi) \to \widehat{g}(\xi)$ *en* $L^\infty_{loc}(\mathbb{R}^N_\xi)$.

(ii) Si $g \in L^2(\mathbb{R}^N)$, *entonces* $\chi_{\varepsilon^{-1}Y'}(\xi)B_1^\varepsilon g(\xi) \to \widehat{g}(\xi)$ *en* $L^2(\mathbb{R}^N_\xi)$.

(iii) Si $g \in H^1(\mathbb{R}^N)$, *entonces* $\|\chi_{\varepsilon^{-1}Y'}(\xi)B_1^\varepsilon g(\xi) - \widehat{g}(\xi)\|_{L^2} \leq c\varepsilon$.

Para demostrarla, se usa una aproximación en celdas de la transformada de Fourier, denominada *transformada de Fourier aproximada.* Consideremos $\{Y_\ell^\varepsilon\}_{\ell\in\mathbb{Z}^N}$ el reticulado de $\mathbb{R}^N$ generado por la celda εY, $Y_\ell^\varepsilon = x_\ell^\varepsilon + \varepsilon Y$ donde $x_\ell^\varepsilon = 2\pi\varepsilon\ell$ es el origen de la celda Y_ℓ^ε. Dado $p > N$, para $g \in W^{1,p}(\mathbb{R}^N)$ con soporte compacto, definimos

$$F^\varepsilon g(\xi) = \sum_{\ell\in\mathbb{Z}^N} g(x_\ell^\varepsilon)e^{-ix_\ell^\varepsilon\cdot\xi} \quad \forall\xi \in \varepsilon^{-1}Y', \quad \text{así} \tag{17}$$

(a) $\varepsilon^N\chi_{\varepsilon^{-1}Y'}F^\varepsilon g(\xi) \to (2\pi)^{-N/2}\widehat{g}(\xi)$ puntualmente para $\xi \in \mathbb{R}^N$.

(b) $\|\varepsilon^N F^\varepsilon g\|_{L^2(\varepsilon^{-1}Y')} \leq c|K|^{\frac{p-2}{2p}}\left\{\|g\|_{L^p(\mathbb{R}^N)} + \varepsilon\|\nabla g\|_{L^p(\mathbb{R}^N)^N}\right\}$ [1].

(c) $\varepsilon^N\chi_{\varepsilon^{-1}Y'}F^\varepsilon g \to (2\pi)^{-N/2}\widehat{g}$ en $L^2(\mathbb{R}^N)$.

En efecto, (a) se prueba usando que g tiene soporte compacto. Utilizando la desigualdad de Morrey (ver [2]) se demuestra (b), y por el Teorema de Egorov se prueba (c).

Demostración Proposición 4. Consideremos el ε-reticulado $\{Y_\ell^\varepsilon\}_{\ell\in\mathbb{Z}^N}$. Escribimos

$$B_1^\varepsilon g(\xi) = \sum_{\ell\in\mathbb{Z}^N}\int_{Y_\ell^\varepsilon} g(x)e^{-ix\cdot\xi}\phi_1^\varepsilon(x;\xi)dx = \sum_{\ell\in\mathbb{Z}^N} g(x_\ell^\varepsilon)\int_{Y_\ell^\varepsilon} e^{-ix\cdot\xi}\phi_1^\varepsilon(x;\xi)dx + r_1^\varepsilon(\xi), \tag{18}$$

$$r_1^\varepsilon(\xi) = \sum_{\ell\in\mathbb{Z}^N}\int_{Y_\ell^\varepsilon}(g(x) - g(x_\ell^\varepsilon))e^{-ix\cdot\xi}\phi_1^\varepsilon(x;\xi)dx. \quad \text{(error de aproximación)} \tag{19}$$

Utilizando la desigualdad de Morrey, se prueba que

$$\|r_1^\varepsilon\|_{L^2(\varepsilon^{-1}Y')} \leq C(K)\varepsilon\|\nabla g\|_{L^p},$$

y dado que $\|\phi_1(\cdot,\varepsilon\xi) - \phi(\cdot,0)\|_{L^2(Y)} \leq c\varepsilon$, de (18) se demuestra (i). Utilizando argumentos de densidad y el Teorema de Convergencia Dominada, se tiene (ii).

Para probar (iii), se considera el espacio $W^{2,p}(\mathbb{R}^N)$, y se escriben las identidades

$$B_1^\varepsilon g(\xi) = \sum_{\ell\in\mathbb{Z}^N}\int_{Y_\ell^\varepsilon}\left[g(x_\ell^\varepsilon) + \nabla g(x_\ell^\varepsilon)\cdot(x - x_\ell^\varepsilon)\right]e^{-ix\cdot\xi}\phi_1^\varepsilon(x;\xi)dx + r_2^\varepsilon(\xi),$$

$$r_2^\varepsilon(\xi) = \sum_{\ell\in\mathbb{Z}^N}\int_{Y_\ell^\varepsilon}\left\{g(x) - g(x_\ell^\varepsilon) - \nabla g(x_\ell^\varepsilon)\cdot(x - x_\ell^\varepsilon)\right\}e^{-ix\cdot\xi}\phi_1^\varepsilon(x;\xi)dx.$$

Luego, se utilizan las mismas técnicas que en (i) y en (ii).

[1] $|K|$ es la medida de Lebesgue de K.

4 Resultados de correctores

Definimos la siguiente función, que llamaremos *Aproximación de Bloch*

$$\theta^\varepsilon \stackrel{\text{def}}{=} \int\limits_{\varepsilon^{-1}Y'} \widehat{u^*}(\xi) e^{ix\cdot\xi} \phi_1^\varepsilon(x;\xi) d\xi, \tag{20}$$

donde $\widehat{u^*}$ es la transformada de Fourier de la solución homogeneizada y ϕ_1^ε es la primera onda de Bloch asociada al operador (3).

Teorema 5 *Los coeficientes $a_{k\ell}$ verifican (2), y pertenecen a $W^{1,\infty}(Y)$. Entonces*

(i) Si $f \in L^2(\mathbb{R}^N)$, se tiene que $(u^\varepsilon - \theta^\varepsilon) \to 0$ en $H^1(\mathbb{R}^N)$.

(ii) Si $f \in H^1(\mathbb{R}^N)$, se tiene que $\|\nabla(u^\varepsilon - \theta^\varepsilon)\|_{L^2(\mathbb{R}^N)^N} \le c\varepsilon^{\frac{1}{2}}$.

Demostración. Utilizando la Proposición 3 nos basta probar, respectivamente

$$\left\| P_1^\varepsilon u^\varepsilon(x) - \theta^\varepsilon \right\|_{H^1(\mathbb{R}^N)} \to 0, \tag{21}$$

$$\left| P_1^\varepsilon u^\varepsilon(x) - \theta^\varepsilon \right|^2_{H^1(\mathbb{R}^N)} \le c\varepsilon \tag{22}$$

Tomamos las antitransformadas de Bloch y consideramos los integrandos

$$(\theta^\varepsilon)^\delta(x) = \int\limits_{\substack{\xi\in\varepsilon^{-1}Y' \\ |\xi|>\delta\varepsilon^{-1}}} \widehat{u^*}(\xi) e^{ix\cdot\xi} \phi_1^\varepsilon(x;\xi) d\xi,\ (P_1^\varepsilon u^\varepsilon)^\delta(x) = \int\limits_{\substack{\xi\in\varepsilon^{-1}Y' \\ |\xi|>\delta\varepsilon^{-1}}} B_1^\varepsilon u^\varepsilon(\xi) e^{ix\cdot\xi} \phi_1^\varepsilon(x;\xi) d\xi.$$

Utilizando la regularidad de u^ε, u^* y que $\lambda_1(\eta) \ge c|\eta|^2$ para todo $\eta \in Y'$, sigue que

$$|(P_1^\varepsilon u^\varepsilon)^\delta|^2_{H^1(\mathbb{R}^N)} \le c\delta^{-2}\varepsilon^2 \|f\|^2_{L^2(\mathbb{R}^N)}, \quad \|(\theta^\varepsilon)^\delta\|^2_{H^1(\mathbb{R}^N)} \le c\varepsilon^2 \|u^*\|^2_{H^2(\mathbb{R}^N)}.$$

Para demostrar (21), (22), basta probar, respectivamente, que

$$\int\limits_{|\xi|\le\delta\varepsilon^{-1}} [1 + \lambda_1^\varepsilon(\xi)] |B_1^\varepsilon u^\varepsilon(\xi) - \widehat{u^*}(\xi)|^2 d\xi \to 0, \tag{23}$$

$$\int\limits_{|\xi|\le\delta\varepsilon^{-1}} \lambda_1^\varepsilon(\xi) |B_1^\varepsilon u^\varepsilon(\xi) - \widehat{u^*}(\xi)|^2 d\xi \le c\varepsilon \tag{24}$$

Utilizando las siguientes relaciones, ya conocidas (ver [5]),

$$\lambda_1^\varepsilon(\xi) B_1^\varepsilon u^\varepsilon(\xi) = B_1^\varepsilon f(\xi), \quad q_{k\ell}\xi_k\xi_\ell \widehat{u^*}(\xi) = \widehat{f}(\xi)\ \forall \xi \in \mathbb{R}^N,$$
$$|\lambda_1^\varepsilon - \xi^t \lambda''(0)\xi| \le c\varepsilon|\xi|^4 \qquad \forall |\varepsilon\xi| = |\eta| \le \delta,$$

y gracias a la Proposición 4, como $B_1^\varepsilon u^\varepsilon \to u^*$ en $L^2_{loc}(\mathbb{R}^N)$, se concluye la demostración. □

Teorema 6 *Bajo las hipótesis del Teorema 5, se tiene que*

(i) Si $f \in H^{-1}(\mathbb{R}^N)$, *entonces* $\|\theta^\varepsilon - u^*\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon$.

(ii) Si $f \in L^2(\mathbb{R}^N)$, *entonces* $\left\|\theta^\varepsilon - u^* - \varepsilon\chi^k \frac{\partial u^*}{\partial x_k}\right\|_{H^1(\mathbb{R}^N)} \leq c\varepsilon$.

(iii) Si $f \in H^1(\mathbb{R}^N)$, $\left\|\theta^\varepsilon - u^* - \varepsilon\chi^k \frac{\partial u^*}{\partial x_k} - \varepsilon^2\left(\chi^{k\ell} - \beta^{(2)}_{k\ell}\right)\frac{\partial^2 u^*}{\partial x_k \partial x_\ell}\right\|_{H^1(\mathbb{R}^N)} \leq c\varepsilon^2$.

En definitiva, hemos obtenido los primeros tres términos en el anstaz o expansión asintótica de la así llamada Aproximación de Bloch. Estos proporcionan, respectivamente, la solución homogeneizada, y el primer y segundo corrector a esta solución.

Demostración. En (i), como $f \in H^{-1}(\mathbb{R}^N)$, entonces $u^* \in H^1(\mathbb{R}^N)$. Como se tiene que $\|\phi_1(y;\varepsilon\xi) - \phi_1(y;0)\|_{L^2(Y)} \leq c|\varepsilon\xi|$, entonces por Parseval

$$\|\theta^\varepsilon - u^*\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon^2 \int\limits_{\varepsilon^{-1}Y'} |\xi|^2 |\widehat{u^*}|^2 d\xi + \int\limits_{\mathbb{R}^N - \varepsilon^{-1}Y'} |\widehat{u^*}|^2. \quad (25)$$

Es claro que $\int\limits_{\mathbb{R}^N - \varepsilon^{-1}Y'} |\widehat{u^*}|^2 d\xi \leq c\varepsilon^2 \int\limits_{\mathbb{R}^N - \varepsilon^{-1}Y'} |\xi|^2 |\widehat{u^*}|)d\xi \leq c\varepsilon^2 |u^*|_{H^1(\mathbb{R}^N)}$,

por integrar fuera de un entorno de 0. Entonces volviendo a (25), sigue (i).

A partir de los resultados de la Proposición 2, ya que escribimos el desarrollo de Taylor de ϕ_1 con respecto de η, en $\eta = 0$,

$$\phi_1(\cdot;\eta) = p^{(0)} + ip^{(0)}\chi^k(\cdot)\eta_k - p^{(0)}\left(\chi^{k\ell}(\cdot) - \beta^{(2)}_{k\ell}\right)\eta_k\eta_\ell + 0(|\eta|^3),$$

entonces, gracias a la regularidad que se tiene para cada caso, se demuestra (ii) y (iii). □

Nota 1 *Por* (i) *del Teorema 5 y* (ii) *del Teorema 6, tenemos*

$$\left\|u^\varepsilon(x) - u^* - \varepsilon\chi^k \frac{\partial u^*}{\partial x_k}\right\|_{H^1(\mathbb{R}^N)} \leq c\varepsilon.$$

Por lo tanto, se tiene que $\varepsilon\chi^k(\frac{x}{\varepsilon})\frac{\partial u^*}{\partial x_k}(x)$ *es el primer corrector. Así,* θ^ε *equivale al primer corrector, pero además por el apartado* (ii) *del Teorema 5 lo mejora.*

5 Conclusiones

El problema (4) se reduce a la ecuación algebraica

$$\lambda_1^\varepsilon(\xi)B_1^\varepsilon u^\varepsilon(\xi) = B_1^\varepsilon f(\xi) \quad \forall\xi \in \varepsilon^{-1}Y',$$

ya que todos los coeficientes de Bloch mayores que 1 se pueden despreciar, véase la Proposición 3. Entonces, si queremos encontrar los términos correctores, sólo tenemos que expandir asintóticamente el primer coeficiente de Bloch de u^ε en función de f y del primer autovalor λ_1^ε. Además, en las demostraciones de los Teoremas 5 y 6, se utilizó y probó que fuera de un entorno de 0, los coeficientes de Bloch son irrelevantes para nuestro estudio. Así, por la analiticidad del primer autovalor y autofunción de Bloch, la expansión asintótica de u^ε se encuentra utilizando los desarrollos de Taylor de λ_1 y ϕ_1, que conocemos por las Proposiciones 1 y 2.

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SEVERAL QUESTIONS CONCERNING THE CONTROL OF PARABOLIC SYSTEMS

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Abstract

This paper is devoted to recall several recent results concerning the null controllability of some parabolic systems. Among others, we will consider the classical heat equation, the Burgers, Navier-Stokes and Ginzburg-Landau equations, etc.

Key words: *Controllability, linear and nonlinear partial differential equations, parabolic systems.*

AMS subject classifications: *93B05, 35K15*

1 Introduction

Let us first recall some general ideas that can be applied to a large family of (linear and nonlinear) evolution problems.

Suppose that we are considering an abstract *state equation* of the form

$$\begin{cases} y_t - A(y) = Bv, \quad t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{1}$$

which governs the behavior of a physical system. It is assumed that

- $A : D(A) \subset H \mapsto H$ is a (generally nonlinear) operator,
- $y : [0,T] \mapsto H$ is the *state*, i.e. the variable that can be used to identify the properties of the system,
- $v : [0,T] \mapsto U$ is the *control*, i.e. the variable we can choose in order to get good properties,
- $B \in \mathcal{L}(U;H)$ and
- $y^0 \in H$ (for simplicity, we assume that U and H are Hilbert spaces).

Suppose that the state equation is well-posed in the sense that, for each $y^0 \in H$ and each $v \in L^2(0,T;U)$, it possesses exactly one solution. Then the *null controllability* problem for (1) can be stated as follows:

For each $y^0 \in H$, find $v \in L^2(0,T;U)$ such that the corresponding solution of (1) satisfies $y(T) = 0$.

For each system of the form (1), the null controllability problem leads to several interesting questions. Among them, let us mention the following:

- First, are there controls v such that $y(T) = 0$?
- Then, if this is the case, which is the *cost* we have to pay to drive y to zero? In other words, which is the minimal norm of a control $v \in L^2(0,T;U)$ satisfying this property?
- How can these controls be computed?

The controllability of differential systems is a very relevant area of research and has been the subject of many papers the last years. In particular, in the context of partial differential equations, the null controllability problem was first analyzed in [31, 32, 28, 29, 24, 27]. For semilinear systems of this kind, the first contributions have been given in [33, 9, 17].

In this contribution, I will recall some recent results concerning the null controllability of some relevant parabolic systems. More precisely, we will consider the classical heat equation, the Burgers equation and the Navier-Stokes and Ginzburg-Landau systems.

2 The heat equation. Controllability, observability and Carleman estimates

We will first consider the controlled heat equation, complemented with initial and Dirichlet boundary conditions:

$$\begin{cases} y_t - \Delta y = v1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{2}$$

Here (and also in the following Sections), $\Omega \subset \mathbb{R}^N$ is a nonempty bounded domain, $\omega \subset\subset \Omega$ is a (small) nonempty open subset (1_ω is the characteristic function of ω) and $y^0 \in L^2(\Omega)$.

It is well known that, for every $y^0 \in L^2(\Omega)$ and every $v \in L^2(\omega \times (0,T))$, there exists a unique solution y to (2), with

$$y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega)).$$

In this context, the null controllability problem reads:

For each $y^0 \in L^2(\Omega)$, find $v \in L^2(\omega \times (0,T))$ such that the corresponding solution of (2) satisfies

$$y(x,T) = 0 \ \text{ in } \ \Omega. \tag{3}$$

Together with (2), for each $\varphi^1 \in L^2(\Omega)$, we can introduce the associated adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & (x,t) \in \Omega \times (0,T), \\ \varphi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \varphi(x,T) = \varphi^1(x), & x \in \Omega. \end{cases} \tag{4}$$

Then, it is well known that the null controllability of (2) is in practice equivalent to the following property:

There exists $C > 0$ such that

$$\|\varphi(\cdot,0)\|^2_{L^2} \leq C \iint_{\omega\times(0,T)} |\varphi|^2 \, dx\, dt \qquad \forall \varphi^1 \in L^2(\Omega). \tag{5}$$

This is called an observability estimate for the solutions of (4). We thus find that, in order to solve the null controllability problem for (2), it suffices to prove (5).

The estimate (5) is implied by the so called *global Carleman inequalities.* These have been introduced in the context of the controllability of PDEs by Fursikov and Imanuvilov, see [24, 17]. When they are applied to the solutions of the adjoint systems (4), they take the form

$$\iint_{\Omega\times(0,T)} \rho^2 \, |\varphi|^2 \, dx\, dt \leq K \iint_{\omega\times(0,T)} \rho^2 \, |\varphi|^2 \, dx\, dt \qquad \forall \varphi^1 \in L^2(\Omega), \tag{6}$$

where $\rho = \rho(x,t)$ is an appropriate weight, depending on Ω, ω and T and the constant K only depends on Ω and ω.[1]

Combining (6) and the dissipativity of the backwards heat equation (4), it is not difficult to deduce (5) for some C only depending on Ω, ω and T.

As a consequence, we have:

Theorem 1 *The linear system* (2) *is null controllable. In other words, for each $y^0 \in L^2(\Omega)$, there exists $v \in L^2(\omega \times (0,T))$ such that the corresponding solution of* (2) *satisfies* (3).

There are many generalizations and variants of this result that provide the null controllability of other similar linear state equations:

- Time-space dependent (and sufficiently regular) coefficients can appear in the equation, other boundary conditions can be used, boundary control (instead of distributed control) can be imposed, etc. For a review of recent applications of Carleman inequalities to the controllability of parabolic systems, see [12].

[1] In order to prove (6), we have to use a weight ρ decreasing to zero, as $t \to 0$ and also as $t \to T$, for instance exponentially.

- The controllability of Stokes-like systems can also be analyzed with these techniques. This includes systems of the form

$$y_t - \Delta y + (a \cdot \nabla)y + (y \cdot \nabla)b + \nabla p = v1_\omega, \quad \nabla \cdot y = 0, \tag{7}$$

  where a and b are regular enough; see for instance [13].

- Other linear parabolic (non-scalar) systems can also be considered, etc.

As mentioned above, an interesting question related to theorem 1 concerns the cost of null controllability. One has the following result from [15]:

Theorem 2 *For each $y^0 \in L^2(\Omega)$, let us set*

$$C(y^0) = \inf\{\, \|v\|_{L^2(\omega\times(0,T))} : \textit{the solution of } (2) \textit{ satisfies } y(x,T) = 0 \textit{ in } \Omega \,\}.$$

Then we have the following estimate

$$C(y^0) \leq \exp\left(C\left(1 + \frac{1}{T}\right)\right) \|y^0\|_{L^2}, \tag{8}$$

where the constant C only depends on Ω and ω.

Remark 1 Notice that theorem 1 ensures the null controllability of (2) for *any* ω and T. This is a consequence of the fact that, in a parabolic equation, the information is transmitted at infinite speed. This is not the case for the wave equation. Indeed, null controllability does not always hold, for hyperbolic equations. Contrarily, the couple (ω, T) has to satisfy appropriate geometrical assumptions; see [29] and [4] for more details. ■

3 Positive and negative controllability results for the one-dimensional Burgers equation

In this Section, we will be concerned with the null controllability of the following system for the viscous Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_\omega, & (x,t) \in (0,1)\times(0,T), \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1). \end{cases} \tag{9}$$

Some controllability properties of (9) have been studied in [17] (see Chapter 1, theorems 6.3 and 6.4). It is shown there that, in general, a stationary solution of (9) with large L^2-norm cannot be reached (not even approximately) at any time T. In other words, with the help of one control, the solutions of the Burgers equation cannot go anywhere at any time.

For each $y^0 \in L^2(0,1)$, let us introduce

$$T(y^0) = \inf\{\, T > 0 : (9) \text{ is null controllable at time } T \,\}.$$

Then, for each $r > 0$, let us define the quantity

$$T^*(r) = \sup\{\, T(y^0) : \|y^0\|_{L^2} \le r \,\}.$$

Our main purpose is to show that $T^*(r) > 0$, with explicit sharp estimates from above and from below. In particular, this will imply that (global) null controllability at any positive time does not hold for (9).

More precisely, let us set $\phi(r) = (\log \frac{1}{r})^{-1}$. We have the following result from [10]:

Theorem 3 *One has*

$$C_0\phi(r) \le T^*(r) \le C_1\phi(r) \quad as \quad r \to 0, \tag{10}$$

for some positive constants C_0 and C_1 not depending of r.

Remark 2 *The same estimates hold when the control v acts on system (9) through the boundary only at $x = 1$ (or only at $x = 0$). Indeed, it is easy to transform the boundary controlled system*

$$\begin{cases} y_t - y_{xx} + yy_x = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0, \quad y(1,t) = w(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{11}$$

into a system of the kind (9). The boundary controllability of the Burgers equation with two controls (at $x = 0$ and $x = 1$) has been analyzed in [21]. There, it is shown that even in this more favorable situation null controllability does not hold for small time. It is also proved in that paper that exact controllability does not hold for large time.[2]

The proof of the estimate from above in (10) can be obtained by solving the null controllability problem for (9) via a (more or less) standard fixed point argument, using global Carleman inequalities to estimate the control and energy inequalities to estimate the state and being very careful with the role of T in these inequalities.

Let us recall the proof of the estimate from below, that is inspired by the arguments in [1].

Let us show that there exist positive constants C_0 and C_0' such that, for any sufficiently small $r > 0$, we can find initial data y^0 satisfying $\|y^0\|_{L^2} \le r$ with the following property: for any state y associated to y^0, one has

$$|y(x,t)| \ge C_0' r \quad \text{for some} \quad x \in (0,1) \quad \text{and any} \quad t : 0 < t < C_0\phi(r).$$

Thus, let us set $T = \phi(r)$ and let $\rho_0 \in (0,1)$ be such that $(0,\rho_0) \cap \omega = \emptyset$. Notice that this is not restrictive, since it is always possible to work in a suitable open subset $\tilde{\omega} \subset \omega$.

[2]Let us remark that the results in [21] do not allow to estimate $T(r)$; in fact, the proofs are based in contradiction arguments.

We can suppose that $0 < r < \rho_0$. Let us choose $y^0 \in L^2(0,1)$ such that $y^0(x) = -r$ for all $x \in (0, \rho_0)$ and let us denote by y an associated solution of (9).

Let us introduce the function $Z = Z(x,t)$, with

$$Z(x,t) = \exp\left\{-\frac{2}{t}\left(1 - e^{-\rho_0^2(\rho_0 - x)^3/(\rho_0/2 - x)^2}\right) + \frac{1}{\rho_0 - x}\right\}. \tag{12}$$

Then one has $Z_t - Z_{xx} + ZZ_x \geq 0$.

Let us now set $w(x,t) = Z(x,t) - y(x,t)$. It is immediate that

$$\begin{cases} w_t - w_{xx} + ZZ_x - yy_x \geq 0, & (x,t) \in (0,\rho_0) \times (0,T), \\ w(0,t) \geq 0, \quad w(\rho_0,t) = +\infty, & t \in (0,T), \\ w(x,0) = r, & x \in (0,\rho_0). \end{cases} \tag{13}$$

Consequently, $w^-(x,t) \equiv 0$. Indeed, let us multiply the differential equation in (13) by $-w^-$ and let us integrate in $(0,\rho_0)$. Since w^- vanishes at $x = 0$ and $x = \rho_0$, after some manipulation we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^{\rho_0} |w^-|^2\,dx + \int_0^{\rho_0} |w_x^-|^2\,dx \\ &\qquad = \int_0^{\rho_0} w^-(ZZ_x - yy_x)\,dx \leq C\int_0^{\rho_0} |w^-|^2\,dx. \end{aligned} \tag{14}$$

Hence,

$$y \leq Z \quad \text{in } (0,\rho_0) \times (0,T). \tag{15}$$

Let us set $\rho_1 = \rho_0/2$ and let $\widetilde{r}$ be a regular function satisfying the following: $\widetilde{r}(0) = \widetilde{r}(\rho_1) = 0$; $\widetilde{r}(x) = r$ for all $x \in (\delta\rho_1, (1-\delta)\rho_1)$ and some $\delta \in (0, 1/4)$; $-r \leq -\widetilde{r}(x) \leq 0$;

$$|\widetilde{r}_x| \leq Cr \ \text{ and } \ |\widetilde{r}_{xx}| \leq C \ \text{ in } \ (0,\rho_1), \tag{16}$$

where $C = C(\rho_1)$ is independent of r.

Let us introduce the solution u of the auxiliary system

$$\begin{cases} u_t - u_{xx} + uu_x = 0, & (x,t) \in (0,\rho_1) \times (0,T), \\ u(0,t) = Z(\rho_1,t), \quad u(\rho_1,t) = Z(\rho_1,t), & t \in (0,T), \\ u(x,0) = -\widetilde{r}(x), & x \in (0,\rho_1). \end{cases} \tag{17}$$

We need the following lemma, whose proof can be found in [10]:

Lemma 4 *One has*

$$|u| \leq Cr \ \text{ and } \ |u_x| \leq Cr^{1/2} \quad \text{in } (0,\rho_1) \times (0,\phi(r)), \tag{18}$$

where C is independent of r.

Taking into account (15) and that $u_x, y \in L^\infty((0,\rho_1)\times(0,T))$ (see lemma 4 below), a standard application of Gronwall's lemma shows that

$$y \le u \ \text{ in } \ (0,\rho_1)\times(0,T). \tag{19}$$

On the other hand, we see from (18) that $u_t - u_{xx} \le C^* r^{3/2}$ in $(0,\rho_1)\times(0,\phi(r))$ for some $C^* > 0$. Let us consider the functions p and q, given by $p(t) = C^* r^{3/2} t - r$ and $q(x,t) = c(e^{-(x-(\rho_1/4))^2/4t} + e^{-(x-3(\rho_1/4))^2/4t})$. It is then clear that $b = u - p - q$ satisfies

$$b_t - b_{xx} \le 0 \ \text{ in } \ (\rho_1/4, 3\rho_1/4)\times(0,\phi(r)),$$

$$b(\rho_1/4,t) \le Z(\rho_1,t) - C^* r^{3/2} t + r - c(1 + e^{-\rho_1^2/(16t)}) \ \text{ for } \ t \in (0,\phi(r)),$$

$$b(3\rho_1/4,t) \le Z(\rho_1,t) - C^* r^{3/2} t + r - c(1 + e^{-\rho_1^2/(16t)}) \ \text{ for } \ t \in (0,\phi(r)),$$

$$b(x,0) = 0 \ \text{ for } \ x \in (\rho_1/4, 3\rho_1/4).$$

Obviously, in the definition of q the constant c can be chosen large enough to have $Z(\rho_1,t) - C^* r^{3/2} t + r - c(1 + e^{-\rho_1^2/(16t)}) < 0$ for any $t \in (0,\phi(r))$. If this is the case, we get $u \le p + q$ and, in particular,

$$u(\rho_1/2,t) \le (p+q)(\rho_1/2,t) = 2ce^{-\rho_1^2/(64t)} + C^* r^{3/2} t - r.$$

Therefore, we see that there exist C_0 and C_0' such that $u(\rho_1/2,t) < -C_0' r$ for any $t \in (0, C_0\phi(r))$.

This proves the first inequality in (10) and, consequently, ends the proof of theorem 3.

4 Other more realistic nonlinear equations and systems

There are a lot of more realistic nonlinear equations and systems from mechanics that can also be considered. First, we have the well known Navier-Stokes equations:

$$\begin{cases} y_t + (y\cdot\nabla)y - \Delta y + \nabla p = v1_\omega, \quad \nabla\cdot y = 0, & (x,t) \in Q, \\ y = 0, & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{20}$$

Here and below, Q and Σ respectively stand for the sets

$$Q = \Omega\times(0,T) \ \text{ and } \ \Sigma = \partial\Omega\times(0,T),$$

where $\Omega \subset \mathbb{R}^N$ is a nonempty bounded domain, $N = 2$ or $N = 3$ and (again) $\omega \subset\subset \Omega$ is a nonempty open set.

The controllability of this system has been analyzed in [13] and [14].[3] Essentially, these results establish the local exact controllability of the solutions

[3]The main ideas come from [18, 25]; some additional results will appear soon in [22] and [19]; for other control results concerning the Navier-Stokes equations, see [6, 7].

of (20) to uncontrolled trajectories (this is, more or less, the analog of the *positive* controllability result in theorem 3).

Similar results have been given in [20] for the Boussinesq equations

$$\begin{cases} y_t + (y \cdot \nabla)y - \Delta y + \nabla p = \theta k + v1_\omega, \quad \nabla \cdot y = 0, \\ \theta_t + y \cdot \nabla\theta - \Delta\theta = u1_\omega, \end{cases} \tag{21}$$

complemented with initial and Dirichlet boundary conditions for y and θ (see [14] for a controllability result with a reduced number of scalar controls).

Let us also mention [3, 23], where the controllability of the MHD and other related equations has been analyzed.

Another system is considered in [11]:

$$\begin{cases} y_t + (y \cdot \nabla)y - \Delta y + \nabla p = \nabla \times w + v1_\omega, \quad \nabla \cdot y = 0, \\ w_t + (y \cdot \nabla)w - \Delta w - \nabla(\nabla \cdot w) = \nabla \times y + u1_\omega. \end{cases} \tag{22}$$

Here, $N = 3$. These equations govern the behavior of a micropolar fluid, see [30]. As usual, y and p stand for the velocity field and pressure and w is the microscopic velocity of rotation of the fluid particles. Again, the local exact controllability of the solutions to the trajectories is established.

Notice that this case involves a nontrivial difficulty. The main reason is that w is a nonscalar variable and the equations satisfied by its components w_i are coupled through the second-order terms $\partial_i(\nabla \cdot w)$. This is a serious inconvenient and an appropriate strategy has to be applied in order to deduce the required Carleman estimates.

For these systems, the proof of the controllability can be achieved arguing as in the first part of the proof of theorem 3. This is the general structure of the argument:

- First, consider a linearized similar problem and the associated adjoint system and rewrite the original controllability problem in terms of a fixed point equation.
- Then, prove a global Carleman inequality and an observability estimate for the adjoint system. This provides a controllability result for the linearized problem.
- Prove appropriate estimates for the control and the state (this needs some kind of *smallness* of the data); prove an appropriate compactness property of the state and deduce that there exists at least one fixed point.

There is an alternative method that relies on the implicit function theorem. It corresponds to another strategy introduced in [17]:

- First, rewrite the original controllability problem as a nonlinear equation in a space of admissible "state-control" pairs.

- Then, prove an appropriate global Carleman inequality and a regularity result and deduce that the linearized equation possesses at least one solution. Again, this provides a controllability result for a related linear problem.
- Check that the hypotheses of a suitable implicit function theorem are satisfied and deduce a local result.

At present, no negative result is known to hold for these nonlinear systems (apart from the one-dimensional Burgers equation).

5 Some remarks on the Ginzburg-Landau equation

This Section is concerned with the controllability of the Ginzburg-Landau equation. The system under consideration is the following:

$$\begin{cases} m_t - \alpha m \times m_t - \Delta m + \dfrac{|m|^2 - 1}{\varepsilon}\, m = v 1_\omega, & (x,t) \in Q, \\ \dfrac{\partial m}{\partial n} = 0, & (x,t) \in \Sigma, \\ m(x,0) = m^0, & x \in \Omega. \end{cases} \tag{23}$$

Here, $\Omega \subset \mathbb{R}^3$ is a regular bounded open set, $m = (m_1, m_2, m_3)$ is the *magnetization* field, $\varepsilon > 0$ is a parameter, $\alpha \geq 0$ is a physical constant and it is assumed that m^0 is a measurable initial field satisfying $|m^0(x)| \equiv 1$. For the motivation of the system satisfied by m, see for instance [5].

In this framework, an interesting controllability problem is the following:

> *Given a stationary solution $m^* = m^*(x)$ and an initial field $m^0 = m^0(x)$ with $|m^*(x)| \equiv |m^0(x)| \equiv 1$, find a control $v \in L^2(\omega \times (0,T))^3$ such that the associated solution of* (23) *satisfies*
>
> $$m(x,T) = m^* \quad \text{in} \quad \Omega.$$

By introducing the new variable y, with $m = m^* + y$, this can be rewritten in terms of a null controllability problem. Indeed, let us consider the system

$$\begin{cases} y_t - \alpha(y + m^*) \times y_t - \Delta y + G_\varepsilon(x,y)y = v 1_\omega, & (x,t) \in Q, \\ \dfrac{\partial y}{\partial n} = 0, & (x,t) \in \Sigma, \\ y(x,0) = m^0(x) - m^*(x), & x \in \Omega, \end{cases} \tag{24}$$

where

$$G_\varepsilon(x,y)y \equiv \frac{|m^*(x) + y|^2 - 1}{\varepsilon}\,(m^*(x) + y) - \frac{|m^*(x)|^2 - 1}{\varepsilon}\, m^*(x).$$

Then the problem is:

Given a stationary solution $m^* = m^*(x)$ and an initial field $m^0 = m^0(x)$ with $|m^*(x)| \equiv |m^0(x)| \equiv 1$, find a control $v \in L^2(\omega \times (0,T))^3$ such that the associated solution of (24) *satisfies*

$$y(x,T) = 0 \quad in \quad \Omega.$$

A partial (positive) answer to this problem is given in [8].[4]

More precisely, it is shown there that there exists $\kappa = \kappa(\Omega, \omega, T, \alpha, \varepsilon)$ such that, whenever $\|m^* - m^0\|_{L^2} \leq \kappa$, the existence of such controls can be ensured.

Remark 3 *For any fixed v, the solutions of (23) converge in some sense as $\varepsilon \to 0$ to a solution of the so called Landau-Lifshitz equation:*

$$\alpha m_t = m \times (\Delta m - m_t + v1_\omega), \quad |m| = 1. \tag{25}$$

Consequently, it would be very interesting to be able to solve the previous problem with controls v uniformly bounded with respect to ε. However, this is apparently a difficult question.

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FRACTIONAL TIME-DERIVATIVE OF SOME EVOLUTION PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

We study the problem of the existence of fractional time-derivative of some nonlinear partial differential equations of fluid mechanics. We use a result due to J. Simon on Nikolskii spaces and the classical Hardy-Littlewood maximal Theorem to obtain results similar to those given by J.-L Lions using the Fourier transform.

Key words: *fractional time-derivative, Nikolskii spaces, evolution partial differential equations.*
AMS subject classifications: *35B65, 35Q10, 76D05.*

1 Introduction

The first work that recognized the importance of the existence of fractional time-derivative in partial differential equations was carried out by Lions [7].

One of the main techniques to show the existence of solutions in evolution partial differential equations is the Galerkin method. This method requires uniform a priori estimates of the approximate solutions and to take the limit in the nonlinear terms, the strong convergence of the approximate solutions is needed.

To obtain this convergence, the so called "compactness theorems"are usual. One of the well known "compactness theorems"is Aubin-Lions' Lemma [[8], p. 58]. To apply this lemma, some information on the time-derivatives of the approximate solutions is needed. Usually this requires, among other things, to consider the special basis for Galerkin approximation. When one takes other

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basis, it is not possible to obtain the information on the time-derivative of the approximate solutions, which from computational view point is very important. There exist also problems where even by using the special basis it is not possible to obtain such information, for example for nonhomogeneous fluids (see [1], [14]) or for diffusion fluids (see [2]). Consequently, generalizations of the Aubin-Lions Lemma are desirable.

By using the Fourier transform, Lions proved in [7] a compactness theorem of the Aubin-Lions kind, where the required information is not on the usual time-derivative, but on a fractional time-derivative. However, the result is only valid in the L^2-context. Later, Simon [[14], p. 1097] extended these results using Nikolskii spaces and Lions [[8], p. 77] proved that the approximate solutions of the Navier-Stokes equations constructed with an arbitrary basis have uniformly bounded fractional time-derivatives $D_t^\gamma u^k$ for all $0 \leq \gamma < 1/4$ (this result being true for n-dimensional bounded domains, $n = 2, 3, 4$). Later, for the Shinbrot's conjecture that the weak solution of the Navier-Stokes equations has a fractional time-derivative $D_t^\gamma u$ for $0 \leq \gamma < 1/2$, in [11] Shinbrot provided an incorrect prove (he pointed out this fact in [[12], p. 179]).

In Zhang [16], by using the classical maximality Theorem due to Hardy-Littlewood together with results of Simon [14], it was proved that this conjecture is true in the two-dimensional case and partially in the three and fourth dimensional cases (see Section 3).

The main purpose of this work is to provide an abstract result (see Proposition 2.6) that can be used to obtain and estimate the fractional time-derivatives and to show how to use this result in some evolution partial differential equations from fluids mechanics. The ideas developed in this work can be adapted to study other evolution partial differential equations.

The paper is organized as follows. In Section 2, we give some preliminary results and we prove our abstract result. Section 3 is devoted to apply this result to the classical Navier-Stokes equations and the generalized Boussinesq system.

2 Preliminaries and results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. we denote by $L^p(\Omega)$ the usual Lebesque spaces and by $\|\cdot\|_{L^p}$ the L^p-norm on Ω; in the case $p = 2$, we simply denote the L^2-norm by $|.|$ and the corresponding inner product by $(\cdot, \cdot)$. When B is a Banach spaces, we denote by $L^q(0, T; B)$ the Banach space of the B-valued functions defined in the interval $(0, T)$ that are L^q-integrable in the sense of Bochner. We will denote the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D) \,/\, \|\partial^\alpha f\|_{L^q} < \infty, |\alpha| \leq m\},$$
$$m = 0, 1, 2, \ldots, 1 \leq q \leq \infty,\ D = \Omega \text{ or } D = (0, T) \times \Omega,\ 0 < T < \infty$$

with the usual norm. We denote by $H^m(D) = W^{m,2}(D)$ and $H_0^m(D)$=closure of $C_0^\infty(D)$ in $H^m(D)$. Also, we consider $H^s(\Omega)$, with $s \in \mathbb{R}$, as defined in Lions [8].

Let I be an interval of $\mathbb{R}$ and let B be a Banach space. The fractional Sobolev spaces are defined for $0 < s < 1,\ 1 \le p < \infty$ by

$$W^{s,p}(I;B) = \{f \in L^p(I;B) \,/\, \|f\|^p_{W^{s,p}} = \int_{I\times I} \left(\frac{\|f(y)-f(x)\|_B}{|y-x|^s}\right)^p \frac{dydx}{|y-x|} < \infty\},$$

when $p = 2$, we observe that $W^{s,2}$ coincides with H^s as defined in Lions [8].

For any $h \ge 0$ we set $I_h = \{t \in I \,/\, t + h \in I\}$ a decreasing sequence of subintervals of I and denote by $u_h(\cdot)$ the function $u_h(t) = u(t + h) - u(t)$. Given $f \in L^p(I;B)(1 \le p < \infty)$, then f, f_h and $f_h + f$ are all defined in I_h.

The Nikolskii spaces are defined for $0 < s < 1,\ 1 \le p < \infty$ by

$$N^{s,p}(I;B) = \{f \in L^p(I;B) \,/\, \rho(f) = \sup_{h>0} \frac{1}{h^s} \left(\int_{I_h} \|f_h(t)\|^p_B dt\right)^{1/p} < \infty\}.$$

We observe that the condition $\rho(f) < \infty$ is equivalent to

$$\|f\|_{L^p(0,T;B)} \le c\, h^s, \quad \forall\, h \in I,$$

where $c > 0$ is a positive constant.

The Nikolskii space $N^{s,p}(I;B)$ together with the following norm

$$\|f\|_{N^{s,p}(I;B)} = \|f\|_{L^p(I;B)} + \rho(f)$$

is a Banach space. An important result that relates the Sobolev and Nikolskii spaces is the following due to Simon [[13], p. 141, Corollary 24].

Proposition 1 *We assume $s > r$ $(0 < r < s < 1,\ 1 \le p < \infty)$. Then*

$$W^{s,p}(I;B) \hookrightarrow N^{s,p}(I;B) \hookrightarrow W^{r,p}(I;B).$$

Remark 1 *In the case $p = 2$, we have*

$$W^{s,2}(I;B) = H^s(I;B) \hookrightarrow N^{s,2}(I;B) \hookrightarrow W^{r,2}(I;B) = H^s(I;B),$$

consequently, if $f \in N^{s,2}(I;B),\ 0 < s < 1$, then $D^r_t f \in L^2(I;B),\ \ \forall\, r < s$, i.e., f has "fractional derivative"of order r in $L^2(I;B)$.

The following compactness theorem is given in Simon [13].

Theorem 2 *Let $X \hookrightarrow E \hookrightarrow Y$ Banach spaces, we assume that the imbedding $X \hookrightarrow E$ is compact. Then the following imbedding are compact:*
(i) $L^q(0,T;X) \cap \{\varphi \,/\, \varphi_t \in L^1(0,T;Y)\} \hookrightarrow L^q(0,T;E)$ if $1 \le q \le \infty$,
(ii) $L^\infty(0,T;X) \cap \{\varphi \,/\, \varphi_t \in L^r(0,T;Y)\} \hookrightarrow C([0,T];E)$ if $1 < r \le \infty$,
(iii) For every function given $k \in L^1(0,T)$, $k \ge 0$ and $1 < r \le \infty$,

$$L^\infty(0,T;X) \cap \{\varphi \,/\, \|\varphi_t\|_Y - k \in L^r(0,T)\} \hookrightarrow C([0,T];E),$$

iv) $L^q(0,T;X) \cap N^{s,q}(0,T;Y) \hookrightarrow L^q(0,T;E)$ if $s > 0,\ 1 \le q \le \infty$.

Remark 2 *The Aubin-Lions compactness theorem [[8], p. 58] is a particular case of (i). Moreover, the Theorem 5.2 in Lions [8] is a particular case of (iv).*

The following result is classic and is fundamental in our next arguments.

Lemma 3 (Hardy-Littlewood Lemma) *Suppose that $\tilde{f}$ is integrable over $(0,a)$. For every $t,\ 0<t\leq a$, we set*

$$\theta(t) = \sup_{\zeta}\frac{1}{t-\zeta}\int_{\zeta}^{t}\tilde{f}(s)ds, \quad \textit{where} \quad 0\leq\zeta<t,$$

$$\tilde{\theta}(t) = \sup_{\zeta}\frac{1}{\zeta-t}\int_{t}^{\zeta}\tilde{f}(s)ds, \quad \textit{where} \quad t<\zeta\leq a \quad \textit{and} \quad G(t)=\max\{\theta(t),\tilde{\theta}(t)\}.$$

Then, if $\tilde{f}\in L^r(0,a), r>1$, we have $G\in L^r(0,a)$ and

$$\int_0^a G^r(t)dt \leq 2\left(\frac{r}{r-1}\right)^r\int_0^a \tilde{f}^r(t)dt.$$

Subsequently, we give our following results

Proposition 4 *Let X and Y Banach spaces . We assume that $X\hookrightarrow Y$, $w\in L^2(0,T;X)$ and*

$$\int_0^{T-h}\|w_h(t)\|_Y^2dt \leq \int_0^{T-h}\|w_h(t)\|_X\int_t^{t+h}g(\tau)\,d\tau\,dt. \tag{1}$$

Then,
a) If $g\in L^1(0,T)$ has that $w\in N^{1/4,2}(0,T;Y)$ and $\|w\|_{N^{\frac{1}{4},2}(0,T;Y)}\leq C\|w\|^{1/2}_{L^2(0,T;X)}$
where C is a constant positive that only depend on $\|g\|_{L^1(0,T)}$.
b) If $g\in L^2(0,T)$ has that $w\in N^{1/2,2}(0,T;Y)$ and $\|w\|_{N^{\frac{1}{2},2}(0,T;Y)}\leq C\|w\|^{1/2}_{L^2(0,T;X)}$
where C is a constant positive that only depend on $\|g\|_{L^2(0,T)}$.

Proof.
a) By using the Fubini's Theorem in the right-hand side of (1), we get

$$\int_0^{T-h}\int_t^{t+h}g(\tau)\|w_h(t)\|_X d\tau\,dt = \int_0^T g(\tau)\int_{\overline{\tau-h}}^{\overline{\tau}}\|w_h(t)\|_X dt\,d\tau, \tag{2}$$

where $\overline{\tau}=0$ if $\tau\leq 0$, $\overline{\tau}=\tau$ if $0\leq\tau\leq T-h$, $\overline{\tau}=T-h$ if $\tau>T-h$. By other hand,

$$\begin{aligned}\int_{\overline{\tau-h}}^{\overline{\tau}}\|w_h(t)\|_X dt &\leq \left(\int_{\overline{\tau-h}}^{\overline{\tau}}1^2dt\right)^{1/2}\left(\int_{\overline{\tau-h}}^{\overline{\tau}}\|w_h(t)\|_X^2dt\right)^{1/2} \\ &\leq 2\,h^{1/2}\|w\|_{L^2(0,T;X)}.\end{aligned} \tag{3}$$

Consequently, setting (3) in (2), we get

$$\int_0^{T-h}\int_t^{t+h} g(\tau)\|w_h(t)\|_X d\tau\, dt \le 2\,h^{1/2}\,\|w\|_{L^2(0,T;X)}\int_0^T g(\tau)\,d\tau \le c_1\,h^{1/2}\|w\|_{L^2(0,T;X)} \tag{4}$$

where $c_1 = 2\,\|g\|_{L^1(0,T)}$. Then, from (4) and (1), we obtain

$$\frac{1}{h^{1/2}}\int_0^{T-h}\|w_h(t)\|_Y^2 dt \le c_1\,\|w\|_{L^2(0,T;X)},$$

which implies (a) with $C = \sqrt{c_1}$.

b) In this case, we will estimate better the right-hand side of (1).

We observe that

$$\begin{aligned}\int_0^{T-h}\int_t^{t+h} g(\tau)\|w_h(t)\|_X d\tau\, dt &= h\int_0^{T-h}[\,\|w_h(t)\|_X\frac{1}{h}\int_t^{t+h} g(\tau)\,d\tau\,]\,dt\\ &\le h\int_0^{T-h}\|w_h(t)\|_X\theta(t)\,dt \end{aligned}\tag{5}$$

where $\theta(t) = \sup_{h>0}\dfrac{1}{h}\displaystyle\int_t^{t+h} g(\tau)\,d\tau$.

By using the Cauchy-Schwarz inequality in the right-hand side of (5), we obtain

$$\begin{aligned}\int_0^{T-h}\int_t^{t+h} g(\tau)\|w_h(t)\|_X d\tau\, dt &\le h\left(\int_0^{T-h}\|w_h(t)\|_X^2 dt\right)^{1/2}\left(\int_0^{T-h}\theta^2(t)\,dt\right)^{1/2}\\ &\le h\,\|w\|_{L^2(0,T;X)}\left(\int_0^{T-h}\theta^2(t)\,dt\right)^{1/2}.\end{aligned}\tag{6}$$

Applying the Hardy-Littlewood's Lemma to the second integral of (6), with $r = 2$, $a = T-h$ e $\tilde{f} = g$, we have

$$\int_0^{T-h}\theta^2(t)\,dt \le 8\int_0^{T-h} g^2(t)\,dt \le 8\|g\|_{L^2(0,T)}^2. \tag{7}$$

Consequently, from (1) together (6) and (7), we get

$$\frac{1}{h}\int_0^{T-h}\|w_h(t)\|_Y^2 dt \le \sqrt{8}\,\|g\|_{L^2(0,T)}\|w\|_{L^2(0,T;X)} \equiv c_2\,\|w\|_{L^2(0,T;X)}$$

where $c_2 = \sqrt{8}\,\|g\|_{L^2(0,T)}$ and this implies (b) with $C = \sqrt{c_2}$. □

Taking into account Remark 1, we have the following result.

Corollary 5 *Under the hypotheses of the Proposition 4, we have*

$$D_t^\alpha w \in L^2(0,T;Y)$$

with $0 \le \alpha < 1/4$ *if (a) is true and* $0 \le \alpha < 1/2$ *if (b) is true.*

The following result is an extension of the Proposition 4 and the proof is analogous.

Proposition 6 *Let X and Y Banach spaces, assume $X \hookrightarrow Y$, $w \in L^p(0,T;X)(1<p<\infty)$, $1 \le r < p$ and*

$$\int_0^{T-h} \|w_h(t)\|_Y^p dt \le \int_0^{T-h} \|w_h(t)\|_X^r \int_t^{t+h} g(\tau)\, d\tau\, dt.$$

Then,

a) If $g \in L^1(0,T)$, we have $w \in N^{\frac{p-r}{p^2},p}(0,T;Y)$ and $\|w\|_{N^{\frac{p-r}{p^2},p}(0,T;Y)} \le C\|w\|^{\frac{r}{p}}_{L^p(0,T;X)}$ where C is a constant positive that only depend on $\|g\|_{L^1(0,T)}$.

b) If $g \in L^{\frac{p}{p-r}}(0,T)$, we have $w \in N^{\frac{1}{p},p}(0,T;Y)$ and $\|w\|_{N^{\frac{1}{p},p}(0,T;Y)} \le C\|w\|^{\frac{r}{p}}_{L^p(0,T;X)}$ where C is a constant positive that only depend on $\|g\|_{L^{\frac{p}{p-r}}(0,T)}$.

3 Applications

3.1 The Navier-Stokes Equations

Our first application is in the classical Navier-Stokes equations. We refer to Ladyzhenskaya [6], Lions [8] or Temam [15] to more information.

In the domain $Q = (0,T)\times\Omega$, we consider the following system of equations that describes the motion of a viscous incompressible fluid:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \ \text{ in } \ Q,\\ &\operatorname{div}\mathbf{u} = 0 \ \text{ in } \ Q,\\ &\mathbf{u}(t,x) = \mathbf{0} \ \text{ on } \ (0,T)\times\partial\Omega \ \text{ and } \ \mathbf{u}(0,x) = \mathbf{u}_0(x) \ \text{ on } \ \Omega. \end{aligned} \tag{8}$$

Here, $\mathbf{u}(t,x) \in \mathbb{R}^n$ denotes the velocity of the fluid at a point $x \in \Omega$ and time $t \in [0,T]$, $p(t,x) \in \mathbb{R}$ the hydrostatic pressure and $\mathbf{f}(t,x) \in \mathbb{R}^n$, is a given external field.

For reference, let

$$\begin{aligned} C^\infty_{0,\sigma}(\Omega) &= \{\mathbf{v} \in (C_0^\infty(\Omega))^n \ ; \ \operatorname{div}\mathbf{v} = 0 \ \text{ in } \ \Omega\},\\ \mathbf{H} &= \text{closure of } C^\infty_{0,\sigma}(\Omega) \text{ in } \mathbf{L}^2(\Omega) - \text{norm},\\ \mathbf{V}_s &= \text{closure of } C^\infty_{0,\sigma}(\Omega) \text{ in } \mathbf{H}^s(\Omega) - \text{norm}. \end{aligned}$$

By simplicity, we denote $\mathbf{V} = \mathbf{V}_1$, $\mathbf{L}^2 = \mathbf{L}^2(\Omega)$ and $\mathbf{H}^m = \mathbf{H}^m(\Omega)$. Let P be the orthogonal projection from $\mathbf{L}^2$ on $\mathbf{H}$, then

$$\mathbf{H}^\perp = \{\phi \in \mathbf{L}^2 \ ; \ \text{exist } p \in H^1 \text{ with } \phi = \nabla p\}.$$

The Stokes Operator $A = -P\Delta$ is defined on $D(A) = \mathbf{V}\cap\mathbf{H}^2$. It is well known that A is a positive definite, self-adjoint operator and is characterized by the relation

$$(A\mathbf{w},\mathbf{v}) = (\nabla\mathbf{w},\nabla\mathbf{v}), \quad \forall\,\mathbf{w}\in D(A), \ \forall\,\mathbf{v}\in\mathbf{V}.$$

The weak formulation of problem (8) is

$$\begin{aligned} &\frac{d}{dt}(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle, \quad \forall\, \mathbf{v} \in \mathbf{V}\cap\mathbf{L}^n \\ &\mathbf{u}(0) = \mathbf{u}_0, \end{aligned} \tag{9}$$

where $a(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial v_j}{\partial x_i}\frac{\partial w_j}{\partial x_i} dx$ and $b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{n} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx$.

Definition 1 *Let $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{f} \in L^2(0,T;\mathbf{V}^*)$. We will say that $\mathbf{u}$ is a weak solution of the (8) if and only if $\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$ and satisfies (9).*

The following result on existence of weak solutions is well known (see Lions [8], p. 74-79).

Theorem 7 *If $\mathbf{f} \in L^2(0,T;\mathbf{V}^*)$, $\mathbf{u}_0 \in \mathbf{H}$, then there exist a weak solution $\mathbf{u}$ of (8). Furthermore,*

$$\mathbf{u} \in C([0,T];\mathbf{V}^*_{(n-2)/4}) \cap C_{\mathrm{w}}([0,T];\mathbf{H}). \tag{10}$$

If $n = 2$, the weak solution $\mathbf{u}$ is unique and

$$\mathbf{u} \in C([0,T];\mathbf{H}). \tag{11}$$

Remark 3 *It should be noted that the precise meaning of $\mathbf{u} \in C([0,T];B)$ in (3.3) (with $B = \mathbf{V}^*_{(n-2)/4}(\Omega)$) or in (3.4) (with $B = \mathbf{H}$) is that $\mathbf{u}$ is almost everywhere equal to a continuous function from $[0,T] \to B$.*

Lions [8] proved the above theorem in two ways. One of them was by using the special basis for Galerkin approximations valid for any $n \geq 2$. The second method was using any arbitrary basis of the separable Hilbert space $V \cap \mathbf{L}^n$. But this proof is valid only for $n \leq 4$. The second proof is clearly more interesting from the computational view point, in this proof he prove the existence of fractional time-derivative for the approximations, $D_t^\gamma \mathbf{u}^k$ for $0 \leq \gamma < 1/4$ and consequently for $\mathbf{u}$.

In fact, we have.

Proposition 8 *For the weak solution $\mathbf{u}$ obtained in Theorem 7, $D_t^\gamma \mathbf{u} \in L^2(0,T;\mathbf{H})$ for $0 \leq \gamma < 1/4$ if $n = 3,4$ and $0 \leq \gamma < 1/2$ if $n = 2$.*

Proof. We observe that the following estimates should be proved first for the approximations $\mathbf{u}^k$ and then carried to $\mathbf{u}$ in the limit, Since that is a standard procedure and all the computations with the approximate solutions are exactly the same as those to be done formally with the solution, to simplify the notation, we will work directly with $\mathbf{u}$ in the rest of paper. Since $\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$, integrating the weak formulation (9) from t to $t+h$, we obtain

$$\begin{aligned} (\mathbf{u}_h(t),\mathbf{v}) &= \int_t^{t+h} \left(\langle \mathbf{f}(\tau),\mathbf{v}\rangle - a(\mathbf{u}(\tau),\mathbf{v}) + b(\mathbf{u}(\tau),\mathbf{v},\mathbf{u}(\tau)) \right) d\tau \\ &\leq \|\mathbf{v}\|_{\mathbf{V}} \int_t^{t+h} \left(g_1(\tau) + g_2(\tau) \right) d\tau \end{aligned} \tag{12}$$

where $g_1(\tau) = \|\mathbf{f}(\tau)\|_{\mathbf{V}^*} + \|\mathbf{u}(\tau)\|_{\mathbf{V}}$ and $g_2(\tau) = \|\mathbf{u}(\tau)\|_{\mathbf{L}^p}\|\mathbf{u}(\tau)\|_{\mathbf{L}^q}$ with $1/p + 1/q = 1/2$.

Setting $\mathbf{v} = \mathbf{u}_h(t)$ in (12) and integrating from 0 to $T-h$, we get

$$\int_0^{T-h} \|\mathbf{u}_h(t)\|^2 dt \leq \int_0^{T-h} \|\mathbf{u}_h(t)\|_{\mathbf{V}} \int_t^{t+h} (\, g_1(\tau) + g_2(\tau)\,)\, d\tau\, dt. \tag{13}$$

We observe that $g_1 \in L^2(0,T)$ for any $n \geq 2$, $g_2 \in L^2(0,T)$ only if $n = 2$, and $g_2 \in L^1(0,T)$ if $n = 3, 4$. Consequently, $g(t) = g_1(t) + g_2(t)$ belong to $L^2(0,T)$ if $n = 2$ and to $L^1(0,T)$ if $n \geq 3$. Then, applying Proposition 4 in (13), we obtain the desired result. □

Remark 4 *If we assume that* $\mathbf{u} \in L^4(0,T;\mathbf{L}^4)$*, then* $g \in L^2(0,T)$ *for* $n = 3, 4$ *and consequently we obtain the result of Zhang [16]. But, this is not known for the weak solutions.*

The Proposition 4 is also useful for strong solution. First, we recall the following theorem, see for instance Ladyzhenskaya [6], Heywood [5], Temam [15].

Theorem 9 *If* $\mathbf{u}_0 \in D(A)$ *and* $\mathbf{f}$, $\mathbf{f}_t \in L^2(0,T;\mathbf{L}^2)$*, then the problem (8) has an unique strong solution. Moreover, this solution satisfies*

$$\mathbf{u} \in L^\infty(0,T^*;D(A)), \quad \mathbf{u}_t \in L^\infty(0,T^*;\mathbf{H}) \cap L^2(0,T^*;\mathbf{V}), \quad \mathbf{u}_{tt} \in L^2(0,T^*;\mathbf{V}^*),$$

where $T^* = T$ *if* $n = 2$ *and* $0 < T^* \leq T$ *if* $n = 3$.

By using the Proposition 4, we can prove the following result.

Proposition 10 *The strong solution given by the Theorem 9, satisfies*

$$D_t^\gamma \mathbf{u}_t \in L^2(0,T^*;\mathbf{H}), \quad \forall\, 0 \leq \gamma < 1/2. \tag{14}$$

Proof. In fact, differentiating the weak formulation (9) with respect to t, we have

$$(\mathbf{u}_{tt}, \mathbf{v}) = (\mathbf{f}_t, \mathbf{v}) - (\mathbf{u}_t \cdot \nabla \mathbf{u}, \mathbf{v}) - (\mathbf{u} \cdot \nabla \mathbf{u}_t, \mathbf{v}) - (\nabla \mathbf{u}_t, \nabla \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{V}. \tag{15}$$

Integrating (15) from t to $t + h \leq T^*$, we get

$$((\mathbf{u}_t)_h(t), \mathbf{v}) \leq \|\mathbf{v}\|_{\mathbf{V}} \int_t^{t+h} g(\tau)\, d\tau \tag{16}$$

where $g(\tau) = \|\mathbf{f}_t(\tau)\|_{\mathbf{V}^*} + c\,\|\mathbf{u}_t(\tau)\|\,\|A\mathbf{u}(\tau)\| + c\,\|A\mathbf{u}(\tau)\|\,\|\mathbf{u}_t(\tau)\|_{\mathbf{V}} + \|\mathbf{u}_t(\tau)\|_{\mathbf{V}}$.

Setting $\mathbf{v} = (\mathbf{u}_t)_h(t)$ in (16) and integrating the result from 0 to $T^* - h$, we obtain

$$\int_0^{T^*-h} \|(\mathbf{u}_t)_h(t)\|^2\, dt \leq \int_0^{T^*-h} \|(\mathbf{u}_t)_h(t)\|_{\mathbf{V}} \int_t^{t+h} g(\tau)\, d\tau\, dt. \tag{17}$$

We observe that Theorem 9 implies $g \in L^2(0,T^*)$, consequently the Proposition 4 and (17) implies (14). □

Remark 5 *Analogous results are proved for the micropolar, magnetohydrodynamic equations, see [3], [10].*

Remark 6 *In the case of strong solution the dimension of* $\mathbb{R}^n$ *(*$n = 2$ *or 3) is not important as in the case of weak solutions.*

An open question important in the theory of Navier-Stokes equations is the uniqueness of weak solutions for $n \geq 3$. We believe that this problem is equivalent to existence of the fractional time-derivative $D_t^\gamma \mathbf{u}$ of weak solution for $0 \leq \gamma < 1/2$.

3.2 The Generalized Boussinesq Equations

Our second application is in the Generalized Boussinesq equations. These system of equations describe the motion of a viscous, incompressible heat conducting fluids with viscosity and thermal conductivity depending on the temperature.

Let $\mathbf{u}(t,x) \in \mathbb{R}^n, \theta(t,x) \in \mathbb{R}, p(t,x) \in \mathbb{R}$ be the velocity, the temperature and the pressure of the fluid respectively. The motion of the fluid is described by the initial boundary-valued problem:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot (\mu(\theta) \nabla \mathbf{u}) + \operatorname{grad} p = \alpha\, \theta\, \mathbf{g}, \\ &\operatorname{div} \mathbf{u} = 0 \quad \text{on} \quad (0,T) \times \Omega, \\ &\mathbf{u}(t,x) = 0 \quad \text{on} \quad (0,T) \times \partial\Omega \quad \text{and} \quad \mathbf{u}(0,x) = \mathbf{u}_0(x) \quad \text{on} \quad \Omega, \end{aligned} \tag{18}$$

where $\mathbf{g}(t,x) \in \mathbb{R}^n$ denotes the gravitational field, $\mu(\theta)$ is the Kinematic viscosity and Ω is a bounded open subset of $\mathbb{R}^n$, $n = 2, 3$.

The conservation of internal energy is described by the initial boundary-value problem:

$$\begin{aligned} &\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta - \nabla \cdot (\chi(\theta) \nabla \theta) = f \quad \text{on} \quad (0,T) \times \Omega \\ &\theta(t,x) = 0 \quad \text{on} \quad (0,T) \times \partial\Omega \\ &\theta(0,x) = \theta_0(x) \quad \text{on} \quad \Omega, \end{aligned} \tag{19}$$

Where $f(t,x) \in \mathbb{R}$ is an external force and $\chi(\theta)$ is the thermal conductivity.

In the system (19) we have used a standard argument to reduce the case of a non-homogeneous boundary condition to one homogeneous. Also, we observe that the usual Boussinesq equations correspond to the case where $\mu(\theta)$ and $\chi(\theta)$ are positive constants.

For the derivation and physical discussion of equations (18)-(19) see Drazin and Reid [4].

The weak formulation of problem (18)-(19) is

$$\frac{d}{dt}(\mathbf{u}, \mathbf{v}) + b\,(\mathbf{u}, \mathbf{u}, \mathbf{v}) + (\mu(\theta)\nabla \mathbf{u}, \nabla \mathbf{v}) = \langle \alpha\, \theta\, \mathbf{g}, \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in \mathbf{V}, \tag{20}$$

$$\frac{d}{dt}(\theta, \varphi) + \tilde{b}\,(\mathbf{u}, \theta, \varphi) + (\chi(\theta)\nabla\theta, \nabla\varphi) = \langle f, \varphi \rangle \quad \forall\, \varphi \in H_0^1, \tag{21}$$

where we used the notation given in subsection 3.1 and

$$\tilde{b}\,(\mathbf{u},\theta,\varphi)=\sum_{j=1}^{n}\int_{\Omega}u_j\frac{\partial\theta}{\partial x_j}\varphi\,dx.$$

Definition 2 *Let* $\mathbf{u}_0\in\mathbf{L}^2$, $\theta_0\in L^2$, $f\in L^2(0,T;L^2)$, $\mathbf{g}\in L^\infty(0,T;\mathbf{L}^p)$ $(p>3/2$ *if* $n=3$ *and* $p>1$ *if* $n=2)$. *We will say that* $(\mathbf{u},\theta)$ *is a weak solution of (18)-(19) if and only if* $\mathbf{u}\in L^2(0,T;\mathbf{V})\cap L^\infty(0,T;\mathbf{H})$, $\theta\in L^2(0,T;H_0^1)\cap L^\infty(0,T;L^2)$ *and (20), (21) is verified.*

We assume that $\mu(\cdot)$ and $\chi(\cdot)$ are continuous functions, and

$$0<\mu_0\le\mu(\sigma)\le\mu_1<\infty;\quad 0<\chi_0\le\chi(\sigma)\le\chi_1<\infty\quad\text{for all }\ \sigma\in\mathbb{R}.\tag{22}$$

The following result on existence of weak solutions was proved by Lorca and Boldrini [9] using the special basis for Galerkin approximations as in Lions [8].

Theorem 11 *Under the hypothesis (22) and the conditions of Definition 2, the problem (18)-(19) has a weak solution.*

We can proved the above theorem using the Theorem 2. In first time, we prove the following lemma.

Lemma 12 *Let* $\{\mathbf{w}^k\}$ *and* $\{\varphi^k\}$ *arbitrary basis of the Hilbert spaces* $\mathbf{V}$ *and* H_0^1, *respectively. We consider the Galerkin approximations* $\mathbf{u}^k(t,x)=\sum_{i=1}^{k}\alpha_i(t)\mathbf{w}^i(x)$ *and* $\theta^k(t,x)=\sum_{i=1}^{k}\beta_i(t)\varphi^i(x)$ *of* $\mathbf{u}$ *and* θ *respectively, solutions of*

$$\frac{d}{dt}(\mathbf{u}^k,\mathbf{v})+b\,(\mathbf{u}^k,\mathbf{u}^k,\mathbf{v})+(\mu(\theta^k)\nabla\mathbf{u}^k,\nabla\mathbf{v})\;=\;\langle\alpha\,\theta^k\,\mathbf{g},\mathbf{v}\rangle\quad\forall\,\mathbf{v}\in\mathbf{V}^k,\tag{23}$$

$$\frac{d}{dt}(\theta^k,\varphi)+\tilde{b}\,(\mathbf{u}^k,\theta^k,\varphi)+(\chi(\theta^k)\nabla\theta^k,\nabla\varphi)\;=\;\langle f,\varphi\rangle\quad\forall\,\varphi\in H_k,\tag{24}$$

where $\mathbf{V}^k=span\{\mathbf{w}^1,\ldots,\mathbf{w}^k\}$ *and* $H_k=span\{\varphi^1,\ldots,\varphi^k\}$. *Then,* $\mathbf{u}^k\in L^\infty(0,T;\mathbf{H})\cap L^2(0,T;\mathbf{V}),\theta^k\in L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1)$ *uniformly in* k.

Proof. The same observations done in the proof of the Proposition 8 is valid. Setting $\mathbf{v}=\mathbf{u}$ in (23) and $\varphi=\theta$ in (24), we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2+\|\mu(\theta)^{1/2}\nabla\mathbf{u}\|^2=(\alpha\,\theta\,\mathbf{g},\mathbf{u}),$$
$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2+\|\chi(\theta)^{1/2}\nabla\theta\|^2=(f,\theta).$$

By using (22) and the Hölder inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2+\mu_0\|\nabla\mathbf{u}\|^2\le\alpha\,\|\mathbf{g}\|_{\mathbf{L}^p}\|\theta\|_{L^{\frac{2p}{p-1}}}\|\mathbf{u}\|_{\mathbf{L}^{\frac{2p}{p-1}}},$$
$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2+\chi_0\|\nabla\theta\|^2\le\|f\|\,\|\theta\|.$$

The Sobolev imbedding together with Young's inequality imply

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \frac{\mu_0}{2}\|\nabla\mathbf{u}\|^2 \leq \alpha^2\, c\, \|\mathbf{g}\|^2_{\mathbf{L}^p}\|\nabla\theta\|^2, \tag{25}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \frac{\chi_0}{2}\|\nabla\theta\|^2 \leq c\,\|f\|^2. \tag{26}$$

Integrating (25) and (26) with respect to t, we get

$$\|\mathbf{u}(t)\|^2 + \mu_0\int_0^t \|\nabla\mathbf{u}(s)\|^2 ds \leq c\|\mathbf{g}\|^2_{L^\infty(0,T;\mathbf{L}^p)}\int_0^t \|\nabla\theta(s)\|^2 ds + \|\mathbf{u}_0\|^2, \tag{27}$$

$$\|\theta(t)\|^2 + \chi_0\int_0^t \|\nabla\theta(s)\|^2 ds \leq c\|f\|^2_{L^2(0,T;L^2)} + \|\theta_0\|^2. \tag{28}$$

By using (28) in (27), we get

$$\|\mathbf{u}(t)\|^2 + \chi_0\int_0^t \|\nabla\theta(s)\|^2 ds \leq c_1\, c_2 + c_3$$

where $c_1 = c\,\|\mathbf{g}\|^2_{L^\infty(0,T;\mathbf{L}^p)}$, $c_2 = c\,\|f\|^2_{L^2(0,T;L^2)} + \|\theta_0\|^2$, $c_3 = \|\mathbf{u}_0\|^2$. Consequently, $\mathbf{u}^k \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$, and $\theta^k \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1)$ uniformly in k. □

Proposition 13 *$D_t^\gamma\mathbf{u}^k \in L^2(0,T;\mathbf{H})$, $D_t^\gamma\theta^k \in L^2(0,T;L^2)$ for $0 \leq \gamma < 1/4$ if $n = 3$ and $0 \leq \gamma < 1/2$ if $n = 2$.*

Proof. Integrating (20) from t to $t+h$, we get

$$\begin{aligned}(\mathbf{u}_h(t),\mathbf{v}) &= \int_t^{t+h}\left[(\alpha\theta(\tau)\mathbf{g}(\tau),\mathbf{v}) - (\mu(\theta(\tau))\nabla\mathbf{u}(\tau),\nabla\mathbf{v}) + b(\mathbf{u}(\tau),\mathbf{v},\mathbf{u}(\tau))\right]d\tau \\ &\leq \|\mathbf{v}\|_V\int_t^{t+h}\left[f_1(\tau) + f_2(\tau)\right]d\tau\end{aligned}$$

where $f_1(\tau) = \alpha\,\|\mathbf{g}\|_{L^\infty(0,T;\mathbf{L}^p)}\|\theta(\tau)\|_{H_0^1} + \mu_1\|\mathbf{u}(\tau)\|_{\mathbf{V}}$, $f_2(\tau) = \|\mathbf{u}(\tau)\|_{\mathbf{L}^p}\|\mathbf{u}(\tau)\|_{\mathbf{L}^q}$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The rest of analysis is exactly equals to (8). For temperature we work in the same manner. □

Proof of Theorem 11

From Lemma 12 and Theorem 2-(iv), we have that there exist subsequences of $\mathbf{u}^k$, θ^k and $\mathbf{u}$, θ such that

$$\mathbf{u}^k \to \mathbf{u} \text{ strongly in } L^2(0,T;\mathbf{H}) \text{ and } \theta^k \to \theta \text{ strongly in } L^2(0,T;L^2), \tag{29}$$

$$\mathbf{u}^k \to \mathbf{u} \text{ weakly in } L^2(0,T;\mathbf{V}) \text{ and } \theta^k \to \theta \text{ weakly in } L^2(0,T;H_0^1). \tag{30}$$

Then the convergence in (29) imply

$$\mathbf{u}^k \to \mathbf{u} \ \text{ a.e. on } \ [0,T]\times\Omega \quad \text{and} \quad \theta^k \to \theta \ \text{ a.e. on } \ [0,T]\times\Omega.$$

Now, let $\phi \in C^1([0,T])$ with $\phi(T)=0$. Multiplying (23) by $\phi(t)$ and integrating the result, we obtain

$$-\int_0^T (\mathbf{u}^k,\mathbf{w}^j)\phi'(t)dt + \int_0^T (\mu(\theta^k)\nabla\mathbf{u}^k,\nabla\mathbf{w}^j)\phi(t)dt - \int_0^T b\,(\mathbf{u}^k,\mathbf{u}^k,\mathbf{w}^j)\phi(t)dt$$
$$= \alpha\int_0^T (\theta^k\mathbf{g},\mathbf{w}^j)\phi(t)dt + (P_k\mathbf{u}_0,\mathbf{w}^j)\phi(0), \tag{31}$$

for all $\mathbf{w}^j \in \mathbf{V}^k$ and where $P_k : \mathbf{L}^2 \to \mathbf{V}^k$.

Taking the limit in (31) as $k\to\infty$, we get

$$-\int_0^T (\mathbf{u},\mathbf{w}^j)\phi'(t)dt + \int_0^T (\mu(\theta)\nabla\mathbf{u},\nabla\mathbf{w}^j)\phi(t)dt - \int_0^T b\,(\mathbf{u},\mathbf{u},\mathbf{w}^j)\phi(t)dt$$
$$= \alpha\int_0^T (\theta\mathbf{g},\mathbf{w}^j)\phi(t)dt + (\mathbf{u}_0,\mathbf{w}^j)\phi(0). \tag{32}$$

In fact, the convergence of the trilinear form $b\,(\cdot,\cdot,\cdot)$ is well known and for the other nonlinear term, we have that

$$\int_0^T (\mu(\theta^k)\nabla\mathbf{u}^k,\phi(t)\nabla\mathbf{w}^j)dt \quad = \quad \int_0^T (\nabla\mathbf{u}^k,\mu(\theta^k)\phi(t)\nabla\mathbf{w}^j)dt$$
$$\longrightarrow \quad \int_0^T (\nabla\mathbf{u},\mu(\theta)\phi(t)\nabla\mathbf{w}^j)dt,$$

since $\nabla\mathbf{u}^k \to \nabla\mathbf{u}$ weakly in $L^2(0,T;\mathbf{H})$ and $\mu(\theta^k)\phi(t)\mathbf{w}^j \to \mu(\theta)\phi(t)\mathbf{w}^j$ strongly in $L^2(0,T;\mathbf{L}^2)$.

Finally, applying arguments of density and continuity, we have that (32) is true for all $\mathbf{v}\in\mathbf{V}$ and consequently $\mathbf{u}$ satisfies (20). Analogously, we can to prove that θ satisfies (21).

Remark 7 *When we use the Proposition 4 in the case of nonhomogeneous fluids in the two-dimensional domain, we obtain better results than the given by Simon [14].*

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NUMERICAL METHODS FOR A COUPLED NONLINEAR SCHRÖDINGER SYSTEM

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Abstract

We study a numerical method based on a Crank-Nicolson scheme for a coupled nonlinear Schrödinger system that describes some physical phenomena such as the propagation in birefringent optical fibers, Kerr-like photo refractive media in optics and Bose-Einstein condensates. This numerical scheme preserves the densities and the energy of the solution, and it is proved the convergence of its sequence of solutions.

Key words: *Schrödinger equations, asymptotically free solution, Crank-Nicolson scheme.*
AMS subject classifications: *35Q55, 35B40, 65N06*

1 Introduction

In this paper we consider the initial value problem for the coupled nonlinear Schrödinger system

$$i\,u_t + u_{xx} + |u|^2\,u + \beta\,|v|^2\,u = 0 \tag{1}$$
$$i\,v_t + v_{xx} + |v|^2\,v + \beta\,|u|^2\,v = 0 \tag{2}$$
$$u(x,\,0) = u_0(x) \tag{3}$$
$$v(x,\,0) = v_0(x) \tag{4}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$. $u = u(x,\,t)$ and $v = v(x,\,t)$ are complex unknown functions and β is a real positive constant which depends on the anisotropy of the fiber. If $\beta = 0$ the equations (1)-(4) are two copies of a single nonlinear Schrödinger equation which is integrable; when $\beta = 1$, (1)-(4) is known as the Manakov system [16] which is also integrable. In all the other cases the situation is much more complicated from different points of view. The study of the propagation of pulses in nonlinear optical fiber has been of great interest in the last years. In 1981, I. P. Kaminow [13] showed that single-mode optical fibers are not really "single-mode" but actually bimodal due to the presence of birefringence which can deeply influence the way in which an optical evolves during the propagation

along the fiber. In a single mode optical fiber, when third order nonlinear effects are included, pulse propagation is described by the nonlinear Schrödinger equation. However, a single-mode optical fiber is not exactly single mode, it is actually bimodal due to the presence of birefringence tends to split a pulse into two pulses in the two polarization directions, but nonlinear effects can trap then together against splitting. Menyuk [17, 18] showed that the two polarization components in a birefringent optical fiber are governed by the two coupled nonlinear Schrödinger equations (1)-(4). The Cauchy problem for the system (1)-(4) was firstly studied by E. S. P. Siqueira [19, 20] for initial data $u_0 \in H^1(\mathbb{R})$ and $v_0 \in H^1(\mathbb{R})$, then it was proved that using the techniques developed in [4, 5] the solution $u \in C(\mathbb{R} : H^1(\mathbb{R})) \cap C^1(\mathbb{R} : H^{-1}(\mathbb{R}))$ and $v \in C(\mathbb{R} : H^1(\mathbb{R})) \cap C^1(\mathbb{R} : H^{-1}(\mathbb{R}))$. This Schrödinger system has been extensively studied by many authors [13, 15, 16, 17, 18] and references therein. Recently, J. C. Ceballos *et al.* [6] proved following the idea of N. Hayashi *et al.* [11, 12] the gain in regularity for the coupled system using the operator $J = x + 2\,i\,t\partial_x$ that commutes with the Schrödinger operator. Indeed they proved that all solutions of finite energy to (1)-(4) are smooth for $t \neq 0$ provided that the initial functions $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ decay rapidly enough as $|x| \to \infty$. For the Nonlinear Schrodinger equation

$$i\,u_t = \triangle u - \lambda\,|u|^{p-1}\,u \tag{5}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $p > 1$, and the constant λ is positive, W. Strauss [21] showed that if p is large enough, then a substantial class of solutions of (5) are asymptotically free(definition 2.1). Indeed, W. Strauss proved that the only asymptotically free solution to (5) is identically zero when $1 < p \leq 1 + 1/n$ for $n \geq 2$, and when $1 < p \leq 2$ for $n = 1$. In 1984, J. Barab [1] using the idea of R. Glassey [10] extended the theorem of W. Strauss and proof that the only smooth asymptotically free solution of (5) is identically zero when $n \geq 1$ and $1 < p \leq 1+2/n$. In this work, we will show numerically that *there does not exist any finite energy asymptotically free solution of the coupled system* (1)-(4). It is well-known that the one-dimensional system (1)-(4) is integrable by inverse scattering if $\beta = 0$ or $\beta = 1$. For instance, if we take $\beta = 0$, then we recall that the long-time dynamics of the solutions of the scalar nonlinear Schrödinger equation (1) is defined in terms of the scattering data of the associated Lax operator [14] which is the Zakharov-Shabat operator [22]. If the initial data is large enough in $L^1(\mathbb{R})$-norm, e. g. for $u_0 = A\,sech(x)$ for $A > 1/\sqrt{2}$, there are isolated eigenvalues of the Lax operator, which contribute to the nonlinear Schrödinger equation solitons in the limit as $t \to \infty$. However, if the initial data is small enough in $L^1(\mathbb{R})$-norm, e. g. for small values of A, no isolated eigenvalues occur and the solution u disperses away as $t \to \infty$, similar to the solutions of the linear Schrödinger equation.

2 Numerical scheme. Semi-discretization in time

Let T be the computation time and N the points for the time discretization, thus we define a time step $\delta t = T/N$. Equations (1)-(4) are discretized at time

$t_n = n\delta t$ and $t_{n+\frac{1}{2}} = (n+\frac{1}{2})\delta t$ using the Crank-Nicolson scheme proposed by Delfour-Fortin-Payre [9]. It takes the semi-discretized form

$$\begin{aligned} & i\frac{u^{n+1}-u^n}{\delta t} + \partial_{xx}\left(\frac{u^{n+1}+u^n}{2}\right) \\ & + \left(\frac{|u^{n+1}|^2 + |u^n|^2 + \beta\left(|v^{n+1}|^2 + |v^n|^2\right)}{2}\right)\frac{u^{n+1}+u^n}{2} \;=\; 0, \qquad (6) \\ & i\frac{v^{n+1}-v^n}{\delta t} + \partial_{xx}\left(\frac{v^{n+1}+v^n}{2}\right) \\ & + \left(\frac{|v^{n+1}|^2 + |v^n|^2 + \beta\left(|u^{n+1}|^2 + |u^n|^2\right)}{2}\right)\frac{v^{n+1}+v^n}{2} \;=\; 0, \qquad (7) \end{aligned}$$

with the initial data $u^0(x) = u_0(x)$ and $v^0(x) = v_0(x)$. This scheme is studied for the scalar case (a single Schrödinger equation) with different discretizations of the operator $\Delta = \partial_{xx}$ in [7, 8, 9]. In [2] the author propose a relaxation scheme which is a different version of the discretization of the nonlinear part.

The scheme (6)-(7) has the property to conserve the energy like in the continuous case (see Lemma 3.1). Indeed, multiplying (6) by $\overline{u^{n+1}+u^n}$ and (7) by $\overline{v^{n+1}+v^n}$, integrating with respect to the space and taking the imaginary part, we get

$$\int_{\mathbb{R}} |u^{n+1}|dx = \int_{\mathbb{R}} |u^n|dx, \qquad \text{and} \qquad \int_{\mathbb{R}} |v^{n+1}|dx = \int_{\mathbb{R}} |v^n|dx. \qquad (8)$$

Moreover, we have

Proposition 1 *Let* $(u^n, v^n)_{n\in\mathbb{N}}$ *be a numerical solution of* (6)-(7). *Let* $(0, 0) \neq (u^0, v^0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, *then for all* $n \in \mathbb{N}$

$$\begin{aligned} & ||u_x^{n+1}||^2_{L^2(\mathbb{R})} + ||v_x^{n+1}||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u^{n+1}||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v^{n+1}||^4_{L^4(\mathbb{R})} \\ & -\beta\int_{\mathbb{R}} |u^{n+1}|^2\,|v^{n+1}|^2\,dx = ||u_x^n||^2_{L^2(\mathbb{R})} + ||v_x^n||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u^n||^4_{L^4(\mathbb{R})} \\ & \qquad -\frac{1}{2}\,||v^n||^4_{L^4(\mathbb{R})} - \beta\int_{\mathbb{R}} |u^n|^2\,|v^n|^2\,dx \qquad (9) \end{aligned}$$

and

$$||u_x^n||^2_{L^2(\mathbb{R})} + ||v_x^n||^2_{L^2(\mathbb{R})} \leq c \qquad (10)$$

with $c = c\left(||u_0||_{H^1(\mathbb{R})},\ ||v_0||_{H^1(\mathbb{R})}\right)$ *independent of* $n \in \mathbb{N}$.

Proof. The arguments are very similar to the proof of [3, Lemma 3.2]. Differentiating (6) and (7) with respect to x and multiplying by $\overline{u^{n+1}+u^n}$ and

$\overline{v^{n+1}+v^n}$ respectively, integrating in space and taking the imaginary part, we obtain (9). Thus, the rest is a copy of the proof of [3, Lemma 3.2]. □

From the conservation law (8) and Proposition 1 we deduce the estimate of the numerical solution of (6)-(7) in $L^\infty(0,T,H^1(\mathbb{R}))$. On the other hand, from the existence and uniqueness of the solution of (1)-(2) (see Ceballos *et al.* [6]), we can pass to the limit obtaining:

Corollary 2 *Let* $(u^n,\, v^n)_{n\in\mathbb{N}}$ *be the numerical solution of* (6)-(7) *and* $(u,\, v)$ *be the smooth solution of* (1)-(2). *Let* $(u_0,\, v_0) = (u^0,\, v^0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, *then*

$$(u^n,\, v^n) \to (u,\, v) \quad in \quad L^\infty(0,T,L^2_{loc}(\mathbb{R})),$$

as $n \to \infty$.

3 Full discretization

For the spatial approximation, we restrict the infinite domain $\mathbb{R}$ to a large enough bounded interval (L_1, L_2) and take a homogeneous Dirichlet condition on the boundary. We consider that the approximated domain is large enough when the value of the solution at the boundary is less than 10^{-6}. The numerical tests show that this is enough to avoid a damage of the qualitative behavior of the solution.

We mesh the bounded domain in K intervals, we denote $\delta x = K/(L2 - L1)$ the space step, and u_j^n the value of the approximation of the solution at the point $j\delta x$. For the approximation of $\Delta = \partial_{xx}$ we consider the centered finite difference $\partial_{xx}u \approx \dfrac{u_{j+1} - 2u_j + u_{j-1}}{\delta x^2}$. Thus, the full discrete scheme can be read as

$$\begin{aligned}
\left\{ i\, Id + A\left(|u^{n+1}|^2 + |u^n|^2 + \beta\left(|v^{n+1}|^2 + |v^n|^2\right)\right)\right\} u^{n+1} & \\
= -\frac{\delta t}{2\delta x^2}\left(u^n_{j+1} - 2u^n_j + u^n_{j-1}\right) & \qquad (11)\\
+\left\{ i - \frac{\delta t}{4}\left[|u^{n+1}|^2 + |u^n|^2 + \beta\left(|v^{n+1}|^2 + |v^n|^2\right)\right]\right\} u^n & \\
\left\{ i\, Id + A\left(|v^{n+1}|^2 + |v^n|^2 + \beta\left(|u^{n+1}|^2 + |u^n|^2\right)\right)\right\} v^{n+1} \; = & \\
-\frac{\delta t}{2\delta x^2}\left(v^n_{j+1} - 2v^n_j + v^n_{j-1}\right) & \qquad (12)\\
+\left\{ i - \frac{\delta t}{4}\left[|v^{n+1}|^2 + |v^n|^2 + \beta\left(|u^{n+1}|^2 + |u^n|^2\right)\right]\right\} v^n &
\end{aligned}$$

where $A(\xi) = \dfrac{\delta t}{2\delta x^2}\begin{pmatrix} 2+\xi_1/2 & -1 & 0 & \cdots \\ -1 & 2+\xi_2/2 & -1 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & -1 & 2+\xi_J/2 \end{pmatrix}$, with $\xi \in \mathbb{R}^K$.

The scheme (11)-(12) is defined by a nonlinear system of $K \times K$ equations.

We solve it using a fixed point algorithm. In [3], it is numerically proven the conservativity of the densities and the energy of this scheme.

4 Numerical experiments

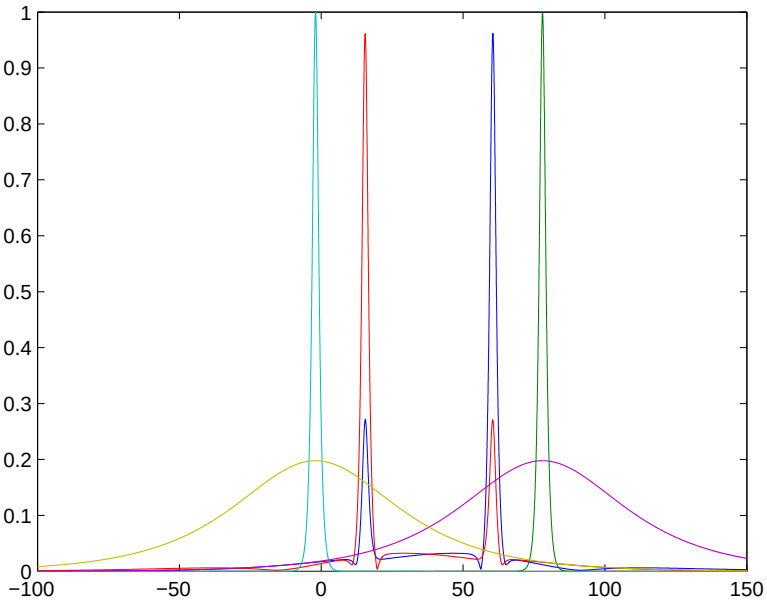

Figure 1: The soliton of the uncoupled system ($\beta = 0.0$) v/s the coupled system ($\beta = 2.0$) and the free solution of the linear Schrodinger equation; $\Omega = (-50, 100)$ and $T = 10$ with $N = 2000$ and $K = 2000$.

Rescaling equations (1)-(2), we test the numerical scheme on the nonlinear system

$$i\,u_t + u_{xx} + 2|u|^2\,u + \beta\,|v|^2\,u = 0$$
$$i\,v_t + v_{xx} + 2|v|^2\,v + \beta\,|u|^2\,v = 0$$

with the initial data

$$u(x,0) = \frac{i\exp(2ik_1x)}{\cosh(x+2)}, \qquad v(x,0) = \frac{i\exp(2ik_2x)}{\cosh(x+2)}, \qquad x \in \mathbb{R}.$$

The exact solution for this initial data is known when $\beta = 0$

$$u_{ex}(x,t) = \frac{i\exp(2ik_1x + (1-4k_1^2)t}{\cosh(x+2-4k_1t)}, \qquad v_{ex}(x,t) = \frac{i\exp(2ik_1x + (1-4k_2^2)t}{\cosh(x+2-4k_2t)},$$

where $x \in \mathbb{R}$ and $t > 0$.

We choose $k_1 = 1.0$ and $k_2 = 0.0$, and the final time $T = 10$. We take $\Omega = (-50, 100)$ to avoid any numerical reflections due to the boundaries.

Figure 1 shows the behavior of the solutions of the uncoupled system (when $\beta = 0.0$) comparing them with the free solutions and the behavior of solutions of the coupled system ($\beta = 2.0$). These simulation were done for $\Omega = (-20, 50)$, $T = 10.0$, $N = 2000$ and $K = 2000$. We remark, as least for this two numerical examples, that the solutions of the nonlinear system, coupled or uncoupled, can not decay as the free solutions, that is on $\mathcal{O}(t^{-1/2})$, which is in agreement with the statement of the Main Theorem and its proof.

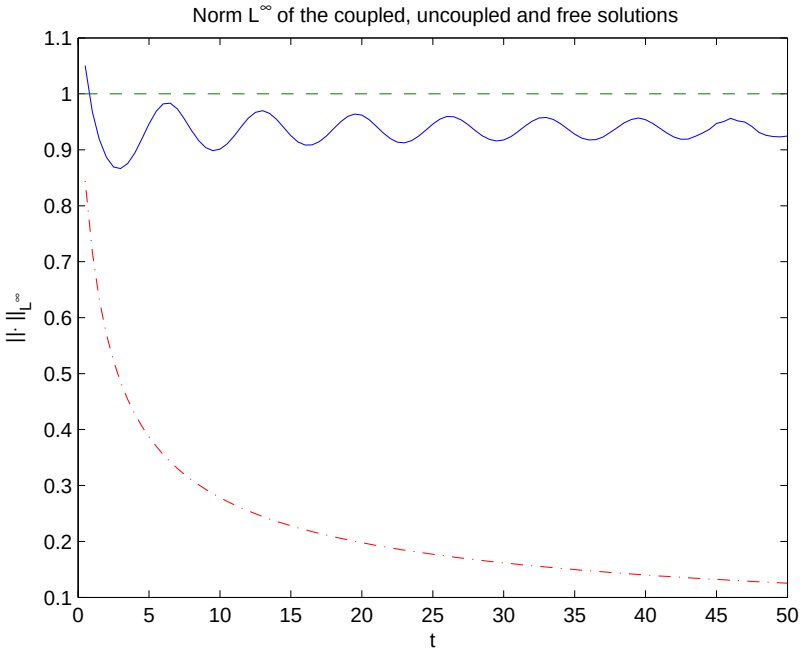

Figure 2: The $L^\infty-$norm of the solution for the free system ('$-\cdot$'), the nonlinear uncoupled system with $\beta = 0.0$ ('$--$') and the nonlinear coupled system with $\beta = 2.0$ ('continuous line').

In Figure 2 we can see the behavior of the $L^\infty-$norm for the different solutions. We compare the $L^\infty-$norm of the solution for the nonlinear uncoupled system, that is for $\beta = 0.0$, which is constant in time ($\|u\|_{L^\infty} = \|v\|_{L^\infty} = 1.0$), with the coupled system, that is for $\beta = 2.0$, which oscillate in time, and with the solutions of the free linear Schrödinger equation, which decays as $t^{-1/2}$. In this case, $\Omega = (-50, 250)$ and $T = 50$ with $N = 50000$ and $K = 50000$.

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THE NAVIER-STOKES EQUATIONS. A CHALLENGE TO NEWTONIAN DETERMINISM

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Abstract

The question of global existence and uniqueness of solutions of the time-dependent Navier-Stokes equations is reviewed. Special emphasis is put on its philosophical implications and the historical perspective. An attempt is made to approach the core of the problem while keeping technicalities to a minimum.

Key words: *Navier-Stokes equations, global existence, uniqueness, regularity.*
AMS subject classifications: *35Q30, 76D03, 76D05.*

One of the most valued features of science is its ability to predict the future. For instance, celestial mechanics is able to predict eclipses with great precision and well in advance of their taking place. More related to the subject of this article is the case of meteorology; in this case, predictions are not as precise and long-lasting as one would like, but they are still quite remarkable.

In these examples, and many others of the same kind, the possibility of predicting the future evolution of a system is based upon knowing in detail the laws that govern that evolution, as well as its present state. Mathematically, the state of a system is described by means of a more or less large collection of numeric variables, and the laws that govern the temporal evolution of these variables take usually the form of differential equations. These equations specify a condition that must be satisfied at every moment in time and that determines the rate of variation of the different variables as a function of their value at that very moment. In the case of celestial mechanics the state variables are the positions and velocities of the different bodies, and the differential equations are provided by Newton's laws, namely that the mass of a body times its acceleration is equal to the force that acts on it, and that this force can be

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calculated from the positions of the different bodies by means of the formula that we know as the law of universal gravitation.

The Navier-Stokes equations are simply the differential equations that govern another form of motion, namely the motion of a fluid, for instance air or water. In fact, these equations are still expressing Newton's law —mass times acceleration is equal to force—, although here one is not considering a finite set of particles but a continuum of matter. Another difference from celestial mechanics is that the Navier-Stokes equations take into account the friction forces, which act so as to slow the motion down. In this article we restrict ourselves to the special case of an incompressible fluid, i. e. a fluid of constant density, which is approximately the case of liquids.

It goes without saying that a quantitative knowledge about fluid motions is important in many fields of science and technology, from physiology to airplane industry.

As we have suggested, one expects that the Navier-Stokes equations will share with the equations of celestial mechanics the property of determining the future evolution from the present state. However, certain difficulties arise when one faces the task of providing a proper mathematical proof of such a statement. In the case of celestial mechanics, and many other differential equations, such a proof can indeed be given. But the Navier-Stokes equations resist tenaciously to it. In spite of the earnest efforts that have been made about it, until now one has not been able to prove mathematically neither the supposed determinism of these equations nor the absence of it.

The difficulties that arise in this mathematical problem are not deprived of physical meaning. Already at the end of the nineteenth century, some accurate experimenters had pointed out that in certain situations the motions of fluids exhibited an apparent lack of determinism. This experimental phenomenon was given the name of turbulence, since what is observed is not so different from the ordinary meaning of this term. On the other hand, such an apparent lack of determinism could be simply a consequence of an insufficient precision in the specification of the initial state. In fact, today it is well known that the exact solutions of a differential equation can be perfectly deterministic at the same time that they exhibit a highly sensitive dependence on the initial state, so that in practice one observes an apparently non-deterministic behaviour. Towards 1960 the meteorologists became well aware of such a possibility, thanks to the computing power provided by electronic machines, and some years later they christened it as the "butterfly effect".

In this regard, it must be remarked that the problem about the determinism of the Navier-Stokes equations could be more severe than a simple butterfly effect: According to the present state of affairs, it could be that the exact solutions of the Navier-Stokes equations were genuinely indetermined! Distinguishing between one thing and the other is probably not important for practical purposes, but it is certainly important for science as an instrument for understanding the world.

The importance of the problem has been recognized by several people and institutions that have engaged in giving a list of mathematical challenges for the twenty-first century [72–81]. In particular, it is one of the seven "Millennium Problems" for which the *Clay Mathematics Institute* (*CMI*) has established a prize of one million dollars. Let us remark, however, that the specific problem stated by the *CMI* (see [78]) is not concerned exactly with the question of determinism. In fact, the notion of determinism is about the existence and *uniqueness* of solutions, whereas the *CMI* statement is about their existence and *regularity*. As we will see, a positive answer to the latter question implies a positive answer to the former, i. e. the question of determinism, but the converse implication is not true. In this connection we fully agree with Ol'ga Ladyzhenskaya [31] and other authors in that the really important question is that of determinism, and this point of view will prevail in the present exposition.

1 Differential equations and determinism: the case of celestial mechanics

In order to help understand the meaning of the Navier-Stokes equations and the difficulties with the question of determinism, it will be good to keep in mind the case of celestial mechanics.

In that case one wants to describe the motion of celestial bodies. Naturally, such a description must start by giving a list of the **bodies** under consideration; symbolically, this can be done by means of an index α that takes integer values from 1 to N, the number of bodies. Of course, one must pay attention to the **position** of each body α. In the following we will denote it by $\boldsymbol{x}_\alpha$ and we will assume that it is specified by cartesian coordinates $\boldsymbol{x} = (x, y, z)$. Another essential component of the description is the variable **time** t.

The motion of the bodies is certainly described by the functions $t \mapsto \boldsymbol{x}_\alpha(t)$, one for every α. However, in order to attain a deterministic description one needs to go into more details. To begin with, it is essential to consider the **velocity** $\boldsymbol{u}$, that is, the derivative of the position with respect to time, $\boldsymbol{u} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$. Like position, we assume that the velocity is specified by the corresponding cartesian coordinates $\boldsymbol{u} = (u, v, w)$, and the velocity of each body α will be denoted by $\boldsymbol{u}_\alpha$. If at time t we know not only the position of a body but also its velocity, then we can figure out an estimate of the position of that body after a short interval of time. Similarly, the fact that velocity can change with time leads to considering its own derivative $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$, that is, the **acceleration**: if at time t we know the velocity and the acceleration, then we can figure out an estimate of the velocity after a short interval of time. But the acceleration can also change with time, so that in principle one should consider also its own derivative, and so on towards derivatives of progressively higher orders.

At first sight it looks like we have gone into an endless spiral. But it is not so: as Robert Hooke and Isaac Newton discovered towards 1679–80, it turns out that nature has certain laws that determine the accelerations of celestial bodies

out of their positions! This changes things in a crucial way: if at time t we know the positions and velocities of the different bodies, then these laws determine also their accelerations, and therefore we can get an estimate of the positions and velocities after a short interval of time; but from here we can now repeat the same process over and over towards progressively larger times.

The laws in question are known today as Newton's second law of motion and the law of universal gravitation. Newton's second law has a very wide scope and it can be viewed as saying that the product of the mass times the acceleration of a body is equal to the sum of the **forces** acting on that body, *and* that these forces are determined by some function of the positions of all bodies, and possibly also of their velocities. The law of universal gravitation is simply a specification of this function in the case of celestial mechanics.

So, the deterministic description of the motion of celestial bodies can be summarized in a set of equations of the following form:

$$\begin{cases} \mathrm{d}\boldsymbol{x}_\alpha/\mathrm{d}t \;=\; \boldsymbol{u}_\alpha, \\ \mathrm{d}\boldsymbol{u}_\alpha/\mathrm{d}t \;=\; \boldsymbol{f}_\alpha(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)\,/\,m_\alpha, \end{cases} \tag{1}$$

where m_α denotes the mass of the body α, and $\boldsymbol{f}_\alpha$ gives the gravitational force on the body α as a function of the relative positions of all bodies. Specifically, this function is given by the following formula:

$$\boldsymbol{f}_\alpha(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) \;=\; \sum_{\beta\neq\alpha} G\, m_\alpha\, m_\beta\, |\boldsymbol{x}_\beta - \boldsymbol{x}_\alpha|^{-3}\, (\boldsymbol{x}_\beta - \boldsymbol{x}_\alpha), \tag{2}$$

where G is a universal constant and $|\boldsymbol{v}|$ denotes the length of a vector $\boldsymbol{v}$.

Conceptually, the system of equations (1) has the form

$$\mathrm{d}z/\mathrm{d}t \;=\; F(z). \tag{3}$$

In the case under consideration z represents the collection of all positions $\boldsymbol{x}_\alpha$ and velocities $\boldsymbol{u}_\alpha$, i. e. a collection of $6N$ numbers, and F represents the collection of functions that, starting from these numbers, determine the right-hand sides of (1). Using a more general terminology, z represents the **state** of the system under consideration, which can change with time. Equation (3) says that at every moment t this state is varying in a specific way $\mathrm{d}z/\mathrm{d}t$ which is determined by the own value of z at that very moment.

But what we are interested in is the possibility of determining the future out of the present, that is seeing whether $z(0)$ determines $z(t)$ for $t > 0$. If the derivative is interpreted as a quotient of infinitesimal quantities, one can say that equation (3) embodies that determinism for infinitesimal values of t. A fundamental task of the theory of differential equations deals with going over from here to finite values of t, and seeing whether this can be achieved for arbitrarily large values of t. Traditionally, one distinguishes three aspects of the problem, namely the *existence*, the *uniqueness*, and the *globality* of the desired solution of (3). When z is specified by a finite collection of numbers, like in celestial mechanics, we are in the field of the so-called ordinary differential

equations, for which there are fairly complete results about these questions. Applied to the equations of celestial mechanics, these results guarantee the determinism of the motion as long as the bodies do not collide with each other (notice that the right-hand side of (2) is not well defined when some $\boldsymbol{x}_\beta$ coincides with $\boldsymbol{x}_\alpha$).

2 The equations of motion of an incompressible fluid

2.1. In accordance with the molecular constitution of matter, one can view a fluid as a multitude of small bodies interacting with each other. From that point of view, the case of a fluid would not be so different from that of celestial mechanics. However, there is a circumstance which changes things quite a lot: now the number of bodies in play is extremely large, of the order of 10^{23} (this quantity corresponds to about 3 grams of water). Obviously, with such figures it is totally out of the question to think about putting into practice the approach of the preceding section. For instance, even if we had at our disposal all the computer storage presently existing on our planet, this would only allow to specify the initial state of a few micrograms of water.

In front of this, one cannot but resort to a less detailed point of view. We will do this by looking at the fluid not as a discrete collection of molecules, but as a continuous distribution of matter over space. In contrast to the case of celestial mechanics, now the variable α that indexes the "particles" in interaction will not be restricted to a finite set, but we will allow it to take values in a continuous set. Contrary to what it may seem, the resulting model will be much easier to deal with than one concerned with a finite but very large number of interacting particles.

Like in the case of celestial mechanics, in principle the motion of the fluid is specified by a function of α and t, namely the position of each particle α at time t. Because of the continuous character of the variable α, now we will prefer to write $\boldsymbol{x}(\alpha, t)$ instead of $\boldsymbol{x}_\alpha(t)$, but this is just a question of notation. A quite natural hypothesis that we will adopt is that two different particles cannot be in the same position at the same time; in other words, the mapping $\alpha \mapsto \boldsymbol{x}(\alpha, t)$ is one-to-one at every time t. We will also assume that the fluid permanently fills a certain region (i. e. an open and connected subset) of space; this region and its boundary will be denoted respectively by Ω and $\partial\Omega$. In such conditions, the function $\boldsymbol{x}(\alpha, t)$ defines a change of variables $(\alpha, t) \leftrightarrow (\boldsymbol{x}, t)$ that leads to an alternative description where the independent variables are not the particle α and the time t, but the **position** $\boldsymbol{x} \in \Omega$ and the **time** t. Among the variables that depend on $\boldsymbol{x}$ and t it is especially relevant the **velocity** $\boldsymbol{u}$; by definition, $\boldsymbol{u}(\boldsymbol{x}, t)$ means the velocity at time t *of the particle* that is occupying the position $\boldsymbol{x}$ at that very moment t.

Since we are interested in the question of whether the future of the fluid is determined from its present, the independent variables $\boldsymbol{x}$ and t play different roles. So, when dealing with functions of $\boldsymbol{x}$ and t we will tend to look at them not as such but rather as functions of $\boldsymbol{x}$ that change with time t, which

is essentially the same, but more in the spirit of our interests. Symbolically, instead of functions $(\boldsymbol{x},t)\mapsto \boldsymbol{f}(\boldsymbol{x},t)$, we will tend to think in terms of functions $t\mapsto \boldsymbol{f}(t)$ where each value $\boldsymbol{f}(t)$ is itself a function of $\boldsymbol{x}$ (namely the function $\boldsymbol{x}\mapsto \boldsymbol{f}(\boldsymbol{x},t)$). In particular, the velocity will be looked at as a vector field that changes with time (here we use the classical terminology where the functions of position are called "fields").

2.2. As it will be seen shortly, the character of the problem calls for an extensive usage of the tools of infinitesimal calculus, especially those that deal with functions of several variables, like partial derivatives, volume integrals and surface integrals. Typically, these tools presume certain regularity hypotheses on the functions involved, as well as on the regions and surfaces of integration. In this connection, it is important to distinguish between the regularity hypotheses that refer to the data of the problem, like the region Ω and its boundary $\partial\Omega$, and those that refer to the unknowns, especially the velocity field $\boldsymbol{u}$. In the latter case, we must consider them as working hypotheses that await confirmation.

To a great extent, the difficulty of the problem lies precisely in the fact that some of these working hypotheses do not get such a confirmation.

In some cases this is not so bad, since it is solved by means of a suitable generalization of the notions involved (in complete analogy to how the equation $x^2=2$ leads to extending the notion of number, passing from rational numbers to "real" ones). Among the generalized notions that suit our problem are especially relevant the Lebesgue integral and the weak derivatives. In fact, the Navier-Stokes equations have played an important historical role as a motivation for the development of these now standard techniques, especially the weak derivatives.

In other cases the problem is more serious: neither a generalization is found that eliminates the need for the hypothesis in question, nor a proof is found of its fulfilment.

In this connection, this exposition has the spirit of not entering into such technicalities unless it becomes absolutely necessary; let us say that "every function is as regular as needed unless there is a reasonable evidence to the contrary". In particular, we will not be especially precise about the regularity hypotheses to be imposed on the data of the problem, since the existing results show no substantial improvement no matter how much regular the data are assumed to be. On the other hand, concerning the unknowns, for the moment we will adopt the working hypothesis that they are as regular as needed, and later on we will make the amendments that turn out necessary.

2.3. As we have already mentioned, we will restrict our attention to the case of a homogeneous **incompressible** fluid. By definition, this means that the **density** ρ remains constant, that is, independent of $\boldsymbol{x}$ and t. Combined with the principle of **mass conservation**, this hypothesis entails a crucial constraint on the velocity field. In the following we will enter into some details to see how

this constraint is derived. This will serve to introduce certain basic notions and techniques that will reappear later on in more involved circumstances.

A first fundamental idea is that mass is not a property of single points $\boldsymbol{x}$, but it refers to more "substantial" parts of Ω. In the following, such parts will be denoted generically by ω, and the set of all possible choices for them will be denoted as $\mathcal{P}(\Omega)$. For most of our purposes it will suffice to take as $\mathcal{P}(\Omega)$ the set of all balls and cubes contained in Ω.

Let us consider the amount of mass contained in such a region $\omega \in \mathcal{P}(\Omega)$. Because of the hypothesis that the fluid has a constant density, independent of both the position $\boldsymbol{x}$ and the time t, this amount of mass will be always the same, namely the volume of ω times the density of the fluid. Notice that this statement is just a consequence of the fact that we are assuming a constant density as well as a fixed region ω; we still have not used the law of conservation of mass. Now we invoke this law, according to which mass can neither be created nor destroyed. In particular, it cannot appear nor disappear inside ω; however, it can certainly get in or out of ω because of the motion of the fluid. So, independently of its being known to vanish, the variation of the mass contained in ω must be equal to the net flux of mass through $\partial\omega$. Therefore, we infer the vanishing of this flux, that is of the surface integral $\int_{\partial\omega} \rho\, u_\perp \mathrm{d}S$, where $u_\perp$ represents (for every $\boldsymbol{x} \in \partial\omega$) the number that gives the projection of the vector $\boldsymbol{u}$ on the direction perpendicular and exterior to $\partial\omega$. In order to arrive at this integral it suffices to realize that during a short interval of time of length $\mathrm{d}t$ the particles that go out of ω through a small piece of surface of area $\mathrm{d}S$ form a cylinder of volume $u_\perp \mathrm{d}S\,\mathrm{d}t$. So, that integral must be zero, or equivalently, since ρ is a non-null constant, the velocity field must satisfy the condition

$$\int_{\partial\omega} u_\perp \mathrm{d}S = 0, \qquad \forall\omega \in \mathcal{P}(\Omega). \tag{4}$$

At this point, another fundamental tool enters into play, namely the divergence theorem. According to it, a surface integral of the form $\int_{\partial\omega} u_\perp \mathrm{d}S$ is always equal to the volume integral $\int_\omega \nabla\cdot\boldsymbol{u}\,\mathrm{d}V$, where $\nabla\cdot\boldsymbol{u}$ represents the quantity $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$, which is known as the divergence of the vector field $\boldsymbol{u}$. By applying this result, condition (4) takes the following form:

$$\int_\omega \nabla\cdot\boldsymbol{u}\ \mathrm{d}V = 0, \qquad \forall\omega \in \mathcal{P}(\Omega). \tag{5}$$

The next step is based on the arbitrary character of the region ω. By considering progressively smaller regions ω around a point $\boldsymbol{x}$, it is not difficult to conclude that the function that appears under the integral sign must vanish at every point of Ω:

$$\nabla\cdot\boldsymbol{u} = 0, \qquad \forall\boldsymbol{x} \in \Omega. \tag{6}$$

So, the divergence of the velocity must vanish everywhere. The vector fields that satisfy such a condition are said to be **solenoidal**. As we have seen, this

condition is a local version of the integral condition (4), which expresses the law of conservation of mass in the particular case of an incompressible fluid.

2.4. Another fundamental equation for describing the motion of a fluid derives from similar considerations in connection with momentum instead of mass. The details are here more involved, but the argumentation follows essentially the same lines. In the same way as the amount of mass contained in ω is given by the scalar $\int_\omega \rho \, \mathrm{d}V$, the amount of momentum is given by the vector $\int_\omega \rho \boldsymbol{u} \, \mathrm{d}V$. Like in the case of mass, the motion of the fluid entails also a transport of momentum through $\partial\omega$. However, the momentum need not remain constant, neither in a fixed region of space, like ω, nor in a material part of the fluid (a fixed set of particles α, whose positions change with time). Even so, Newton's second law ensures that the time-variation of the momentum of a material part of the fluid is determined by the forces that act on it. These forces divide into two kinds: those that act over a distance, like gravity, and those that act by contact, which are associated with the notions of pressure and friction. If we are not dealing with a material part of the fluid but with a fixed region of space, then we must take into account also the flux of momentum through the boundary.

Altogether one arrives at the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\omega \rho \, \boldsymbol{u} \, \mathrm{d}V\right) + \int_{\partial\omega} \rho \, \boldsymbol{u} \, u_\perp \, \mathrm{d}S = \int_\omega \boldsymbol{f} \, \mathrm{d}V + \int_{\partial\omega} \mathsf{T}\boldsymbol{e}_\perp \, \mathrm{d}S, \qquad \forall \omega \in \mathcal{P}(\Omega). \quad (7)$$

The first term of this equation is the derivative of the momentum contained in ω, whereas the other three terms correspond respectively, from left to right, to the transport through $\partial\omega$, the distance forces and the contact ones. The distance forces are represented simply by a vector field $\boldsymbol{f}$ with dimensions of force per unit of volume, whereas the contact forces require a more elaborate description: $\boldsymbol{e}_\perp$ represents the unit vector in the direction perpendicular and exterior to $\partial\omega$, and $\mathsf{T}\boldsymbol{e}_\perp$ represents a vector, with dimensions of force per unit of area, which is obtained from $\boldsymbol{e}_\perp$ by means of a linear map T that generally depends on the position $\boldsymbol{x}$ (and the time t); this linear map is called the stress tensor and in cartesian coordinates it is given by a symmetric matrix.

Equation (7) expresses Newton's second law. Analogously to the role of the law of universal gravitation in celestial mechanics, here one must also provide some supplementary information for determining $\boldsymbol{f}$ and T at every point $\boldsymbol{x}$. Concerning $\boldsymbol{f}$, we will assume simply that we directly know its dependence on $\boldsymbol{x}$ and t. Concerning T, the question is more involved, not only on account of the tensorial character of this variable, but rather because its dependence on $\boldsymbol{x}$ and t is, partially, through the velocity field $\boldsymbol{u}$. In fact, the forces of friction between two bodies in contact depend on their relative velocities. For standard fluids, the law that relates the stress tensor T with the velocity field $\boldsymbol{u}$ has the following form (for brevity, we directly write a formula specialized for the incompressible case):

$$\mathsf{T} = -p\,\mathsf{I} + \mu\,(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{t}}). \quad (8)$$

The two terms of this formula correspond respectively to the notions of pressure and friction. The **pressure** p is a scalar variable that depends directly on $\boldsymbol{x}$ and t; this variable appears as a coefficient of I, which represents the identity matrix. The friction forces are given by the second term, where $\nabla \boldsymbol{u}$ represents the matrix that contains the derivatives of (u, v, w) with respect to (x, y, z), $(\nabla \boldsymbol{u})^{\mathsf{t}}$ represents the matrix obtained by transposing the preceding one, and μ is a positive coefficient, known as (the dynamic coefficient of) **viscosity**, whose value depends on the material of the fluid. For more details about the meaning of equations (7) and (8) and their different terms the reader is referred to [67].

Now we will pass from the integral equation (7), valid for an arbitrary region ω, to a differential equation valid at every point $\boldsymbol{x}$. This is done in complete analogy with the preceding section, that is, by making use of the divergence theorem and the arbitrary character of the region ω. Besides that, here we will perform some additional minor operations, like putting $\mathrm{d}(\int_\omega \rho \boldsymbol{u}\,\mathrm{d}V)/\mathrm{d}t = \int_\omega \rho\,(\partial \boldsymbol{u}/\partial t)\,\mathrm{d}V$ (which holds if $\boldsymbol{u}$ is regular enough), dividing by the constant ρ, and using equations (8) and (6). Altogether it results in a vector equation that can be written in the following way:

$$\partial \boldsymbol{u}/\partial t + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} = \boldsymbol{f} - \nabla p + \nu\,\Delta \boldsymbol{u}, \qquad \forall \boldsymbol{x} \in \Omega. \tag{9}$$

Here, ∇p represents the gradient of p, i. e. the vector of coordinates $(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z})$, Δ represents the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, and $\boldsymbol{u} \cdot \nabla$ represents the differential operator $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. The two last operators are applied separately to each component of $\boldsymbol{u}$. Finally, ν denotes the quotient μ/ρ, which is known as (the kinematic coefficient of) viscosity, and $\boldsymbol{f}$ and p are the same as before but divided by the constant ρ (alternatively, we can imagine that the units have been chosen in such a way that the value of ρ is equal to one). Notice that the left-hand side of (9) is nothing else than the acceleration of the particle which is passing through the point $\boldsymbol{x}$, i. e. the derivative of $\boldsymbol{u}$ with respect to t when one fixes α (not $\boldsymbol{x}$); in fact, the chain rule applied to the composition $(\alpha, t) \mapsto (\boldsymbol{x}, t) \mapsto u$ ensures that this derivative is equal to $\partial u/\partial t + \nabla u \cdot \partial \boldsymbol{x}/\partial t = \partial u/\partial t + \boldsymbol{u} \cdot \nabla u$, and similarly with the other components of $\boldsymbol{u}$. Notice also that the pressure appears in the equation only through its gradient, so that two pressure fields with a constant difference between them can be considered equivalent to each other.

Equation (9) was obtained in the first half of the nineteenth century by several authors: Claude Navier (1822), Augustin Cauchy (1822), Siméon Poisson (1829), Adhémar Barré de Saint-Venant (1843), and George Stokes (1845) [2–6]. Nowadays we know it by the name of the first and last of these authors, but the other three also arrived at it more or less independently of each other. The reader interested in the historical details is referred to [68]. The principal merit of these authors was finding the way to describe friction, by means of the stress tensor T, since in the absence of friction the corresponding equations ($\nu = 0$) had been obtained already in 1755 by Leonhard Euler [1].

2.5. With a view towards determining the future of the fluid from its present, equations (9) and (6) must be supplemented with some additional conditions on $\boldsymbol{u}$ that specify what happens at the limits of the region Ω.

In this connection, one often considers the case where Ω represents a container whose walls are at rest. In this case, a viscous fluid is bound to satisfy the following **non-slip condition**:

$$\boldsymbol{u} = \boldsymbol{0}, \qquad \forall \boldsymbol{x} \in \partial\Omega. \tag{10}$$

This condition can be decomposed into two parts which are concerned respectively with the normal and tangential components of $\boldsymbol{u}$ at $\partial\Omega$, which we will denote respectively by $u_\perp$ and $\boldsymbol{u}_{/\!/}$ ($\boldsymbol{u} = u_\perp \boldsymbol{e}_\perp + \boldsymbol{u}_{/\!/}$):

$$u_\perp = 0, \qquad \forall \boldsymbol{x} \in \partial\Omega, \tag{10}_\perp$$

$$\boldsymbol{u}_{/\!/} = \boldsymbol{0}, \qquad \forall \boldsymbol{x} \in \partial\Omega. \tag{10}_{/\!/}$$

Conditions $(10)_\perp$ and $(10)_{/\!/}$ lie on different grounds: the first one obeys to purely kinematical reasons, whereas the second is a consequence of viscosity. As it will be seen below, they play different roles in the mathematics of the problem. The vector fields that satisfy condition $(10)_\perp$ are said to be "parallel to the boundary".

In the case of an unbounded region Ω, which arises in a natural way when studying small-scale phenomena ocurring away from any boundary, one needs also some **condition at infinity**, i. e. a specification of the behaviour of the velocity $\boldsymbol{u}$ when $\boldsymbol{x}$ goes away towards infinity. In this connection, it often suits a condition of the form

$$\boldsymbol{u} \to \boldsymbol{0}, \qquad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{11}$$

On the other hand, some mathematical aspects of the problem call for conditions more restrictive than (11), like some condition of the following sort:

> In the limit $|\boldsymbol{x}| \to \infty$ the vector field $\boldsymbol{u}$ tends to zero at least as fast $|\boldsymbol{x}|^{-\alpha}$, and its spatial derivatives of order $k \leq m$ satisfy similar conditions with the exponent $\alpha + k$ instead of α. (12)

However, such conditions can be considered as conditions of regularity at infinity, so, in the spirit of § 2.2, we will not enter into details about them. Let us remark only that they can be related to the convergence of certain integrals, in particular the one that gives the kinetic energy, namely $\rho \frac{1}{2} \int_\Omega |\boldsymbol{u}|^2 \, \mathrm{d}V$.

Although the preceding paragraphs did not refer explicitly to time, of course all of these equations and conditions are required to hold at every moment t. Let us recall also here that our main interest lies in determining the solution for $t > 0$ from an **initial condition** of the form

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0, \tag{13}$$

where the vector field $\boldsymbol{u}_0$ is suposed to be given.

2.6. The region Ω occupied by the fluid will be supposed to be either three-dimensional or two-dimensional. The latter case is not just an academic exercise, but it corresponds to the case of a three-dimensional region of the form $\{(x,y,z) \mid (x,y) \in \Omega,\ z \in \mathbb{R}\}$ where every particle of the fluid is moving on a plane perpendicular to the direction z, and this motion is independent of z. Such motions are called *plane motions*.

In particular, one can single out the following three cases, with $n = 2$ or 3:

(I) Ω is a bounded region of $\mathbb{R}^n$ and $\boldsymbol{u}$ is required to satisfy condition (10).

(II) Ω is the whole space $\mathbb{R}^n$ and $\boldsymbol{u}$ is required to satisfy condition (11).

(III) Ω is the torus $\mathbb{R}^n/\mathbb{Z}^n$; or equivalently: Ω is the whole space $\mathbb{R}^n$ and $\boldsymbol{u}$ is required to satisfy a condition of periodicity in n orthogonal directions.

The cases (II) and (III) are less realistic than (I), but they retain the essential difficulties of the problem while simplifying some technical aspects; this simplification is due to the fact that some aspects of the problem lend themselves to explicit computation, mainly because $\partial\Omega = \emptyset$. This relative simplicity is the reason why the "millennium problem" posed by the *Clay Mathematics Institute* is restricted to the cases (II) and (III) (with $n=3$).

On the other hand, we will also specialize to the case where distance forces are not present: $\boldsymbol{f} = \boldsymbol{0}$. Instead, we could consider the general case where the field of distance forces is the gradient of a "potential", $\boldsymbol{f} = \nabla\phi$, which is the case of gravity. But this apparently more general case reduces to $\boldsymbol{f} = \boldsymbol{0}$ by taking $q = p - \phi$ as a new variable instead of p. So, we are concerned with the problem of solving the Navier-Stokes equations (9) (with $\boldsymbol{f} = \boldsymbol{0}$) and (6) together with conditions (10), (11) and (13). The complete system of equations is thus the following:

$$\begin{gathered} \partial\boldsymbol{u}/\partial t = \nu\,\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \nabla p, \qquad \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \qquad \boldsymbol{u}|_{\infty} = \boldsymbol{0}, \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{gathered} \tag{14}$$

where the expression $\boldsymbol{u}|_{\infty} = \boldsymbol{0}$ must be understood as a symbolic representation of the condition at infinity (11). From now on we will refer to these equations also as (14.1)–(14.5), and instead of $(10)_{\perp}$ and $(10)_{/\!/}$ we will also write $(14.3)_{\perp}$ and $(14.3)_{/\!/}$. Notice that in the case (I) there is no way for $\boldsymbol{x} \in \Omega$ to go away to infinity, so that condition (14.4) is then empty. On the other hand, condition (14.3) is empty in the case (II), because then $\partial\Omega = \emptyset$. Finally, in the case (III) Ω is a torus, without any boundary nor the possibility of going away to infinity, so that both (14.3) and (14.4) are then empty.

3 Perspectives of determinism

3.1. Let us begin by noticing that we have somehow so many equations as unknowns. In fact, only two of the equations listed in (14) are required to hold

on the whole of Ω, namely equations (14.1) and (14.2), of which the first one has a vector character whereas the second has a scalar character. Correspondingly, there are also two unknown functions to be determined on the whole of Ω, namely the vector field $\boldsymbol{u}$ and the scalar field p.

As we have already suggested in § 1, we intend to look at system (14) as an "abstract" equation of the form (3), namely $\mathrm{d}z/\mathrm{d}t = F(z)$, where z is an object that represents the varying state of the system. According to what we have just said, it looks like z amounts to specifying the fields of velocity and pressure, $\boldsymbol{u}$ and p. This is true to a certain extent, but there is something which looks awkward: If we know the fields $\boldsymbol{u}$ and p at a certain moment of time, then we know the right-hand side of (14.1), which determines $\partial\boldsymbol{u}/\partial t$. However, equation (14.2), which corresponds somehow to pressure, does not involve $\partial p/\partial t$. The solution to this puzzle is that conditions (14.2), $(14.3)_\perp$ and (14.4) determine p directly from $\boldsymbol{u}$: at every moment in time there is only one pressure field (except for an additive constant) that keeps the evolution of $\boldsymbol{u}$ within those conditions.

This is due to a result that can be traced back to Stokes (1849) [7] and Hermann von Helmholtz (1858) [9] and that can be stated in the following terms: *Every vector field on a bounded region decomposes in a unique way as a sum of a gradient field plus a solenoidal field parallel to the boundary; the same holds for an unbounded region if one restricts oneself to fields vanishing at infinity*. So, for every vector field $\boldsymbol{g}$ on Ω that satisfies $\boldsymbol{g}\,|_\infty = \boldsymbol{0}$ there is a unique gradient field ∇q for which $\boldsymbol{v} = \boldsymbol{g} - \nabla q$ satisfies $\nabla\cdot\boldsymbol{v} = 0$ in Ω, $v_\perp = 0$ in $\partial\Omega$, and $\boldsymbol{v}\,|_\infty = \boldsymbol{0}$. By applying this result to the vector field that appears in equation (14.1), namely $\boldsymbol{g} = \nu\,\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}$, we see that ∇p is indeed determined by the restriction that $\partial\boldsymbol{u}/\partial t$ keeps conditions (14.2), $(14.3)_\perp$ and (14.4).

According to these remarks, the problem that we are dealing with fits into an abstract equation of the form $\mathrm{d}z/\mathrm{d}t = F(z)$, where z reduces to specifying the velocity field $\boldsymbol{u}$.

3.2 Like many other differential equations, the equations of Navier-Stokes do not lend themselves to an explicit resolution. At least, this can be done only in just a few particular cases. However, one can always try to solve them numerically; in other words, one can look for an approximate solution by means of numerical methods.

In this connection it is worth mentioning that the development of numerical methods for partial differential equations was motivated to a certain extent by a field quite related to the one that we are dealing with, and quite relevant to daily life, namely meteorology. Certainly, the weather is a matter of certain physical processes, among which it is especially relevant the motion of the atmosphere as a fluid; so one can look at its evolution in time as governed by a system of partial differential equations that contains certain variants of the equations of fluid dynamics [69]. Already in 1904, the norwegian mathematician Vilhelm

Bjerknes was looking at weather forecast as a question of numerically solving the relevant differential equations from a given initial state [11]. Of course, he lacked the computing tools of today, instead of which he made use of certain graphical techniques. A few years later, in 1920, the british mathematician Lewis Fry Richardson was attacking the same problem by means of the finite-difference method, that is, by resorting to a close net of discrete values of the coordinates of space and time and replacing the partial derivatives by quotients of finite differences. Nevertheless, he still lacked the present day computing technology: in order to work out a six-hours forecast he needed six weeks of intensive calculation! Such figures started to change at the middle of the twentieth century with the arrival of the first computers: in 1953, as a final result of a project promoted by John von Neumann, the computer MANIAC I (Mathematical Analyzer, Numerical Integrator And Computer) of Princeton's Institute for Advanced Study was already able to produce reasonable weather forecasts of 24 hours with a computing time of the order of 10 minutes. Since then, the power of computers has not ceased to increase, and the numerical methods have also been made more efficient, with the result that the numerical solution of the equations of meteorology or the Navier-Stokes equations is nowadays within reach of a personal computer. The reader interested in the state of the art of the numerical solution of the Navier-Stokes equations is referred to [70], which reviews the so-called method of finite elements.

Anyway, the computational meteorologists did not take long to realize that the predictions produced by the computers were very sensitive to small changes in the initial data: slightly different initial states led often to quickly diverging evolutions, so that after a relatively short period of time these evolutions seemed quite unrelated to each other. This sensitivity with respect to the initial data was put forward in a crucial symposium held in Tokyo in 1960, where Edward Lorenz reported several "experimental" instances of it. Shortly afterwards, in 1963, Lorenz himself showed that the phenomenon can already be present in the exact solutions of quite simple systems of ordinary differential equations. Nowadays we know it as "butterfly effect", by the title of a talk given by Lorenz in 1972: "Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

3.3 These numerical and theoretical remarks made by Lorenz bear some resemblance to certain experimental observations on real fluids that had been made already in the nineteenth century, mainly by Osborne Reynolds in 1883 [10].

Reynolds' experiment dealed with the flow of water through a long horizontal pipe. The entrance to the pipe was submerged in a large reservoir where the water was practically at rest. At the other end there was a valve that allowed to control the rate of flow, i. e. the amount of water passing through the pipe per unit of time. The entrance was a well-rounded funnel and near to it there was a capillary that let aniline-dyed water out towards the entrance of the pipe. The whole apparatus was made of glass, which allowed to observe the motion of

the dyed water. When the flow rate was small, the dyed water formed a thread which remained clear-cut and rectilinear along the whole length of the pipe. In contrast, for larger flow rates this clear-cut and rectilinear character lasted only up to a certain distance of the entrance, after which the thread started undulating and wiggling until the dye was eventually dispersed over the whole section of the pipe.

This phenomenon is called **turbulence**, since what is observed is not so different from the ordinary meaning of this term. Generally speaking, turbulence occurs for sufficiently large values of a dimensionless parameter today known as the Reynolds number, namely $Re = Ud/\nu$, where U represents the average velocity of the fluid through the pipe (that is, the rate of flow divided by the area of the section), d is the diameter of the pipe, and ν is the kinematic viscosity of the fluid.

Reynolds' experiment is especially remarkable in that the turbulent motion develops out of a flow that starts being quite straight and apparently should remain in the same way, but it does not. This is highly reminiscent of a lack of determinism or a high sensitivity with respect to the initial conditions. Somehow, the variable time t is here replaced by the distance x from the beginning of the pipe, but these two variables are related to each other through the average velocity U, which is fixed by the experimental settings.

3.4 So, both the experiments with real fluids and the numerical simulations show signs of an apparent lack of determinism, or at least a high sensitivity with respect to the initial conditions.

Concerning the numerical simulations, one must keep in mind that they are not giving the exact solutions of the equations of motion of the fluid. In principle, one expects that the numerical solution will approach the exact one as the mesh of the net becomes smaller. However, in the same way as solutions can show a high sensitivity to small changes in the initial conditions, they could also show a similar phenomenon with respect to small changes in the equations. Therefore, even if we get numerical solutions defined for all times $t \geq 0$, in general it is not clear that they stay near to an exact solution.

Altogether the following question therefore stands:

Question 1 *Is it true that the Navier-Stokes equations determine all the future motion of the fluid out of its initial state?*

In other words: Is it true that for every initial state there is only one solution of the Navier-Stokes equations, and that this solution is determined for all time $t \geq 0$? Or still: Is it true that the "evolution problem" for the Navier-Stokes equations has the properties of *existence, uniqueness and globality* of solutions?

4 The functional point of view

4.1. We have seen that in order to specify the state of motion of an incompressible fluid it suffices to give the velocity field $\boldsymbol{u}$. As we have already mentioned, we think of the dependence of $\boldsymbol{u}$ on the position $\boldsymbol{x}$ as an "internal" component of a certain object, and it is this object as a whole what we view as varying with the time t. This point of view is characteristic of *functional analysis*, where functions defined on some region Ω are viewed as points of certain abstract spaces.

In fact, one can say that the study of the Navier-Stokes equations has played an important role in the development of several parts of functional analysis. For instance, later on we will see that some crucial ideas of the celebrated theory of distributions, which Laurent Schwartz formulated in detail in 1950–51, appear already in the works of Carl Oseen (1910) and Jean Leray (1933–34) about the Navier-Stokes equations (significantly, in 1934–35 the latter was exposing them in some lectures at the Collège de France where the young Schwartz was one of the attendees). Other subjects of functional analysis that have received a definite impulse from the study of the Navier-Stokes equations are, for instance, semigroup theory, differential equations in Banach spaces, and fractional powers of operators.

4.2. In the following we briefly introduce the basic notions that we need from functional analysis as well as their associated notations. Let us remember that the objects of our interest are functions on Ω, especially those with values in $\mathbb{R}^n$, like the velocity field $\boldsymbol{u}$. Such functions form certainly a linear space, but in contrast to $\mathbb{R}^n$ this space does not have a finite basis. Even so, one can still apply several tools of great utility.

To begin with, we will make use of the notion of **norm**, which can be viewed as extending the notion of length of a vector from $\mathbb{R}^n$. A norm is just a suitable way to measure the *magnitude* of a "vector". In the case that interests us, one can choose from multiple options, among which we will single out the sup norm $\|\boldsymbol{u}\|_\infty = \sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{u}(\boldsymbol{x})|$ and the quadratic norm $\|\boldsymbol{u}\|_2 = (\int_\Omega |\boldsymbol{u}(\boldsymbol{x})|^2 \,\mathrm{d}V)^{1/2}$, where $|\boldsymbol{w}|$ denotes the length of a vector $\boldsymbol{w}$ from $\mathbb{R}^n$. The quadratic norm is especially significant in our problem, since its square is essentially the kinetic energy (it differs from it by the factor $\rho/2$).

Matching up with the quadratic norm, it is interesting to consider also the operation that combines two vector fields $\boldsymbol{u}, \boldsymbol{v}$ into the number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \,\mathrm{d}V$, where $\boldsymbol{w} \cdot \boldsymbol{w}'$ denotes the scalar product of two vectors $\boldsymbol{w}$ and $\boldsymbol{w}'$ from $\mathbb{R}^n$. In particular, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = \|\boldsymbol{u}\|_2^2$. This operation is referred to as **scalar product** (or inner product) of the vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$, since its properties are analogous to those of the scalar product of vectors from $\mathbb{R}^n$. By analogy with the latter case, the functional scalar product that we have just defined gives rise to a notion of *orthogonality*, which is especially useful in the problem that we are dealing with. By definition, two vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ are

orthogonal to each other when $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$. On the other hand, sometimes the question is not so much whether $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is zero, but whether it is positive or negative, that is, whether the *angle* formed by $\boldsymbol{u}$ and $\boldsymbol{v}$ is acute or obtuse.

Every norm gives rise to a notion of **distance**, which determines a topology. By definition, the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is the magnitude of their difference, that is $\|\boldsymbol{u} - \boldsymbol{v}\|$. Contrary to what happens in finite-dimensional vector spaces, a sequence of functions can converge in one norm but diverge in another. Similarly, if we have a vector field that varies with time, this variation can be continuous in one norm but not in another. Therefore, the choice of the norm can be crucial when analyzing the different aspects of the problem that we are dealing with.

On the other hand, in order to ensure the existence of solutions for certain equations, it is essential to place oneself in a **complete** space (that is, a space where all Cauchy sequences are convergent). In other words, one must place oneself in some Banach space (i. e. a complete normed space) or a Hilbert one (the case where the norm derives from a scalar product). Unfortunately, completeness conflicts with the regularity that we have adopted as a working hypothesis in § 2.2 : regular functions do not form complete spaces in the norms that interest us (like $\|\boldsymbol{u}\|_\infty$ or $\|\boldsymbol{u}\|_2$). So, one must rely on certain constructions of completion, or equivalently, of closure in certain complete spaces defined in advance.

Among the latter one has the spaces $C(\bar{\Omega})$ and $L_2(\Omega)$. The space $C(\bar{\Omega})$, which uses the sup norm $\|\boldsymbol{u}\|_\infty$, is made up simply by all functions that are continuous and bounded on $\bar{\Omega}$. The space $L_2(\Omega)$, which uses the quadratic norm $\|\boldsymbol{u}\|_2$ and the scalar product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$, is made up by all functions on Ω whose square is integrable in the sense of Lebesgue (here one identifies two functions whenever they differ only in a set of zero measure). In general, it will be clear from the context whether we are dealing with functions with values in $\mathbb{R}$ or $\mathbb{R}^n$; otherwise, we will make it explicit by means of notations like $L_2(\Omega, \mathbb{R}^n)$.

Besides being a natural starting point for the completion constructions that give the spaces where solutions are looked for, regular functions play another fundamental role —somehow dual to the preceding one— in the definition of a weak solution, which will be given later on. In this connection, it will be useful to fix some particular sets of regular functions. Somehow, it amounts to defining what do we mean by a regular function and distinguishing those that satisfy certain conditions that interest us, like vanishing at $\partial\Omega$ and at infinity. On the other hand, the results that are obtained through the use of these sets are relatively independent of the exact way that they are defined. For our purposes, it will suffice to consider the following sets of functions of $\boldsymbol{x} \in \Omega$:

- $\mathcal{D}(\bar{\Omega})$: the set formed by the functions on Ω that are infinitely differentiable, with bounded derivatives of all orders, and vanish when $|\boldsymbol{x}|$ exceeds a certain value (which may depend on the function; the latter condition is empty when Ω is bounded);
- $\mathcal{D}(\Omega)$: the subset of $\mathcal{D}(\bar{\Omega})$ formed by the functions that vanish in

a whole neighbourhood of $\partial\Omega$ (which may depend on the function); such functions are said to have compact support in Ω.

$\mathcal{D}_\sigma(\Omega)$: the subset of $\mathcal{D}(\Omega,\mathbb{R}^n)\simeq\mathcal{D}(\Omega)^n$ formed by solenoidal vector fields (i. e. those with zero divergence).

On the other hand, it will be necessary to consider also regular functions of $\boldsymbol{x}\in\Omega$ and $t\in[0,T)$. In this connection, we will make use of the following sets, where $\mathcal{X}(\Omega)$ stands for either $\mathcal{D}(\bar\Omega)$ or $\mathcal{D}(\Omega)$ or $\mathcal{D}_\sigma(\Omega)$, and the value of T will be clear from the context:

$\widetilde{\mathcal{X}}(\Omega)$: the set formed by the functions of $\boldsymbol{x}\in\Omega$ and $t\in[0,T)$ that are infinitely differentiable with respect to these two variables and belong to $\mathcal{X}(\Omega)$ for every $t\in[0,T)$.

5 Successive approximations and classical solutions

5.1. The first results about the existence and uniqueness of (exact) solutions of the Navier-Stokes were obtained by the swedish mathematician Carl Oseen in a series of articles that started off in 1907. In one of them, published in 1910 [12: § II.3], Oseen obtained solutions for $\Omega=\mathbb{R}^3$ by means of a method that Joseph Liouville (1830), Émile Picard (1890) and other authors had successfully applied to other differential equations and that is today quite standard, especially in the case of ordinary differential equations. It is traditionally known as the *method of successive approximations*, although in principle this name would be applicable to any other method where the solution $\boldsymbol{u}$ is obtained as the limit of some sequence $\boldsymbol{u}_m$ $(m=1,2,...)$. What is characteristic of the method of successive approximations is that this sequence results from an *iteration*, i. e. each approximation $\boldsymbol{u}_m$ $(m>1)$ is determined from the preceding one $\boldsymbol{u}_{m-1}$ (the first one $\boldsymbol{u}_1$ will be completely arbitrary). More specifically, each step is based upon solving a (non-homogeneous) *linear* problem which is related to the non-linear one that interests us. Such a plan still admits of several variants; in fact, the one described below is different from the one used by Oseen.

Let us recall that we would like to solve problem (14), which we rewrite here with a little change in the order of the terms of (14.1):

$$\begin{aligned}\partial\boldsymbol{u}/\partial t-\nu\,\Delta\boldsymbol{u}+\nabla p\;&=\;-(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u},\qquad\nabla\cdot\boldsymbol{u}\;=\;0,\\ \boldsymbol{u}|_{\partial\Omega}\;=\;0,\qquad\boldsymbol{u}|_{\infty}\;&=\;0,\qquad\boldsymbol{u}|_{t=0}\;=\;\boldsymbol{u}_0.\end{aligned}\tag{14}$$

In order to approach the solution of this system, we will base ourselves upon knowing how to solve the following non-homogeneous linear one:

$$\begin{aligned}\partial\boldsymbol{u}/\partial t-\nu\,\Delta\boldsymbol{u}+\nabla p\;&=\;\boldsymbol{f},\qquad\qquad\nabla\cdot\boldsymbol{u}\;=\;0,\\ \boldsymbol{u}|_{\partial\Omega}\;=\;\boldsymbol{0},\qquad\boldsymbol{u}|_{\infty}\;&=\;\boldsymbol{0},\qquad\boldsymbol{u}|_{t=0}\;=\;\boldsymbol{u}_0,\end{aligned}\tag{15}$$

where $\boldsymbol{f}$ is a given function of $\boldsymbol{x}$ and t.

5.2. Problem (15) is akin to the heat equation $\partial u/\partial t-\nu\Delta u=f$ with supplementary conditions analogous to those of (15) (except (15.2), since in the

heat equation u is a scalar). In fact, solving (15) amounts to combining the solution of the heat equation with the Stokes-Helmholtz decomposition that was mentioned in § 3.1. Altogether it turns out that the solution of (15) can be expressed by means of a formula of the kind

$$\boldsymbol{u}(t) = \Gamma(t) * \boldsymbol{u}_0 + \int_0^t \Gamma(t-s) * \boldsymbol{f}(s)\,\mathrm{d}s, \tag{16}$$

where $\Gamma(t)$ represents a $n \times n$ matrix that depends on t and two spatial variables $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and, for a general vector field $\boldsymbol{v}$, $\Gamma(t) * \boldsymbol{v}$ represents the integral operator with kernel $\Gamma(t)$, that is

$$(\Gamma(t) * \boldsymbol{v})(\boldsymbol{x}) = \int_\Omega \Gamma(\boldsymbol{x}, \boldsymbol{y}, t)\, \boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}). \tag{17}$$

For $\Omega = \mathbb{R}^n$, the dependence of Γ on the spatial variables $\boldsymbol{x}, \boldsymbol{y}$ is through their diference $\boldsymbol{x} - \boldsymbol{y}$ (so the integral $\Gamma(t) * \boldsymbol{v}$ corresponds then to the notion of convolution, which is the usual meaning of the symbol "$*$"). However, we will keep the general form $\Gamma(\boldsymbol{x}, \boldsymbol{y}, t)$ so as to keep including the case $\Omega \neq \mathbb{R}^n$ whenever possible.

5.3. Certainly, system (14) can be seen as a particular case of (15) where $\boldsymbol{f} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. By applying formula (16), it follows that $\boldsymbol{u}$ is a solution of (14) if and only if it satisfies the following *integral equation*:

$$\boldsymbol{u}(t) = \Gamma(t) * \boldsymbol{u}_0 - \int_0^t \Gamma(t-s) * ((\boldsymbol{u}(s)\cdot\nabla)\,\boldsymbol{u}(s))\,\mathrm{d}s. \tag{18}$$

Notice that this equation does not involve the derivative of $\boldsymbol{u}$ with respect to t, nor any second derivative of $\boldsymbol{u}$ with respect to $\boldsymbol{x}$. So, equation (18) is *less demanding than* (14) *about the regularity to be assumed a priori* on $\boldsymbol{u}$. On the other hand, as soon as $\boldsymbol{u}$ has a certain minimum of regularity, for instance, if it has bounded and continuous spatial derivatives, then being a solution of (18) automatically implies the regularity required by (14) and being a solution of it. This is due to a regularizing effect produced by Γ.

5.4. In the following we will transform equation (18) into another one that is still less demanding about the regularity to be assumed a priori on $\boldsymbol{u}$. This step is not strictly necessary, but it simplifies the ensuing treatment (in particular, it allows to treat at the same time the cases $n=2$ and $n=3$).

More specifically, we will transform the expression $\Gamma(t-s) * ((\boldsymbol{u}(s)\cdot\nabla)\,\boldsymbol{u}(s))$ of (18) into another one that is equivalent to it whenever $\boldsymbol{u}$ is a solution. In fact, in this case one has $\nabla\cdot\boldsymbol{u} = 0$, so that the expression $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ can be rewritten in the following way (where the different components of a vector are indicated by means of subscript indices): $\sum_k u_k \nabla_k u_j = \sum_k \nabla_k (u_k\, u_j)$. This leaves things ready for an integration by parts inside the integral indicated by the symbol "$*$", which transfers the differential operator ∇_k to $\Gamma(t-s)$.

As a result of these operations, one arrives at a new integral equation that can be written as follows:

$$\boldsymbol{u}(t) = \Gamma(t) * \boldsymbol{u}_0 + \int_0^t \nabla_{\boldsymbol{y}} \Gamma(t-s) * \boldsymbol{u}^{(2)}(s)\,\mathrm{d}s, \tag{19}$$

where $\boldsymbol{u}^{(2)}$ represents the energy tensor, i. e. the symmetric matrix formed by the products $u_i u_j$, and $\nabla_{\boldsymbol{y}}\Gamma$ operates on $\boldsymbol{u}^{(2)}$ in the following way: $\sum_{jk}(\partial\Gamma_{ij}/\partial y_k)u_k u_j$.

As one can see, equation (19) *does not contain any derivative* of the function $\boldsymbol{u}$. Even so, and similarly to what we said about equation (18), Γ and $\nabla_{\boldsymbol{y}}\Gamma$ have a regularizing effect which guarantees that, if $\boldsymbol{u}$ has a minimum of regularity, for instance it is simply bounded and continuous, then being a solution of (19) automatically implies the regularity which is required for going back to equations (18) and (14). Furthermore, one can prove that in these conditions the functions $\boldsymbol{u}$ and p are in fact infinitely differentiable.

5.5. In order to look for the solution of (19), and therefore of (14), it is quite natural to proceed by iteration according to the following formula (the first approximation $\boldsymbol{u}_1$ can be taken simply as $\boldsymbol{u}_1 = \boldsymbol{0}$):

$$\boldsymbol{u}_m(t) = \Gamma(t) * \boldsymbol{u}_0 + \int_0^t \nabla_{\boldsymbol{y}} \Gamma(t-s) * \boldsymbol{u}^{(2)}_{m-1}(s)\,\mathrm{d}s. \tag{20}$$

If everything goes as expected, the sequence $\boldsymbol{u}_m$ will converge towards some limit $\boldsymbol{u}$ and this limit will satisfy (19).

In order to obtain the desired convergence to some limit, it suffices to ensure that $\boldsymbol{u}_m$ is a Cauchy sequence in a suitable complete space. In this connection, the preceding formulation, based upon equation (19), allows to work in a space of the form $C([0,\tau], C(\bar{\Omega}))$ with the norm $\sup_{0\le t\le\tau} \|\boldsymbol{u}(t)\|_\infty$ (this space is certainly equivalent to $C(\bar{\Omega}\times[0,\tau])$ with the norm $\sup_{0\le t\le\tau,\,\boldsymbol{x}\in\Omega} |\boldsymbol{u}(\boldsymbol{x},t)|$).

Unfortunately, in order to achieve the convergence it turns out necessary to restrict τ to be small enough. So, in general this method does not give a global solution (applying the same procedure with $\boldsymbol{u}_0$ replaced by $\boldsymbol{u}(\tau)$ and so on is not ensured to work because the subsequent values of τ could become progressively smaller and their sum could be finite).

What does not present any problem is the uniqueness of the solution obtained. In fact, this property is essentially a consequence of the same mechanism that ensures that $\boldsymbol{u}_m$ is a Cauchy sequence.

5.6. In order to convert the preceding arguments into a proper proof it is essential to have suitable bounds on the entries of the matrix Γ and their derivatives. In this connection, the case $\Omega = \mathbb{R}^n$ has the great advantage that there are explicit formulae for $\Gamma(\boldsymbol{x},\boldsymbol{y},t)$, which were obtained by Oseen in 1907. These formulae allow to arrive at the following result, which is due

to Leray (1934) [20: § III] (part (b) requires some additional arguments related to § 6.2):

Theorem 1 *Let* $\Omega = \mathbb{R}^n$ $(n = 2, 3)$. *If the initial state* $\boldsymbol{u}_0$ *is regular enough, then the solution* $\boldsymbol{u}(t)$ *of* (14) *is determined, that is, it exists and is unique, in a time interval of the form* $0 \le t < T(\boldsymbol{u}_0)$, *where* $T(\boldsymbol{u}_0)$ *can be infinite. Furthermore, there exist two constants* C_1 *and* C_2 *such that:*

(a) *If* $T(\boldsymbol{u}_0) < \infty$ *then* $\|\boldsymbol{u}(t)\|_\infty > \dfrac{C_1\,\nu}{(\nu(T(\boldsymbol{u}_0) - t))^{1/2}}$ *as* $t \to T(\boldsymbol{u}_0)$.

(b) *If* $\dfrac{1}{\nu^3}\,\|\boldsymbol{u}_0\|_\infty\,\|\boldsymbol{u}_0\|_2^2 < C_2$ *then* $T(\boldsymbol{u}_0) = \infty$.

5.7. As one can see in the works of Leray from 1933–34, the method that we have just described allows for certain variations that lead to analogous results in several other norms [18: § IV; 19: § III; 20: § III]. On the other hand, we already mentioned that the first result of this kind was obtained by Oseen in 1910 [12: § II.3].

Remarkably, Leray dealed not only with the case $\Omega = \mathbb{R}^n$, but he was able to apply the preceding method also to the case where Ω is a convex bounded region of $\mathbb{R}^2$ [19: § III]. In general, the main difficulty of the case $\Omega \neq \mathbb{R}^n$ lies in obtaining suitable information about $\Gamma(\boldsymbol{x}, \boldsymbol{y}, t)$. A crucial step in this connection is being able to deal with the case where Ω is a half-space (a half-plane when $n = 2$). By analogy with other partial differential equations, one would expect that Γ_Ω could be constructed from the derivatives of $\Gamma_{\mathbb{R}^n}$ —the matrix function that Oseen had given in 1907—, but in the present case this parallelism does not work. Even so, Leray still recognized the solution of the half-plane problem in another of Oseen's articles, published in 1919 [15]. From here, he was able to construct the kernel Γ in the case of a convex bounded region of $\mathbb{R}^2$ —using complex variable methods—, and he obtained the bounds that allow to arrive at a result similar to theorem 1.

Since 1960 this method has been extended in several directions, especially by Kirill K. Golovkin and Vsevolod A. Solonnikov, who among other things eliminated the condition of convexity on Ω and extended the treatment to the three-dimensional case [32].

On the other hand, towards 1960 two methods appeared that made it easier to obtain results of the type of theorem 1 for $\Omega \neq \mathbb{R}^n$. One of them, which we will consider in more detail in § 10.2, had been introduced in 1957 by Andreĭ A. Kiselëv and Ol'ga A. Ladyzhenskaya. The other, which was introduced at about the same time, is the method based on "semigroup theory", which we will consider in § 10.3. Here we remark only that the latter is not so different from the procedure that we have been describing in this section.

5.8. Theorem 1 and the analogous results that we have just said to apply in the case $\Omega \neq \mathbb{R}^n$ can be summarized as follows:

Answer 1 *The Navier-Stokes equations determine the short-term future of the fluid, but in general it is not clear that they determine the long-term future. On the other hand, the latter does get determined when either the initial state is near enough to rest or the viscosity is large enough.*

6 The direction of the quadratic term · Dissipation of energy · Globality for plane motions

6.1. The reason why the preceding methods are not able to extend the solutions to arbitrarily long times is the quadratic character of the non-linear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$.

In fact, the problem is already present in the following simple ordinary differential equation:

$$\mathrm{d}u/\mathrm{d}t + \nu\, u \,=\, u^2, \qquad u(0) \,=\, u_0, \tag{21}$$

where u represents a number that depends on t, u_0 is its initial value, and ν is a positive parameter that will play a role analogous to the viscosity of a fluid. By making use of the elementary tools of infinitesimal calculus, the solution is easily found to be given by the following formula:

$$u(t) \,=\, \frac{\nu}{1 - \mathrm{e}^{\nu t}\,(u_0 - \nu)/u_0}.$$

By examining this expression, one immediately sees that in the case $u_0 > \nu$ the denominator vanishes for a certain positive value of t, which we will denote as $T(u_0)$. For this value of t there is not a well-defined value of u, and still less a well-defined value of $\mathrm{d}u/\mathrm{d}t$, so the preceding function is a solution of (21) only during the interval $[0, T(u_0))$. In contrast, for $u_0 \le \nu$ the denominator remains positive for all positive values of t, so the existence time $T(u_0)$ is then infinite. These facts are summarized in the following statements, which show a clear parallelism with the results of theorem 1:

(a) If $u_0 > \nu$ then $T(u_0) = \dfrac{1}{\nu}\,\log\dfrac{u_0}{u_0 - \nu}$,
and $u(t) = \dfrac{\nu}{1 - \mathrm{e}^{-\nu(T(u_0)-t)}} > \dfrac{1}{T(u_0) - t} \to \infty$ as $t \to T(u_0)$.

(b) If $u_0 \le \nu$ then $T(u_0) = \infty$
(and if $u_0 < \nu$ then $u(t) \to 0$ as $t \to \infty$).

If equation (21) is changed by inverting the sign of the quadratic term, i. e. u^2 is replaced by $-u^2$, the problem is still there, with the only difference that now the problematic initial states are $u_0 < -\nu$. However, the problem disappears when u^2 is replaced by $-u|u|$: in this case the existence time is infinite for any u_0. So, quadratic terms are not always at odds with global existence. In this example of a single ordinary equation, what matters is the sign of the term in question in comparison with the sign of u. In a higher-dimensional space one cannot talk about the sign of a vector, but one can

talk about its direction, so it will not be a matter of comparing signs but of comparing directions. To this effect it is quite suitable to use a scalar product.

As it will be seen next, the Navier-Stokes equations do not look bad at all from that point of view.

6.2. In fact, let us take equation (14.1), i. e. $\partial \boldsymbol{u}/\partial t - \nu\,\Delta \boldsymbol{u} + \nabla p = -(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}$, and see what happens when we multiply it by $\boldsymbol{u}$ in the sense of the functional scalar product $\langle \boldsymbol{u},\boldsymbol{v}\rangle$. Let us recall that this means taking the scalar product by $\boldsymbol{u}$ at every point of Ω followed by an integration over Ω. In order to simplify matters we will assume that Ω is bounded; however, in the unbounded case the condition at infinity (14.4) allows to arrive at the same conclusions.

To begin with, one immediately sees that the term $\partial \boldsymbol{u}/\partial t$ gives rise to the derivative of the total **kinetic energy** of the fluid:

$$\int_\Omega \boldsymbol{u}\cdot \partial \boldsymbol{u}/\partial t \; \mathrm{d}V \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\int_\Omega |\boldsymbol{u}|^2 \,\mathrm{d}V\right) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\,\|\boldsymbol{u}\|_2^2\right). \tag{22}$$

On the other hand, the viscosity term $\nu\Delta\boldsymbol{u}$ can be transformed by means of Green's formula (which is nothing else than a particular application of the divergence theorem). This formula gives rise to an integral over $\partial\Omega$, but this integral vanishes since $\boldsymbol{u}$ satisfies the non-slip condition (14.3):

$$\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{u}\;\mathrm{d}V = \int_{\partial\Omega} \boldsymbol{u}\cdot\nabla_\perp \boldsymbol{u}\;\mathrm{d}S - \int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{d}V = -\int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{d}V. \tag{23}$$

($|\nabla\boldsymbol{u}|^2$ means the sum of the squares of all derivatives of the form $\nabla_i u_j$).

Concerning the pressure term, the condition of incompressibility $\nabla\cdot\boldsymbol{u}=0$ leads to another application of the divergence theorem, with a final result equal to zero because the velocity field is parallel to the boundary, i. e. $u_\perp = 0$ on $\partial\Omega$:

$$\int_\Omega \boldsymbol{u}\cdot\nabla p\;\mathrm{d}V = \int_\Omega \nabla\cdot(p\,\boldsymbol{u})\;\mathrm{d}V = \int_{\partial\Omega} p\,u_\perp\,\mathrm{d}S \;=\; 0 \tag{24}$$

By the way, this shows that *the vector fields which are solenoidal and parallel to the boundary are orthogonal to gradient vector fields*. So, the Stokes-Helmholtz decomposition that was introduced in § 3.1 is quite analogous to the orthogonal decompositions often used by elementary geometry in $\mathbb{R}^n$.

Finally, let us see what happens with the quadratic term. The expression that must be integrated over Ω is the scalar product of $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ by $\boldsymbol{u}$, that is the quantity $\sum_{i,k} u_k\,(\nabla_k u_i)\,u_i$, which from now on will be denoted as $\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}$. This expression can be rewritten in a way that makes things much simpler. In fact, because of the form taken by the derivative of a square, we see that

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u} \;=\; \sum_{i,k} u_k\,(\nabla_k u_i)\,u_i \;=\; \sum_k u_k \nabla_k (\tfrac{1}{2}\sum_i u_i^2) \;=\; \boldsymbol{u}\cdot\nabla(\tfrac{1}{2}\,|\boldsymbol{u}|^2).$$

So, at the end we are taking the scalar product of $\boldsymbol{u}$ by a gradient, which according to the preceding paragraph is always zero:

$$\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}V = \int_\Omega \boldsymbol{u}\cdot\nabla(\tfrac{1}{2}\,|\boldsymbol{u}|^2)\,\mathrm{d}V \;=\; 0. \tag{25}$$

Altogether one arrives at the conclusion that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\,\|\boldsymbol{u}\|_2^2\right) \;=\; -\int_\Omega \nu\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}V \;\le\; 0. \tag{26}$$

So, the kinetic energy of the fluid decreases at a rate which involves the viscosity coefficient ν. As a limiting case, the energy can remain constant, but this happens only in the state of rest (in order for (26) to hold with an equality sign, $\nabla\boldsymbol{u}$ must vanish all over Ω, which implies that $\boldsymbol{u}$ is a constant, from which the non-slip condition allows to conclude that $\boldsymbol{u}=\boldsymbol{0}$). By integrating the preceding relation in time one obtains the so-called **energy equality**:

$$\tfrac{1}{2}\,\|\boldsymbol{u}(t)\|_2^2 \;+\; \int_{t_0}^{t}\!\!\int_\Omega \nu\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}V\mathrm{d}s \;=\; \tfrac{1}{2}\,\|\boldsymbol{u}(t_0)\|_2^2, \quad \text{whenever} \quad 0\le t_0\le t. \tag{27}$$

Here we have introduced a convention that we will use from now on: *unless we explicitly express the contrary, in any integral of the form* $\int_{\cdot\cdot}^{t}\ldots\mathrm{d}s$ *it will be understood that the expression under the integral sign is evaluated at time* s.

The preceding arguments appeared for the first time in a memoir of Stokes read in 1850 [8: part I, sec. V].

6.3. In particular, the preceding result entails that the quadratic norm $\|\boldsymbol{u}(t)\|_2$ is bounded by a number independent of $t\ge 0$, namely $\|\boldsymbol{u}_0\|_2$. More precisely,

> If $\boldsymbol{u}$ is a solution defined in an interval of the form $0\le t<T$, then $\|\boldsymbol{u}(t)\|_2$ is bounded by a quantity independent of t. (28)

If an analogous result were true for the sup norm, then the globality of solutions would follow. In fact, according to part (a) of theorem 1, if the existence time is finite then the sup norm must increase without limit as we approach that time, which contradicts the bound under consideration. But for the moment we only have a bound on the quadratic norm, which is perfectly compatible with an unbounded sup norm.

On the other hand, equality (27) imposes a bound not only on $\|\boldsymbol{u}(t)\|_2$, but also on the integral $\int_0^t\int_\Omega|\nabla\boldsymbol{u}|^2\,\mathrm{d}V\mathrm{d}s$. Most remarkably, in the two-dimensional case this fact turns out to be sufficient for deriving the boundedness of the sup norm and therefore the globality of the solution:

Theorem 2 *In the two-dimensional case the solution of problem* (14) *is determined for all* $t\ge 0$.

This result was proved by Leray in 1933 for $\Omega = \mathbb{R}^2$ [18: § IV], and by Ladyzhenskaya in 1958 for $\Omega \subseteq \mathbb{R}^2$ [26]. In both cases, the proof is crucially based upon certain integral inequalities quite related to the celebrated immersion theorems of Sergeĭ L. Sobolev, which the latter stated in 1936–50. More specifically, Leray based his proof on the following inequality [18: p. 74], where ω stands for an arbitrary disc in $\Omega = \mathbb{R}^2$:

$$\left(\int_{\partial\omega} |f|^2 \, \mathrm{d}S \right)^2 \leq 8 \int_\Omega |f|^2 \, \mathrm{d}V \int_\Omega |\nabla f|^2 \, \mathrm{d}V. \tag{29}$$

On the other hand, Ladyzhenskaya based herself in the following one [26c: p. 428], where Ω represents a bounded but otherwise arbitrary region of $\mathbb{R}^2$, and f is restricted to vanish on $\partial\Omega$:

$$\int_\Omega |f|^4 \, \mathrm{d}V \leq \frac{1}{2} \int_\Omega |f|^2 \, \mathrm{d}V \int_\Omega |\nabla f|^2 \, \mathrm{d}V. \tag{30}$$

([26c] gives a less adjusted constant). In both cases, especially the former, these inequalities were derived earlier that the similar results of other authors.

7 Singularities and turbulence · Weak solutions

7.1. The preceding theorem gives a positive answer to question 1 in the two-dimensional case, but not in the three-dimensional one. In the latter case, the only thing that we know by now is what is stated in theorem 1.

According to part (a) of that theorem, the properties of existence and uniqueness of a solution are guaranteed until a certain time T which can be finite, in which case there must be some point $\boldsymbol{X} \in \overline{\Omega}$ such that $|\boldsymbol{u}(\boldsymbol{x}, t)|$ takes arbitrarily large values as $(\boldsymbol{x}, t)$ approaches $(\boldsymbol{X}, T)$. Such a situation is referred to by saying that the solution develops a **singularity** in the point $(\boldsymbol{X}, T)$ (let us remark that Oseen already talked about singularities in 1910 [13], but his definition was concerned with the local behaviour of $\nabla \boldsymbol{u}$ rather than that of $\boldsymbol{u}$; this is due to the fact that he used an integral equation related to (18) instead of (19)).

Instead of question 1, we can therefore raise the following more specific one:

Question 2 *Is it possible for the solutions of the three-dimensional Navier-Stokes equations to develop singularities?*

On the other hand, part (b) of theorem 1 guarantees the absence of singularities whenever the initial state is near enough to rest or the viscosity is large enough. In this connection, it is interesting to notice that the latter conditions are quite analogous to the condition of smallness of the Reynolds number that guarantees the absence of turbulence in the experiments (§ 3.3).

In fact, since the dimensions of $\|\boldsymbol{u}_0\|_2^2$ are the square of a velocity times a volume, the expression that appears at the left-hand side of the inequality of part (a) of theorem 1 is analogous to the cube of a Reynolds number. In the same connection, it is quite interesting also the analogy between the time T at which the solution ceases to be determined and the distance required for turbulence to develop in Reynolds' experiment.

These analogies lead to answering question 2 by means of a conjecture:

Answer 2 (conjecture) *It looks like the solutions of the three-dimensional Navier-Stokes equations might develop singularities. Such singularities would be related to turbulence.*

This was at least the opinion shared by Oseen, Leray and Ladyzhenskaya. In fact, already in 1910 Oseen was expressing himself in the following terms [12: § II.3]:

> "According to our theory, therefore, it seems likely that irregularities may arise at the interior of an incompressible viscous fluid, even when both the external forces and the initial motion are completely regular".

Later on, in 1927, he explicitly conjectured a relation between singularities and turbulence [16: § 7_8, p. 82]:

> "A detailed study of the singularities which can occur in the motion of a viscous fluid is also interesting from another point of view. If singularities can arise, then we must obviously distinguish between two kinds of motion: the regular motions, i. e. motions without singularities, and the irregular motions, i. e. motions with singularities. On the other hand, hydraulics is already distinguishing between two kinds of motions: the laminar motions and the turbulent ones. This leads to conjecturing that the 'laminar' motions of the experiments correspond to the 'regular' motions of the theory, and the 'turbulent' motions of the experiments correspond to the 'irregular' motions of the theory. Only further researches can decide whether this conjecture is true".

Concerning Leray, it suffices to say that he chose the term "turbulent solution" for referring to a generalized notion of solution which, as it will seen in a while, can go past some singularities. In fact, Leray explicitly conjectured that solutions might develop singularities (the "reason" which he alludes to will be seen shortly, in § 7.2) [20: p. 193]:

> "However, it does not seem possible to deduce from this fact that the motion itself remains regular; I even pointed out a reason which makes me believe in the existence of motions that become irregular after a finite time; unfortunately, I have failed to construct an example of such a singularity".

Finally, concerning Ladyzhenskaya we can quote the following passage [30: § 8, p. 273]:

> "But one can not exclude the possibility that at some moment this smoothness will be destroyed (...) At such catastrophic moments the solution may branch (...) We think that such a branching of the solution is possible in the Navier-Stokes equations".

7.2. In an attempt to produce a specific example of a solution which developed a singularity, Leray [18: § III.9, p. 60–61] considered the so-called **self-similar solutions**. This name refers to solutions of the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \phi(t)\,\boldsymbol{U}(\phi(t)\boldsymbol{x}), \qquad p(\boldsymbol{x},t) = \phi(t)^2\,P(\phi(t)\boldsymbol{x}). \tag{31}$$

By introducing these expressions into equations (14.1), (14.2) and (14.4), one easily sees, by separation of variables, that the function ϕ should be of the form

$$\phi(t) = \frac{1}{(2\,\lambda\,(T-t))^{1/2}}, \tag{32}$$

with λ constant, and that the functions $\boldsymbol{U}(\boldsymbol{X})$ and $P(\boldsymbol{X})$ should satisfy the following equations, which involve the same constant λ:

$$\nu\,\Delta\boldsymbol{U} - \lambda\,((\boldsymbol{X}\cdot\nabla)\boldsymbol{U} + \boldsymbol{U}) - \nabla P \,=\, (\boldsymbol{U}\cdot\nabla)\boldsymbol{U}, \quad \nabla\cdot\boldsymbol{U} \,=\, 0, \quad \boldsymbol{U}\,|_\infty = \boldsymbol{0}. \tag{33}$$

If these equations had a solution different from zero for some $\lambda > 0$, then formulae (31) and (32) would give a solution of (14) defined for $t < T$ which would satisfy $\|\boldsymbol{u}(t)\|_\infty = (2\lambda(T-t))^{-1/2}\,\|\boldsymbol{U}\|_\infty$ and would have a singularity at the point $(\boldsymbol{0}, T)$. In this connection Leray remarked that, by means of some computations similar to those of § 6.2 (and suitable hypotheses on the behaviour of $\boldsymbol{U}$ and P at infinity), the scalar product of equation (33.1) by $\boldsymbol{U}$ results in the following equality:

$$\nu \int_\Omega |\nabla\boldsymbol{U}|^2\,\mathrm{d}V \,=\, \frac{1}{2}\,\lambda\,(n-2)\int_\Omega |\boldsymbol{U}|^2\,\mathrm{d}V. \tag{34}$$

In the case $n = 2$ this implies $\boldsymbol{U} = \boldsymbol{0}$, which shows the impossibility of such singularities, in agreement with theorem 2.

In the case $n = 3$ the possibility of such singularities has remained open until 1996, when Jindřich Nečas, Michael Růžička and Vladimír Šverák showed their impossibility under quite pertinent assumptions on $\boldsymbol{U}$, for instance that $\boldsymbol{U}$ is bounded and its square is integrable [47]. A crucial point of the proof lies in seeing that if $\boldsymbol{U}$, P, λ satisfy (33) then $\Pi = P + \frac{1}{2}\,|\boldsymbol{U}|^2 + \lambda\,\boldsymbol{X}\cdot\boldsymbol{U}$ satisfies a certain differential inequality which entails a maximum principle: for any bounded region ω, the maximum of Π on $\overline{\omega}$ is always attained at $\partial\omega$.

7.3. In front of the possibility that the solutions of the Navier-Stokes equations develop singularities, it is natural to insist upon the following question:

Question 3 *Can one give a meaning to the Navier-Stokes equations for vector fields that contain singularities?*

In fact, the presence of a singularity at a point $(\boldsymbol{x}, t)$ means that the velocity $\boldsymbol{u}$ is not well defined in this point (or maybe it is defined but it is not continuous) and still less are defined its derivatives $\nabla \boldsymbol{u}$, $\Delta \boldsymbol{u}$ and $\partial \boldsymbol{u}/\partial t$. Therefore, the differential equations that were supposed to determine the future motion of the fluid cease to make sense at singular points.

In this connection, it must be remembered that in section 5 we already replaced the differential equations by certain integral equations that make sense for a wider class of functions $\boldsymbol{u}$, namely equation (19) or alternatively equation (18). In fact, the integrals which appear in these equations can make sense even in the presence of some kinds of singularities. However, the integral equations (18) and (19) have been derived from the same differential equations which we say that cease to make sense in the presence of singularities. On the other hand, these differential equations come from another kind of integral equations that, again, make sense for a wider class of functions $\boldsymbol{u}$, namely equations (7) and (4) of section 2, which keep account of the amount of momentum and mass contained in an arbitrary part $\omega \in \mathcal{P}(\Omega)$. In front of this, it is quite natural to ask oneself whether the integral equations (18) and (19) can be obtained from the original integral equations (7) and (4) in a direct way, without going through their differential counterparts. This question was dealt with by Oseen himself in one of his articles of 1910 [14], where he showed how to go from one kind of integral equations to the other and vice versa.

Notably, one of the intermediate steps of this derivation [14: eq. (10), p. 10] corresponds essentially to a generalized notion of solution which was later adopted by Leray and which is today a standard tool for the study of partial differential equations of any kind. This generalization of the notion of solution admits of several variants, and the associated terminology is also quite varied: depending on the authors and the variant under consideration, one talks about weak solutions, generalized solutions, variational solutions or distributional solutions. Anyway, the essential idea is always the same.

7.4. In order to explain this idea we will start with a simple example that forms part of our problem, namely the differential equations which expresses the incompressibility of the fluid:

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \forall \boldsymbol{x} \in \Omega. \tag{6}$$

The classical notion of solution needs that certain partial derivatives —in this case $\partial u/\partial x$, $\partial v/\partial y$ and $\partial w/\partial z$— be well defined at each point of Ω. In order to pass to the weak notion of solution, the equation in question is multiplied by a "test function" ϕ and the result is integrated over the region Ω. The crucial step is the next one, which uses integration by parts so as to transfer the derivatives to the test function. In the case of equation (6), this operation

takes the following form:

$$\int_\Omega (\nabla \cdot \boldsymbol{u})\, \phi \, \mathrm{d}V \;=\; \int_{\partial\Omega} u_\perp \phi \, \mathrm{d}S \;-\; \int_\Omega \boldsymbol{u} \cdot \nabla\phi \, \mathrm{d}V.$$

The test functions are required to be regular enough for the arising integrals to make sense; in fact, one often imposes them to be infinitely differentiable. On the other hand, they are often required also to have compact support in Ω; this ensures that the arising integrals on $\partial\Omega$ are always zero, and in the case of an unbounded region it avoids having to worry about what happens at infinity. So, the test functions are often taken from the set $\mathcal{D}(\Omega)$ that we introduced in § 4.2. In the case of equation (6) the preceding operations lead to the following result:

$$\int_\Omega \boldsymbol{u} \cdot \nabla\phi \; \mathrm{d}V \,=\, 0, \qquad \forall \phi \in \mathcal{D}(\Omega), \tag{35}$$

Now, a weak solution of the differential equation (6) means simply a function $\boldsymbol{u}$ that satisfies equation (35). As one can see, this equation does not require any differentiability on $\boldsymbol{u}$, but only that it be integrable on compact subsets of Ω, i. e. on every bounded part of Ω that keeps away from the boundary $\partial\Omega$. The procedure that we have followed immediately implies that any classical solution is also a weak solution. On the other hand, if $\boldsymbol{u}$ is a weak solution, i. e. it satisfies (35), and it has the differentiability that is required by the classical notion, then the preceding sequence of operations can be reversed so as to derive that $\boldsymbol{u}$ is also a classical solution. To this effect, it is crucial that the integral equation (35) be satisfied for an arbitrary test function (and that the set of test functions be rich enough). In fact, the fulfilment of the differential equation (6) at each point $\boldsymbol{x}$ of Ω is obtained by a limit process where the test function ϕ is concentrated at the point $\boldsymbol{x}$.

Following Oseen, it is interesting also to compare the integral equation (35) with that from which we derived the differential equation (6), namely the integral equation (4):

$$\int_{\partial\omega} u_\perp \mathrm{d}S = 0, \qquad \forall \omega \in \mathcal{P}(\Omega) \tag{4}$$

This equation would correspond to the case where the test function ϕ is the characteristic function of the subregion ω. This function does not belong to $\mathcal{D}(\Omega)$, but it can be suitably approximated by functions from $\mathcal{D}(\Omega)$, which allows to go from (4) to (35). Conversely, one can go also from (35) to (4). As it was shown by Oseen, this can be achieved by means of a a close net of discrete values of the spatial coordinates; starting from (35), the central idea is approximating the partial derivatives of ϕ by finite differences and rearranging the terms so as to get what would be a finite-difference approximation of the divergence of $\boldsymbol{u}$; however, this approximation is not compared with the value of $\nabla \cdot \boldsymbol{u}$ at any point, but it is directly related to the flux integral through the boundary of the corresponding cell of the net; finally, the sum of these integrals leads to (4).

7.5. Let us consider now the differential equation that corresponds to Newton's second law. As we have been doing since § 2.6, we assume $\boldsymbol{f} = \boldsymbol{0}$:

$$\partial \boldsymbol{u}/\partial t \;=\; \nu\,\Delta \boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \nabla p, \qquad \forall \boldsymbol{x} \in \Omega. \tag{9}$$

This case is more involved than the preceding one. On the one hand, equation (9) has a vector character. This leads to considering vector test functions which operate on the different terms of the equation by means of the scalar product of $\mathbb{R}^n$ (followed by an integration). In the following they will be denoted by $\boldsymbol{\psi}$. Furthermore, it is interesting to restrict these vector test functions to be solenoidal. In this way, the process that leads to the integral equation which defines a weak solution has the virtue of automatically eliminating the pressure.

On the other hand, equation (9) combines spatial and temporal derivatives, which calls for test functions depending on both $\boldsymbol{x}$ and t. In this connection, we will opt for a variant where the test functions will not vanish in the limits of integration of the time variable. This will have the effect of including the initial condition (14.5) in the same integral equation that defines a weak solution of (9).

As a matter of fact, the calculations that lead from (9) to the weak formulation are quite analogous to those of § 6.2. The main difference is that there the equation was multiplied by the same function $\boldsymbol{u}$, whereas here we multiply it by the test function $\boldsymbol{\psi}$. These calculations lead to the following result:

$$\begin{aligned}\int_\Omega \boldsymbol{u}(t)\cdot\boldsymbol{\psi}(t)\,\mathrm{d}V \;&=\; \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\psi}(0)\,\mathrm{d}V \\ &\quad + \int_0^t \int_\Omega \Big(\boldsymbol{u}\cdot(\partial\boldsymbol{\psi}/\partial t) - \nu\,\nabla\boldsymbol{u}\!:\!\nabla\boldsymbol{\psi} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\psi}\Big)\,\mathrm{d}V\,\mathrm{d}s, \\ &\qquad \forall\boldsymbol{\psi}\in\widetilde{\mathcal{D}}_\sigma(\Omega), \quad \forall t\in[0,T),\end{aligned} \tag{36}$$

where we use the notations $\nabla\boldsymbol{u}:\nabla\boldsymbol{v} = \sum_{i,k}\nabla_k u_i\,\nabla_k v_i$ and $\boldsymbol{v}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w} = \sum_{i,k} v_k\,\nabla_k u_i\;w_i$, as well as the convention adopted at the end of § 6.2 in connection with $\int_{\cdot\cdot}^t \ldots \mathrm{d}s$. From here one can still perform a second integration by parts so as to complete the transferring of derivatives to the test function (in the last term one uses the equation $\nabla\cdot\boldsymbol{u} = 0$). The result of this second integration by parts is the following:

$$\begin{aligned}\int_\Omega \boldsymbol{u}(t)\cdot\boldsymbol{\psi}(t)\,\mathrm{d}V \;&=\; \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\psi}(0)\,\mathrm{d}V \\ &\quad + \int_0^t \int_\Omega \Big(\boldsymbol{u}\cdot(\partial\boldsymbol{\psi}/\partial t) - \nu\,\boldsymbol{u}\cdot\Delta\boldsymbol{\psi} - \boldsymbol{u}\cdot\nabla\boldsymbol{\psi}\cdot\boldsymbol{u}\Big)\,\mathrm{d}V\,\mathrm{d}s, \\ &\qquad \forall\boldsymbol{\psi}\in\widetilde{\mathcal{D}}_\sigma(\Omega), \quad \forall t\in[0,T),\end{aligned} \tag{37}$$

If $\boldsymbol{u}$ is regular enough (and satisfies the equation $\nabla\cdot\boldsymbol{u} = 0$), then one can follow the reverse path from (37) to (9). In this connection we will

remark only that the pressure appears as a consequence of the Stokes-Helmholtz decomposition (§ 3.1) and its orthogonal character. Concerning the equivalence between (9) and (36) or (37), it is interesting to remark also that the latter equations can be considered as an expression of the so-called *principle of virtual works*, which classical mechanics has been using since the eighteenth century in connection with Newton's second law.

The main difference between equations (36) and (37) is that the latter does not involve any derivative of $\boldsymbol{u}$. This is useful for certain purposes, but for other ones (for example, to give a proper meaning to the non-slip condition) it is advisable to assume a little more regularity in what concerns the dependence of $\boldsymbol{u}$ with respect to $\boldsymbol{x}$. On the other hand, we do not want to require it to be differentiable in a classical sense, since this would not allow to include singularities. In front of such a situation, it turns out to be very suitable the notion of **weak derivative** (which Leray called "quasi-derivative"): the weak derivative of a function f with respect to x_k can be defined simply as a function g such that f is a weak solution of the equation $\nabla_k f = g$. By applying this idea, we will assume that there exist some functions $\nabla_k u_i$ which possibly cannot be identified exactly with the classical derivatives of u_i with respect to x_k, but at any rate they are related to $\boldsymbol{u}$ in the following way:

$$\int_0^t \int_\Omega u_i \, (\nabla_k \phi) \, \mathrm{d}V \, \mathrm{d}s \, = \, - \int_0^t \int_\Omega (\nabla_k u_i) \, \phi \, \mathrm{d}V \, \mathrm{d}s,$$

$$\forall \phi \in \widetilde{\mathcal{D}}(\bar{\Omega}), \quad \forall t \in [0,T). \qquad (38)$$

Here we have allowed the test function to vary in $\widetilde{\mathcal{D}}(\bar{\Omega})$, and not only in $\widetilde{\mathcal{D}}(\Omega)$. In this way, equation (38) is saying not only that $\nabla_k u_i$ is the weak derivative of u_i with respect to x_k, but it also incorporates the non-slip condition (14.3).

7.6. Putting together the different pieces that we have introduced throughout § 7.4–7.5, we will define a **weak solution** of (14) on the interval $[0,T)$ as a vector field $\boldsymbol{u}$ together with a tensor field $\nabla \boldsymbol{u}$ (with components $\nabla_k u_i$) which depend on $t \in [0,T)$ and satisfy the following conditions for every value of t in the interval $[0,T)$:

(a) $|\boldsymbol{u}|^2$ is integrable on Ω, and the norm $\|\boldsymbol{u}(s)\|_2$ is bounded independently of $s \in [0,t]$.

(b) $|\nabla \boldsymbol{u}|^2 = \sum_{i,k} |\nabla_k u_i|^2$ is integrable on $\Omega \times [0,t]$.

(c) Equation (38) is satisfied.

(d) Equation (35) is satisfied.

(e) Either equation (36) or alternatively equation (37) are satisfied.

Although equation (37) is in principle more general than (36), one can show that in the presence of (a)–(d) these two equations are equivalent to each other. On the other hand, each of them is also equivalent to an apparently weaker

variant where the test function $\boldsymbol{\psi}$ does not depend on t (see for instance [23: Lemma 4.1]).

As we have been seeing, the equations (38), (35), (36) and (37) translate in some sense the equations (14.1) and (14.2) together with the non-slip condition (14.3) and the initial condition (14.5). On the other hand, the integrability conditions (a) and (b) are slightly stronger than what is needed for giving a proper meaning to equations (35)–(38). In fact, since the test functions vanish outside of a bounded set, it would suffice to restrict the requirement of integrability to bounded sets. However, in requiring the global integrability, conditions (a) and (b) can be understood as incorporating also a generalized version of the condition at infinity (14.4).

Anyway, by making use of the notion of regularity at infinity that was introduced in § 2.5, one sees that if the function $\boldsymbol{u}$ is regular enough then the fact of its being a solution of (14) in the weak sense implies that it is also a solution of (14) in the classical sense.

Altogether we get an affirmative answer to question 3:

Answer 3 *The notion of weak solution gives a meaning to the Navier-Stokes equations for vector fields that may contain singularities.*

As we have already mentioned, the integral equations (18) and (19), especially the latter, can also be applied to velocity fields with some kinds of singularities. However, those integral equations take it for granted the previous construction of the kernel Γ that solves the linear problem. As it was mentioned in § 5.7, in general terms this construction may present certain difficulties (see however § 10.3). In contrast, the notion of weak solution does not depend on such a construction.

7.7. Independently of its being a motivation behind the notion of weak solution, the possibility of solutions leading to singularities raises doubts about the validity of the equations, that is their being in agreement with the physical reality. In fact, let us remember that, besides Newton's second law, the differential equation (9) embodies another less fundamental physical law which is expressed by equation (8). This law postulates a linear relationship between the stress tensor T and the gradient tensor $\nabla \boldsymbol{u}$. The fluids that satisfy such a law are said to be Newtonian. Indeed, Newton was the first one to model the friction of fluids by means of a linear relationship of this kind. However, this linear relationship might be valid only as an approximation suitable for small gradients $\nabla \boldsymbol{u}$, whereas large gradients —which are unavoidable near a singularity— might involve deviations with respect to linearity.

Now, if such deviations go in a suitable direction then one can show that the solutions cannot develop any singularities; in particular, this is the case if one supposes a law of the same form as (8) where μ is not a constant but it grows

fast enough when $|\nabla \boldsymbol{u}|$ increases. Once again, this remark is present already on the work of Leray [19: § III.10, p. 61–62]. Afterwards, several authors have considered a dependence of the form $\mu = \mu_0 + \varepsilon\,|\nabla \boldsymbol{u}|^\alpha$ $(\mu_0, \varepsilon > 0)$ and they have studied how large must α be so as to guarantee the existence, uniqueness and globality of the solutions. This line of research was initiated in 1966 by Ladyzhenskaya, who needed $\alpha \geq 1/2$ (for $n = 3$) [28]. Later on, in 1993, Hamid Bellout, Frederick Bloom and Jindřich Nečas were able to weaken this condition to $\alpha \geq 1/5$ [46].

This non-Newtonian model is not interesting as a realistic one: in practice most fluids agree quite well with the Newtonian model, and those that deviate from it require other more involved models. However, the results that we have just mentioned suggest a new strategy for dealing with the Newtonian case, i. e. the Navier-Stokes equations that we have been considering until now. The idea is to consider a modification which, like the preceding one, gives existence, uniqueness and globality of solutions, and try to approach the Newtonian case while keeping these properties. For instance, in the preceding case it would be a matter of considering the limit $\varepsilon \to 0$.

On the other hand, with the purpose of looking at the Newtonian case, the above modification of the equations can be replaced by other possibilities which can be even less realistic but easier to deal with from a mathematical point of view. By means of such a strategy, Leray was able to prove that the evolution problem for the Navier-Stokes equations has always a weak solution defined for arbitrarily long times; what he was not able to prove is the uniqueness of such a global weak solution. These weak solutions, global but not necessarily unique, are the ones that Leray called "turbulent" solutions. In fact, Leray developed two different methods of this kind; one of them was used in the case where Ω is a convex bounded region of $\mathbb{R}^2$ [19: § IV] —where by now we already have a result of existence, uniqueness and globality (§ 6.3)—, and the other was used in the case $\Omega = \mathbb{R}^3$ [20: § V]. In the following section we will describe the second of these two methods.

8 Global solutions exist, but they might be not unique

8.1 As we have said, the idea is replacing problem (14) by a slightly modified one where one can prove the existence, uniqueness and globality of a solution; this modified problem must depend on a parameter that allows to progressively approach problem (14). Specifically, Leray considered a problem of the following form, where the parameter is ε and the approach to problem (14) corresponds to the limit $\varepsilon \to 0$:

$$\begin{gathered} \partial \boldsymbol{u}/\partial t - \nu\,\Delta \boldsymbol{u} + \nabla p \,=\, -((R_\varepsilon * \boldsymbol{u})\cdot\nabla)\,\boldsymbol{u}, \qquad \nabla\cdot \boldsymbol{u} \,=\, 0, \\ \boldsymbol{u}\,|_{\partial\Omega} \,=\, \boldsymbol{0}, \qquad \boldsymbol{u}\,|_{\infty} \,=\, \boldsymbol{0}, \qquad \boldsymbol{u}\,|_{t=0} \,=\, R_\varepsilon * \boldsymbol{u}_0, \end{gathered} \tag{39}_\varepsilon$$

As one can see, this problem differs from (14) only in that an integral operator of kernel R_ε has been introduced in a couple of places. This operator has a *regularizing* effect: $R_\varepsilon * f$ is defined for functions f with little regularity (it

suffices that f be integrable in a neighbourhood of every point) but the result is always an infinitely differentiable function. Furthermore, the resulting function *approaches* f as $\varepsilon \to 0$, this convergence being true in many different norms. Instead of a scalar function, the operator $R_\varepsilon *$ can be applied also to a vector field, in which case it is understood that it acts separately on each component. As it is remarked by Leray, for $\Omega = \mathbb{R}^n$ the preceding properties can be easily achieved by taking

$$R_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) \;=\; \frac{c_n}{\varepsilon^n}\; \varrho\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{\varepsilon}\right), \tag{40}$$

where $\varrho(r)$ is an infinitely differentiable function $[0,\infty) \to \mathbb{R}$ that vanishes for $r \geq 1$ and is positive for $0 \leq r < 1$ (for instance, $\varrho(r) = \exp(-(1-r^2)^{-1})$ for $r < 1$), and c_n is the constant that results in $\int_{\mathbb{R}^n} R_\varepsilon(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}V(\boldsymbol{y}) \;=\; 1$. From this construction one easily derives the following specific properties of R_ε, where f represents in each case an arbitrary function (as long as if gives a meaning to the objects under consideration) and C_ε is a constant that depends on ε:

$$\|R_\varepsilon * f\|_\infty \;\leq\; C_\varepsilon\, \|f\|_2, \tag{41}$$

$$\|R_\varepsilon * f - f\|_2 \;\to\; 0 \quad \text{as } \varepsilon \to 0, \tag{42}$$

$$\|R_\varepsilon * f\|_2 \;\leq\; \|f\|_2, \tag{43}$$

$$\|R_\varepsilon * f\|_\infty \;\leq\; \|f\|_\infty, \tag{44}$$

$$\nabla_k (R_\varepsilon * f) \;=\; R_\varepsilon * (\nabla_k f). \tag{45}$$

In particular, property (41) will be crucial for our purposes. In fact, let us recall —see for instance § 6.3— that our problem lies somehow in that a finite value of the quadratic norm $\|f\|_2$ is quite compatible with an infinite value of the sup norm $\|f\|_\infty$ (which corresponds to the notion of singularity as defined in § 7.1). However, property (41) ensures that $\|R_\varepsilon * f\|_\infty$ is finite whenever $\|f\|_2$ is finite.

The present considerations refer especially to the case $\Omega = \mathbb{R}^3$. However, the extension to the case $\Omega \subset \mathbb{R}^3$ is only a matter of suitably defining the operator R_ε (see for instance [42]), and having suitable bounds on the solutions of the linear problem (15), the same that were considered in § 5.7. In fact, problem $(39)_\varepsilon$ will be solved by a method analogous to that used in section 5.

8.2 By making use of properties (43)–(45), one easily checks that problem $(39)_\varepsilon$ admits the same treatment that in sections 5 and 6 was applied to problem (14). This leads to results completely analogous to theorem 1 and the energy inequality (27). The solution of $(39)_\varepsilon$ obtained in this way will be denoted as $\boldsymbol{u}^\varepsilon$. With this notation, the corresponding energy inequality reads as follows:

$$\tfrac{1}{2}\, \|\boldsymbol{u}^\varepsilon(t)\|_2^2 \;+\; \int_{t_0}^{t} \int_\Omega \nu\, |\nabla \boldsymbol{u}^\varepsilon|^2 \, \mathrm{d}V\, \mathrm{d}s \;=\; \tfrac{1}{2}\, \|\boldsymbol{u}^\varepsilon(t_0)\|_2^2, \;\text{ whenever } \; 0 \leq t_0 \leq t. \tag{46}$$

While in the case of problem (14) these results where not sufficient to derive the globality of the solution, now this conclusion does follow. In fact, owing to property (41), inequality (46) with $t_0 = 0$ allows to deduce the following bound independent of $t \geq 0$: $\|R_\varepsilon * \boldsymbol{u}^\varepsilon(t)\|_\infty \leq C_\varepsilon \|\boldsymbol{u}^\varepsilon(0)\|_2$, and introducing this bound in the integral equation homologous to (19) shows that $\|\boldsymbol{u}^\varepsilon(t)\|_\infty$ remains bounded for all $t \geq 0$. According to part (a) of theorem 1 (applied to problem $(39)_\varepsilon$), this allows to conclude that the solution $\boldsymbol{u}^\varepsilon$ is defined for all time $t \geq 0$. So, in contrast to (14), problem $(39)_\varepsilon$ has existence, uniqueness and globality of solutions.

8.3. Having seen that, now it is a question of considering the limit $\varepsilon \to 0$ and seeing: (a) whether $\boldsymbol{u}^\varepsilon$ converges to some limit, and (b) whether we can say that this limit satisfies in some sense the Navier-Stokes equations.

Concerning part (a), we advance that we will not get a limit of $\boldsymbol{u}^\varepsilon$ as ε approaches zero in an arbitrary way, but only when it does so through certain sequences ε_n $(n \to \infty)$; unfortunately, the limit of $\boldsymbol{u}^{\varepsilon_n}$ cannot be ensured to be independent of the sequence ε_n. So, this method will be far from giving a uniqueness result.

Concerning part (b), in principle it is a matter of considering the equation solved by $\boldsymbol{u}^{\varepsilon_n}$ and taking limits as $n \to \infty$, that is, as $\varepsilon_n \to 0$. However, when dealing with a differential equation this is a delicate matter: in fact, generally speaking it is quite possible that $\boldsymbol{u}^{\varepsilon_n}$ has a limit but its derivatives, for instance $\partial \boldsymbol{u}^{\varepsilon_n}/\partial t$, do not have any limit. This difficulty is solved, at least partially, by making use of the notion of weak solution.

8.4. The functions $\boldsymbol{u}^\varepsilon$ that were obtained above solve equation $(39)_\varepsilon$ in the classical sense and therefore they solve it also in the weak sense. According to the definition given in § 7.6, this means that the following conditions are satisfied for every $t \in [0, \infty)$: $|\boldsymbol{u}^\varepsilon|^2$ is integrable on Ω and $\|\boldsymbol{u}^\varepsilon(s)\|_2$ is bounded independently of $s \in [0, t]$; $|\nabla \boldsymbol{u}^\varepsilon|^2$ is integrable on $\Omega \times [0, t]$;

$$\int_\Omega \boldsymbol{u}^\varepsilon(t) \cdot \boldsymbol{\psi}(t)\, \mathrm{d}V = \int_\Omega (R_\varepsilon * \boldsymbol{u}_0) \cdot \boldsymbol{\psi}(0)\, \mathrm{d}V + \int_0^t \int_\Omega \Big(\boldsymbol{u}^\varepsilon \cdot (\partial \boldsymbol{\psi}/\partial t) + \nu\, \boldsymbol{u}^\varepsilon \cdot \Delta \boldsymbol{\psi} + (R_\varepsilon * \boldsymbol{u}^\varepsilon) \cdot \nabla \boldsymbol{\psi} \cdot \boldsymbol{u}^\varepsilon \Big)\, \mathrm{d}V\, \mathrm{d}s,$$

$$\forall \boldsymbol{\psi} \in \widetilde{\mathcal{D}}_\sigma(\Omega), \quad \forall t \geq 0; \qquad (47)$$

$$\int_\Omega \boldsymbol{u}^\varepsilon(t) \cdot \nabla \phi\, \mathrm{d}V = 0, \qquad \forall \phi \in \mathcal{D}(\Omega), \quad \forall t \geq 0; \qquad (48)$$

and finally

$$\int_0^t \int_\Omega u_i^\varepsilon\, (\nabla_k \phi)\, \mathrm{d}V\, \mathrm{d}s = -\int_0^t \int_\Omega (\nabla_k u_i^\varepsilon)\, \phi\, \mathrm{d}V\, \mathrm{d}s,$$

$$\forall \phi \in \widetilde{\mathcal{D}}(\bar{\Omega}), \quad \forall t \geq 0. \qquad (49)$$

8.5. In order to establish the existence of sequences ε_n which give a limit to $\boldsymbol{u}^{\varepsilon_n}$ one uses certain results similar to that which ensures that any bounded sequence of real numbers has a convergent subsequence (Bolzano-Weierstrass theorem). In functional spaces one has not exactly the same result, but there are some generalizations. Among those that interest us, the most classical one is due to David Hilbert (1906) and it is based upon the notion of **weak convergence** (of square-integrable functions). We will define this notion in the following way: a sequence of functions f_n defined on a region Ω is weakly convergent towards f when it satisfies

$$\int_\Omega f_n\,\phi\,\mathrm{d}V \;\to\; \int_\Omega f\,\phi\,\mathrm{d}V, \qquad \forall \phi \in \mathcal{D}(\Omega). \tag{50}$$

Instead of the preceding definition, one often asks for an equivalent analogous condition where the test function ϕ varies over the whole space $L_2(\Omega)$. We have preferred version (50) because it emphasizes the connection with the notion of weak solution. Besides the notion of weak convergence, one has also the notion of **strong convergence**: a sequence f_n is said to be strongly convergent towards f when $\|f_n - f\| \to 0$; as it is suggested by the terminology, the strong convergence implies the weak one, but the converse is not true. Anyway, the so-called Hilbert's selection principle ensures the following: if f_n is a sequence of functions whose quadratic norm is bounded independently of n, then one can extract a subsequence f_n' which has the property of being weakly convergent.

In order to apply this principle and other similar ones to our problem, it is crucial to have bounds on $\boldsymbol{u}^\varepsilon$ that do not depend on ε. Such bounds are provided by the energy equality (46). In fact, by putting $t_0 = 0$ and taking into account that $\boldsymbol{u}^\varepsilon(0) = R_\varepsilon * \boldsymbol{u}_0$, property (43) allows to derive the following bounds independent of ε:

$$\|\boldsymbol{u}^\varepsilon(t)\|_2 \;\le\; \|\boldsymbol{u}_0\|_2, \tag{51}$$

$$2\,\nu \int_0^t \int_\Omega |\nabla \boldsymbol{u}^\varepsilon|^2 \,\mathrm{d}V\,\mathrm{d}s \;\le\; \|\boldsymbol{u}_0\|_2^2. \tag{52}$$

From (51), Hilbert's selection principle immediately implies the existence of a sequence $\varepsilon_n \to 0$ such that $\boldsymbol{u}^{\varepsilon_n}(t)$ is weakly convergent. However, by proceeding in this way the sequence ε_n could depend on t, which obviously does not interest us, since we want the limit to depend on t not in an arbitrary way but according to a certain differential equation. On the other hand, it is not so difficult to make the sequence ε_n independent of t. This can be achieved as a consequence of the fact that the values of $\boldsymbol{u}^\varepsilon(t)$ for different times t are related to each other through equation (47) (which is the weak formulation of the corresponding differential equation). In fact, by means of a selection principle not so different from Hilbert's one —which was already known to Stefan Banach (1923)— one can see that (51) entails the existence of a sequence ε_n which gives a limit to the right-hand side of (47) for every $\boldsymbol{\psi} \in \widetilde{\mathcal{D}}_\sigma(\Omega)$ and $t \in [0, \infty)$. Obviously, this implies an analogous convergence

of the left-hand side of (47). From that, together with (48), one obtains the existence of a function $t \mapsto \boldsymbol{u}(t)$ defined for $0 \le t < \infty$ such that $\boldsymbol{u}^{\varepsilon_n}(t)$ converges weakly towards $\boldsymbol{u}(t)$ at every time $t \in [0, \infty)$.

On the other hand, the decreasing character of the function $t \mapsto \|\boldsymbol{u}^{\varepsilon}(t)\|_2$, which follows from (46), allows to achieve the convergence of the quadratic norm $\|\boldsymbol{u}^{\varepsilon_n}(t)\|_2$ to a certain limit $U(t)$ which is also a decreasing function of t. This is achieved by extracting a subsequence, which we will still denote as ε_n, and it involves another selection principle due to Eduard Helly (1921). The quantities $U(t)$ and $\|\boldsymbol{u}(t)\|_2$, namely the limit of the norm and the norm of the weak limit of $\boldsymbol{u}^{\varepsilon_n}(t)$, are forced to satisfy the inequality $\|\boldsymbol{u}(t)\|_2 \le U(t)$, but the equality is satisfied only when the convergence of $\boldsymbol{u}^{\varepsilon_n}(t)$ to $\boldsymbol{u}(t)$ is strong.

Finally, from (52) Hilbert's selection principle allows to achieve also that the different entries of the matrix $\nabla \boldsymbol{u}^{\varepsilon_n}$ converge weakly to some limit as functions of position and time (more specifically, as functions on $\Omega \times [0, t]$ for every $t \in [0, \infty)$). By taking limits on equation (49), one easily checks that the limiting functions satisfy the condition of being the weak derivatives of $\boldsymbol{u}$. More sharply, Leray proves also the existence of a set $\mathcal{S} \subset (0, \infty)$ of zero length in the sense of Lebesgue such that for $t \notin \mathcal{S} \cup \{0\}$ one can extract a subsequence ε'_n that makes $\nabla \boldsymbol{u}^{\varepsilon'_n}(t)$ weakly convergent (as a sequence of functions on Ω); this implies that the convergence of $\boldsymbol{u}^{\varepsilon_n}(t)$ to $\boldsymbol{u}(t)$ is strong (for the sequence ε_n, which does not depend on t). This part is related to another selection principle which is due to Franz Rellich (1930); however, the result of the latter referred to the case where Ω is a bounded region, so Leray had to provide certain additional arguments so as to take care of the unboundedness of $\Omega = \mathbb{R}^3$. On the other hand, one immediately checks the strong convergence of $\boldsymbol{u}^{\varepsilon_n}(0)$ towards $\boldsymbol{u}(0)$.

Summarizing the contents of this paragraph, there exists a sequence $\varepsilon_n \to 0$ and a set $\mathcal{S} \subset (0, \infty)$ of zero length such that, for every time $t \in [0, \infty)$:

(a) $\boldsymbol{u}^{\varepsilon_n}(t)$ converges weakly to a certain limit $\boldsymbol{u}(t)$; this convergence is strong when $t \notin \mathcal{S}$.

(b) $\|\boldsymbol{u}^{\varepsilon_n}(t)\|_2$ converges to a certain limit $U(t) \ge \|\boldsymbol{u}(t)\|_2$; the last inequality becomes an equality when $t \notin \mathcal{S}$.

(c) $\nabla \boldsymbol{u}^{\varepsilon_n}$ converges weakly, as a function on $\Omega \times [0, t]$, to a certain limit which can be identified with $\nabla \boldsymbol{u}$.

8.6. It turns out that these convergences are sufficient to take limits on the equations (49), (48) and (47) satisfied by $\boldsymbol{u}^{\varepsilon_n}$ and deduce that the limiting function $\boldsymbol{u}$ satisfies equations (38), (35) and (37) with $T = \infty$. In other words, $\boldsymbol{u}$ is indeed a weak solution of problem (14) in the interval $0 \le t < \infty$.

On the other hand, it is also interesting to take limits on the energy equality (46). In this case, the weak convergence of $\nabla \boldsymbol{u}^{\varepsilon_n}$ is not enough to ensure that the space-time integral of $|\nabla \boldsymbol{u}^{\varepsilon_n}|^2$ converges towards the analogous integral of $|\nabla \boldsymbol{u}|^2$. However, one can still derive the following **energy**

inequality:

$$\tfrac{1}{2}\,\|\boldsymbol{u}(t)\|_2^2+\int_{t_0}^{t}\int_{\Omega}\nu\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}V\,\mathrm{d}s\;\le\;\tfrac{1}{2}\,\|\boldsymbol{u}(t_0)\|_2^2,\quad\text{if }\;0\le t_0\le t\;\text{ and }\;t_0\notin\mathcal{S}.\quad(53)$$

Notice that this inequality has not been derived from the notion of weak solution, but from the energy equality satisfied by the approximate solutions $\boldsymbol{u}^{\varepsilon}$. As it will be seen more clearly in the following section, *the notion of weak solution does not itself imply the energy inequality,* let alone the corresponding equality. In front of this, and on account of the importance of the energy inequality, it makes sense to introduce a new concept of solution which explicitly requires the fulfilment of that inequality. At this point we will depart from Leray's terminology: instead of "turbulent solutions", we will talk about "globally dissipative solutions". By definition, a **globally dissipative solution** of problem (14) in the interval $[0,T)$ means a weak solution in the sense of § 7.6 which satisfies also the inequality (53), where $\mathcal{S}$ is some subset of $(0,T)$ that has zero length in the sense of Lebesgue.

With this definition, the result of this section can be stated in the following way [20: § V.31]:

Theorem 3 *Let* $\Omega=\mathbb{R}^3$. *For every initial state* $\boldsymbol{u}_0$ *whose square is integrable and whose divergence is zero in the weak sense, problem* (14) *has at least one globally dissipative solution* $\boldsymbol{u}(t)$ *which is defined for all* $t\ge 0$.

9 Semi-regular solutions · Partial regularity of globally dissipative solutions

The preceding section has left open the possibility of having several globally dissipative solutions with the same initial state. On the other hand, we have also raised the question of whether the notion of weak solution implies or not the energy inequality (53), or maybe even the corresponding equality. In the present section we will continue looking at Leray's work on these and other related questions.

9.1. The contrast between the energy inequality obtained in the preceding section and the energy equality obtained in § 6.2 calls for re-examining the arguments of § 6.2 and their applicability to weak solutions. In doing so, one immediately sees that those arguments correspond essentially to applying equation (36) with $\boldsymbol{\psi}=\boldsymbol{u}$. The problem is that this operation is not allowed unless $\boldsymbol{u}$ is regular enough. In this connection, condition (36) explicitly asks $\boldsymbol{\psi}$ to belong to $\widetilde{\mathcal{D}}_{\sigma}(\Omega)$. This condition can be relaxed to a certain extent, but not to the point of allowing to take $\boldsymbol{\psi}=\boldsymbol{u}$ when $\boldsymbol{u}$ is just a weak solution. The main difficulty lies in *ensuring that the integral* $\int_0^t\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}V\,\mathrm{d}s$ *is well defined* (paradoxically enough, we expect this integral to vanish as soon as it makes sense!).

To this effect, the conditions of integrability of $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$ that we included in the definition of a weak solution (conditions (a) and (b) of § 7.6) are sufficient in dimension 2 (owing to inequality (30)) but not in dimension 3, where some additional condition on $\boldsymbol{u}$ is needed. For example, it would suffice to know that the sup norm $\|\boldsymbol{u}(s)\|_\infty$ is bounded independently of s in the interval $[0,t]$. More generally, it suffices that the following integral be well defined:

$$\|\boldsymbol{u}\|^2_{\infty,2,t} := \int_0^t \|\boldsymbol{u}\|^2_\infty \,\mathrm{d}s. \tag{54}$$

Following Leray [20: § IV.23] (see also [19: § III]), we define a **semi-regular solution** of problem (14) in the interval $[0,T)$ as a weak solution in this interval which is a classical solution in the open interval $(0,T)$ and has the property that the integral (54) converges for every $t\in(0,T)$.

With this definition, the preceding arguments allow to state that *semi-regular solutions satisfy the energy equality* (which by the way implies that $\boldsymbol{u}(t)$ converges strongly to $\boldsymbol{u}_0$ as $t\downarrow 0$).

9.2. Let us assume that $\boldsymbol{u}$ is a weak solution of (14) in the interval $[0,T)$ and that it is also a classical solution in the open interval $(0,T)$. Certainly, the latter implies the convergence of the integral $\int_{t_0}^t \|\boldsymbol{u}\|^2_\infty \,\mathrm{d}s$ whenever $0<t_0<t<T$. So, if any, the convergence problems of the integral (54) lie in the limit $s\downarrow 0$. If the initial state $\boldsymbol{u}_0$ is regular enough, then there is no problem at all: in fact, the solutions that were obtained in section 5 are clearly semi-regular since they have the property that $\|\boldsymbol{u}(s)\|_\infty$ is bounded independently of $s\in[0,t]$ (§ 5.5). On the other hand, the convergence of the integral (54) does not exclude the possibility that $\|\boldsymbol{u}(s)\|_\infty\to\infty$ as $s\downarrow 0$, as one expects to happen in the case of initial states that contain singularities (in the sense of § 7.1).

Now, let us assume that, besides having an integrable square (i. e. finite energy) and zero divergence, $\boldsymbol{u}_0$ is required to satisfy the following additional condition:

$$|\nabla\boldsymbol{u}_0|^2 \text{ is integrable} \tag{55}$$

(where the derivatives $\nabla\boldsymbol{u}_0$ are understood in the weak sense). In dimension 3 this condition allows $\boldsymbol{u}_0$ to contain singularities. Even so, Leray showed that under condition (55) the problem (14) has one and only one semi-regular solution. More generally, he arrived at the same conclusion under a variety of hypotheses alternative to (55) —for instance that $|\boldsymbol{u}_0|^p$ be integrable for some $p>3$—, but later on it will be especially relevant the case of condition (55).

In order to obtain solutions of problem (14) for initial states $\boldsymbol{u}_0$ of low regularity, Leray considered the classical solutions with initial state $R_\varepsilon * \boldsymbol{u}_0$, where R_ε is the regularizing kernel that was introduced in § 8.1, and he then passed to the limit $\varepsilon\to 0$. The semi-regular character of the solution, i. e. the convergence of the integral (54), derives from condition (55) through a bound

of the form $\|\boldsymbol{u}(t)\|_\infty \le C\,(\nu t)^{-1/4}\,\|\nabla \boldsymbol{u}_0\|_2$, which is obtained from the integral equations of section 5 by making use of certain general inequalities.

Altogether it gives the following existence result (the uniqueness will be dealt with in the next paragraph) [20: § III.21, IV.24]:

Theorem 4 *Let $\Omega = \mathbb{R}^3$. If the initial state $\boldsymbol{u}_0$ satisfies condition* (55) *then problem* (14) *has a semi-regular solution which is defined in a time interval of the form $0 \le t < T(\boldsymbol{u}_0)$, where $T(\boldsymbol{u}_0)$ can be infinite. Furthermore, there exist two constants C_3 and C_4 such that:*

(a) *If* $T(\boldsymbol{u}_0) < \infty$ *then* $\|\nabla \boldsymbol{u}(t)\|_2 > \dfrac{C_3\,\nu}{(\nu(T(\boldsymbol{u}_0) - t))^{1/4}}$ *as* $t \to T(\boldsymbol{u}_0)$.

(b) *If* $\dfrac{1}{\nu^4}\,\|\nabla \boldsymbol{u}_0\|_2^2\,\|\boldsymbol{u}_0\|_2^2 < C_4$ *then* $T(\boldsymbol{u}_0) = \infty$.

9.3. In § 9.1 we saw that semi-regular solutions satisfy the energy equality. Besides that, they are also especially interesting in connection with the question of uniqueness. In fact, this question can be analysed by means of a method quite similar to the calculations that lead to the energy equality. So, if $\boldsymbol{u}$ and $\boldsymbol{v}$ are two classical solutions of (14) corresponding respectively to the initial states $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$, their difference $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ is a solution of the problem

$$\begin{gathered} \partial \boldsymbol{w}/\partial t - \nu\,\Delta \boldsymbol{w} + \nabla r = -\,(\boldsymbol{w}\cdot\nabla)\,\boldsymbol{u} - (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{w}, \qquad \nabla\cdot\boldsymbol{w} = 0, \\ \boldsymbol{w}\,|_{\partial\Omega} = \boldsymbol{0}, \qquad \boldsymbol{w}\,|_\infty = \boldsymbol{0}, \qquad \boldsymbol{w}\,|_{t=0} = \boldsymbol{w}_0, \end{gathered} \tag{56}$$

where $\boldsymbol{w}_0 = \boldsymbol{u}_0 - \boldsymbol{v}_0$. By proceeding as in § 6.2, that is, by taking the scalar product by $\boldsymbol{w}$ and integrating, one arrives at the following equality:

$$\tfrac{1}{2}\,\|\boldsymbol{w}(t)\|_2^2 + \int_0^t \int_\Omega \nu\,|\nabla \boldsymbol{w}|^2\,\mathrm{d}V\,\mathrm{d}s = \tfrac{1}{2}\,\|\boldsymbol{w}_0\|_2^2 + \int_0^t \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}\,\mathrm{d}V\,\mathrm{d}s. \tag{57}$$

Notice that the term that contained $\boldsymbol{v}$ has disappeared since $\int_\Omega \boldsymbol{v}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\,\mathrm{d}V = 0$ (by the same reason as in equation (25)). From equation (57) one can deduce that

$$\|\boldsymbol{w}(t)\|_2^2 \;\le\; \|\boldsymbol{w}_0\|_2^2\,\exp\left(\frac{1}{2\,\nu}\int_0^t \|\boldsymbol{u}\|_\infty^2\,\mathrm{d}s\right). \tag{58}$$

In particular, if $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same initial state, that is, $\boldsymbol{u}_0 = \boldsymbol{v}_0$, we have $\boldsymbol{w}_0 = \boldsymbol{0}$, from which (58) ensures that $\boldsymbol{w} = \boldsymbol{0}$, that is $\boldsymbol{u} = \boldsymbol{v}$.

As one can see, (58) requires the convergence of the integral (54) that motivated the definition of semi-regular solutions. As a matter of fact, the preceding argument can be reformulated in such a way that it becomes applicable to semi-regular solutions instead of classical ones. Furthermore, in connection with the fact that equation (57) contains $\boldsymbol{u}$ but not $\boldsymbol{v}$, it turns out that *the hypothesis of semi-regularity can be circumscribed to $\boldsymbol{u}$ only, whereas $\boldsymbol{v}$ can be any globally dissipative solution* [20: § VI.32]. The proof is based upon the identity $\|\boldsymbol{w}\|_2^2 = \|\boldsymbol{u} - \boldsymbol{v}\|_2^2 = \|\boldsymbol{u}\|_2^2 + \|\boldsymbol{v}\|_2^2 - 2\,\langle \boldsymbol{v}, \boldsymbol{u}\rangle$, and

it proceeds by dealing with the three terms of the right-hand side by using respectively: (a) the energy equality applied to the semi-regular solution $\boldsymbol{u}$; (b) the energy inequality applied to the globally dissipative solution $\boldsymbol{v}$; and (c) a particular case of the condition which expresses that $\boldsymbol{v}$ is a weak solution of (14), namely the case where the test function is taken to be the regular function $\boldsymbol{u}$ (that is, equation (36) with $\boldsymbol{u}$ and $\boldsymbol{\psi}$ replaced respectively by $\boldsymbol{v}$ and $\boldsymbol{u}$); at this point one uses also the fact that $\boldsymbol{u}$ is a classical solution for $t > 0$. All of this results in a relation that differs from (57) only in that the equality sign is replaced by an *inequality* one. Anyway, this still allows to derive inequality (58) and to conclude that $\boldsymbol{u}_0 = \boldsymbol{v}_0$ implies $\boldsymbol{u} = \boldsymbol{v}$.

So, one can state the following result, where part (a) was obtained in § 9.1:

Theorem 5 *Let $\Omega = \mathbb{R}^3$. A semi-regular solution $\boldsymbol{u}$ of problem* (14) *in the interval $[0, T)$ has the following properties:* (a) *it satisfies the energy equality;* (b) *it is unique; and* (c) *more generally, it coincides in the whole interval $[0, T)$ with any globally dissipative solution that begins at the same initial state and is defined on $[0, T')$ with $T' \geq T$.*

9.4 Most remarkably, theorems 4 and 5 allow to derive quite interesting results of partial regularity for the globally dissipative solutions.

By definition, a globally dissipative solution $\boldsymbol{u}$ is a weak solution, so it has the property that $|\nabla \boldsymbol{u}|^2$ is integrable on $\Omega \times [0, t]$ for any $t > 0$. Now, this implies that $|\nabla \boldsymbol{u}(t_0)|^2$ is integrable on Ω for any $t_0 \in [0, \infty) \setminus \mathcal{S}$, where $\mathcal{S}$ represents a set of zero length. Therefore, by virtue of the two preceding theorems, $\boldsymbol{u}(t_0)$ is the initial state of a unique semi-regular solution which is defined on an interval of the form $[t_0, T)$, and in this interval the globally dissipative solution $\boldsymbol{u}$ must necessarily coincide with this semi-regular solution. From this, one can deduce that $\boldsymbol{u}$ is a classical solution in a collection of disjoint open intervals J_k ($k = 0, 1, 2, \ldots$) whose union differs from $[0, \infty)$ only in a set of zero length. On the other hand, the fact that a globally dissipative solution satisfies the energy inequality (53) can be combined with the upper bound (b) of theorem 4 so as to obtain some consequences about the intervals of regularity and the behaviour of the solution in the limit $t \to \infty$.

Specifically, the following result is obtained [20: § IV.33–34]:

Theorem 6 *Let $\Omega = \mathbb{R}^3$. Let $\boldsymbol{u}$ be a globally dissipative solution of problem* (14) *in the interval $[0, \infty)$. Then $\boldsymbol{u}$ is a classical solution in a collection of disjoint open intervals J_k ($k = 0, 1, 2, \ldots$) whose union differs from $[0, \infty)$ only in a set of zero length in the sense of Lebesgue. Furthermore, the intervals J_k have the following properties, where C_3 is the constant of*

theorem 4 and $|J|$ *represents the length of a subset of* $\mathbb{R}$*:*

$$J_0 = (\vartheta_0, \infty) \quad \textit{with} \quad \vartheta_0 \geq (2C_3)^{-4} \, \|\boldsymbol{u}_0\|_2^4 \, / \, \nu^5, \tag{59}$$

$$\sum_{k>0} |J_k|^{1/2} < (2C_3)^{-2} \, \|\boldsymbol{u}_0\|_2^2 \, / \, \nu^{5/2}. \tag{60}$$

Finally, as $t \to \infty$ *the state of motion* $\boldsymbol{u}(t)$ *approaches the state of rest in the following way: for every* $T > \vartheta_0$ *there exist two constants* M *and* M' *such that*

$$\|\nabla \boldsymbol{u}(t)\|_2 \leq M \, \|\boldsymbol{u}_0\|_2 \, (\nu t)^{-1/2}, \quad \|\boldsymbol{u}(t)\|_\infty \leq M' \, \|\boldsymbol{u}_0\|_2 \, (\nu t)^{-3/4}, \quad \forall t > T. \tag{61}$$

Notice that according to (59) and (61) any globally dissipative solution ends up being a classical solution and tending to zero, at least in the senses of (61). In this connection, Leray left open the question of whether the quadratic norm, that is the energy, converged also to zero. This question was answered affirmatively in 1984, first in the two-dimensional case (where the question is not trivial either) by Tosio Kato [43], and shortly afterwards in the three-dimensional case by Kyûya Masuda [44].

On the other hand, theorem 6 allows the possibility that the set of singular times be something like the celebrated ternary set of Georg Cantor (1883). In this connection, the bound (60) can be interpreted as a restriction on the way that singular times can get near each other: as it was pointed out by Vladimir Scheffer in 1975 [39b], (60) entails that the complement of $\bigcup_k J_k$ has fractional dimension lower than or equal to $1/2$ in the sense of Haussdorff, which is stronger than saying that it has zero length.

So, in general one cannot discard the possibility that, after a while, the globally dissipative solutions that start from a given initial state branch off into a sort of tree. Even so, we are ensured that in the long term all of the branches end up approaching the null solution.

10 Other techniques

10.1. Theorem 3 about the global existence of globally dissipative solutions was extended to the case $\Omega \subset \mathbb{R}^3$ by Eberhard Hopf (1950) [23]. This author based his result upon an idea that had previously been applied to other differential equations, especially in order to obtain approximate solutions. The version that interests us is due mainly to Boris G. Galërkin (1915) and Alessandro Faedo (1949). Applied to our problem, it is a matter of taking the weak formulation (37) and restricting both the unknown $\boldsymbol{u}$ and the test function $\boldsymbol{\psi}$ to certain spaces of finite but progressively larger dimension, so that in the limit one recovers the original problem.

This strategy works because of certain results analogous to those provided by Fourier series. They can be viewed as saying that the functional spaces that we are concerned with admit denumerable bases formed by regular functions. In particular, one can select from $\mathcal{D}_\sigma(\Omega)$ an infinite but denumerable collection

of vector fields $\boldsymbol{\psi}_k$ $(k = 1, 2, ...)$ whose finite linear combinations allow to approach any other element of $\mathcal{D}_\sigma(\Omega)$ within any desired degree of accuracy and in a variety of norms; more generally, one can approach in this way any element of the corresponding completion of $\mathcal{D}_\sigma(\Omega)$. In the case of the quadratic norm, this completion is formed by all vector fields whose square is integrable in the sense of Lebesgue and which satisfy the two following conditions: their divergence is zero in the weak sense; their normal component to the boundary is zero in a certain generalized sense. By means of a standard procedure, one can achieve that the vector fields $\boldsymbol{\psi}_k$ be also orthogonal to each other and have quadratic norm equal to one, i. e. that the scalar products $\langle \boldsymbol{\psi}_k , \boldsymbol{\psi}_l \rangle = \int_\Omega \boldsymbol{\psi}_k \cdot \boldsymbol{\psi}_l \, \mathrm{d}V$ be equal to zero when $k \neq l$ and equal to one when $k = l$. In complete analogy with Fourier series, it follows that any vector field $\boldsymbol{v}$ from that completion in the quadratic norm can be represented by means of a convergent series of the form $\boldsymbol{v} = \sum_{k=1}^{\infty} \xi_k \, \boldsymbol{\psi}_k$, where ξ_k denotes a numerical coefficient which is given by $\xi_k = \langle \boldsymbol{v} , \boldsymbol{\psi}_k \rangle = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\psi}_k \, \mathrm{d}V$. In terms of these coefficients, which can be interpreted as the coordinates of $\boldsymbol{v}$ in the basis formed by the vector fields $\boldsymbol{\psi}_k$ $(k = 1, 2, ...)$, the quadratic norm of $\boldsymbol{v}$ is given by the following convergent numerical series: $\|\boldsymbol{v}\|_2^2 = \sum_{k=1}^{\infty} \xi_k^2$. In what follows, E_n denotes the space formed by the linear combinations of $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, ..., \boldsymbol{\psi}_n$, and P_n denotes the operator of orthogonal projection on E_n $(P_n \boldsymbol{v} = \sum_{k=1}^{n} \xi_k \, \boldsymbol{\psi}_k)$.

So, a natural way to approach the Navier-Stokes equations from finite dimension is the following: one introduces an integer parameter n, which later on will be let to grow towards infinity, and one considers for every value of n a problem completely analogous to (37) where both $\boldsymbol{u}$ and $\boldsymbol{\psi}$ are restricted to take values in the space E_n:

$$\boldsymbol{u}(t) \in E_n, \qquad \forall t \in [0, T); \tag{62.1}_n$$

$$\begin{aligned} \int_\Omega \boldsymbol{u}(t) \cdot \boldsymbol{\psi}(t) \, \mathrm{d}V \; = \; & \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{\psi}(0) \, \mathrm{d}V \\ & + \int_0^t \int_\Omega \Big(\boldsymbol{u} \cdot (\partial \boldsymbol{\psi} / \partial t) - \nu \, \boldsymbol{u} \cdot \Delta \boldsymbol{\psi} - \boldsymbol{u} \cdot \nabla \boldsymbol{\psi} \cdot \boldsymbol{u} \Big) \, \mathrm{d}V \, \mathrm{d}s, \\ & \forall \boldsymbol{\psi} \in \widetilde{E}_n, \quad \forall t \in [0, T), \end{aligned} \tag{62.2}_n$$

Here $\widetilde{E}_n$ represents the subset of $\widetilde{\mathcal{D}}_\sigma(\Omega)$ described by the condition that $\boldsymbol{\psi}(t) \in E_n$, $\forall t \in [0, T)$. As it was already mentioned in the infinite-dimensional case (see § 7.6), one can restrict $\boldsymbol{\psi}$ to be independent of t. Furthermore, since equation $(62.2)_n$ is linear in $\boldsymbol{\psi}$, it suffices to consider the n cases $\boldsymbol{\psi}(t) = \boldsymbol{\psi}_k$ $(k = 1, 2, ..., n)$. On the other hand, by introducing the representations

$$\boldsymbol{u}(t) = \sum_{k=1}^{n} \zeta_k(t) \, \boldsymbol{\psi}_k, \qquad \boldsymbol{u}_0 = \sum_{k=1}^{\infty} \zeta_{k0} \, \boldsymbol{\psi}_k, \tag{63}$$

one easily sees that the problem reduces to a system of ordinary differential

equations of the form

$$\mathrm{d}\zeta_k/\mathrm{d}t = -\nu \sum_{l=1}^{n} a_{kl}\,\zeta_l + \sum_{l,m=1}^{n} b_{klm}\,\zeta_l\,\zeta_m, \quad \zeta_k(0) = \zeta_{k0}, \quad (k=1,2,...,n) \quad (64)$$

for the numerical functions $\zeta_1, \zeta_2, ..., \zeta_n$. More specifically, a_{kl} and b_{klm} are the following constants: $a_{kl} = \int_\Omega \nabla\boldsymbol{\psi}_k : \nabla\boldsymbol{\psi}_l \,\mathrm{d}V$ and $b_{klm} = \int_\Omega \boldsymbol{\psi}_l \cdot \nabla\boldsymbol{\psi}_k \cdot \boldsymbol{\psi}_m \,\mathrm{d}V$. In the following, the solution of (64) will be denoted by $\zeta_k^{(n)}$ and the corresponding solution of $(62)_n$ will be denoted by $\boldsymbol{u}^{(n)}$ (notice that $\boldsymbol{u}^{(n)}(0) = P_n\boldsymbol{u}_0$).

As one can see, system (64) inherits from the Navier-Stokes equations certain quadratic terms that in principle could limit its solutions to a finite existence time. On the other hand, it turns out that the solutions of this finite-dimensional problem satisfy also the energy equality:

$$\tfrac{1}{2}\,\|\boldsymbol{u}^{(n)}(t)\|_2^2 + \int_{t_0}^{t}\int_\Omega \nu\,|\nabla\boldsymbol{u}^{(n)}|^2 \,\mathrm{d}V\,\mathrm{d}s = \tfrac{1}{2}\,\|\boldsymbol{u}^{(n)}(t_0)\|_2^2, \quad \text{for } 0 \le t_0 \le t. \quad (65)$$

This equality follows immediately from equation $(62.2)_n$ by taking $\boldsymbol{\psi} = \boldsymbol{u}^{(n)}$. In this case this is allowed because the solutions of (64) are infinitely differentiable —as one can see by successive differentiation of (64.1)—, which entails, by (63.1), that $\boldsymbol{u}^{(n)}$ satisfies the conditions that are imposed upon $\boldsymbol{\psi}$. As in Leray's method (see § 8.2), the preceding equality allows to infer that $\boldsymbol{u}^{(n)}$ remains defined for all $t \ge 0$. The difference with respect to § 6.3, where we had apparently the same ingredients, is that here everything happens in a space of finite dimension, where two norms are always equivalent to each other, whereas there we had to distinguish between the quadratic norm, whose boundedness is ensured by the energy equality, and the sup norm, whose boundedness would entail the globality of the solution.

Therefore, we are in a situation completely analogous to that of section 8: problem (14) appears as a limit of a sequence $(62)_n$ $(n \to \infty)$ where each of the problems $(62)_n$ has existence, uniqueness and globality of solution and furthermore the solution satisfies the energy equality. Form here one can follow essentially the same process as in section 8, which results in an extension of theorem 3 to the case of a region $\Omega \subset \mathbb{R}^3$.

10.2 The case $\Omega \subset \mathbb{R}^3$ was also considered in 1955–57 by Andreĭ A. Kiselëv and Ol'ga A. Ladyzhenskaya [25], who gave essentially a result of existence and uniqueness of semi-regular solutions similar to theorems 4 and 5 of section 9.

Concerning the result itself, apart from the hypothesis on Ω, the main difference with respect to the results of section 9 is that the solutions of Kiselëv and Ladyzhenskaya do not require the boundedness of the quantity (54), but of some other one, for instance $\|\boldsymbol{u}\|_{4,\infty,t} := \sup_{0\le s\le t} (\int_\Omega |\boldsymbol{u}(s)|^4 \,\mathrm{d}V)^{1/4}$ (or two other alternatives). Anyway, the fundamental idea is essentially the same: on the one hand, the boundedness of this integral has the virtue of entailing the

uniqueness of such solutions, and on the other hand, one can prove the existence of such a solution, possibly extended to a finite interval only, if the initial state is regular enough. In view of this, we will refer to the solutions of [25] as "semi-regular solutions of Kiselëv and Ladyzhenskaya".

Concerning the methods used by these authors, the uniqueness is proved in essentially the same way as in § 9.3. However, the existence is obtained by methods quite different from those of § 9.2. On the one hand, the semi-regular solutions are not obtained as limits of the classical solutions of section 5, but they use Galërkin's method, like in the preceding paragraph. On the other hand, and more fundamentally, the bounds on the solutions are neither derived from the integral equations of section 5, but they are obtained by means of calculations similar to those that lead to the energy equality. More specifically, besides the bound implied by this equality, Kiselëv and Ladyzhenskaya use another one which is obtained by differentiating equation (14.1) with respect to time, taking its scalar product by $\partial \boldsymbol{u}/\partial t$, and integrating over Ω. The subsequent calculations are crucially based upon being able to bound the integral $\int_\Omega |\boldsymbol{u}|^4 \, \mathrm{d}V$ in terms of $\int_\Omega |\boldsymbol{u}|^2 \, \mathrm{d}V$ and $\int_\Omega |\nabla \boldsymbol{u}|^2 \, \mathrm{d}V$ (and similarly for $\partial \boldsymbol{u}/\partial t$ instead of $\boldsymbol{u}$). In this connection, in the three-dimensional case one can use an inequality similar to (30) but with different exponents, namely:

$$\int_\Omega |f|^4 \, \mathrm{d}V \;\leq\; \frac{1}{3\sqrt{3}} \left(\int_\Omega |f|^2 \, \mathrm{d}V \right)^{1/4} \left(\int_\Omega |\nabla f|^2 \, \mathrm{d}V \right)^{3/4}, \tag{66}$$

where f is restricted to vanish outside a bounded subset of $\mathbb{R}^3$ (to be precise, the original article [25c] does not use inequality (66) but a slightly weaker variant). Altogether it shows that the numerical function $t \mapsto \|\partial \boldsymbol{u}(t)/\partial t\|_2$ satisfies a certain differential inequality from which one can derive the bounds in question for a certain time interval, possibly finite.

Subsequent to [25], in 1958 it was published the work [26] which we already mentioned in § 6.3, where Ladyzhenskaya brought this method to the two-dimensional case and she noticed that the fact of having inequality (30) instead of (66) allowed to extend the solutions to arbitrarily long times. Shortly afterwards, in 1958–62, these works of Kiselëv and Ladyzhenskaya were refined by several authors, especially Jacques L. Lions, Giovanni Prodi, Ciprian Foiaş and James Serrin.

A remarkable feature of this method is its independence from the integral equations of section 5. In particular, it obviates any calculations about the kernel $\Gamma(t)$ (which we already said in § 5.7 that they become especially difficult in the case $\Omega \subset \mathbb{R}^n$). In this sense, and also in a more intrinsic one, the preceding method is comparable to the so-called "functional", "direct", or "energy" methods that several authors had developed previously in the case of partial differential equations of elliptic type.

Having said that, it is also true that in order to prove the regularity of the obtained solutions, for instance, to ensure that they are classical solutions

for $t > 0$, then one can hardly avoid an accurate study of the kernel $\Gamma(t)$ in the lines that were mentioned in § 5.7 and that we resume in the following paragraph.

10.3. As we have mentioned in § 5.7, in the case $\Omega \subset \mathbb{R}^n$ it is not easy to obtain detailed information about the kernel $\Gamma(t)$, which limits to a certain extent the possibilities of the successive approximation method described in section 5. However, this statement must be revised in a more optimistic direction. To be precise, what is quite difficult if not impossible in the case $\Omega \subset \mathbb{R}^n$ is obtaining an explicit expression for the function $\Gamma(\boldsymbol{x}, \boldsymbol{y}, t)$. Certainly, without such an expression the method of successive approximations cannot be considered as a method of effective calculation (of approximate solutions). However, for obtaining results of existence and uniqueness (of exact solutions) there is no need to know an explicit expression of the kernel Γ, but it suffices to be able to bound the magnitude of $\Gamma(t) * \boldsymbol{v}$, measured in a certain norm, in terms of the value of t and of the magnitude of $\boldsymbol{v}$, measured in a norm possibly different from the preceding one. Now, there are certain constructions that appreciably facilitate this task and provide a functional-analytic framework where the method of successive approximations becomes quite manageable, even in the case $\Omega \subset \mathbb{R}^n$. These constructions have quite a general character and they are dealt with in the so-called *semigroup theory*.

10.3.1. Semigroup theory can be viewed as the general study of the evolution problem associated with a linear differential equation in a Banach space. More specifically, one studies a problem of the form

$$\mathrm{d}z/\mathrm{d}t + Az = 0, \qquad z(0) = z_0, \tag{67}$$

where the unknown z is a function of $t \geq 0$ with values in a Banach space $\mathbb{E}$, and A is a linear operator which is defined on a certain subset $\mathbb{X}$ of the space $\mathbb{E}$. The resulting theory allows to study also the non-homogeneous case

$$\mathrm{d}z/\mathrm{d}t + Az = f, \qquad z(0) = z_0, \tag{68}$$

where f represents a given function of t. In its turn, this case provides the base for applying the method of successive approximations to non-linear problems of the form

$$\mathrm{d}z/\mathrm{d}t + Az = B(z), \qquad z(0) = z_0. \tag{69}$$

In the following we will use the following notations: $\|\cdot\|$ represents the norm of the space $\mathbb{E}$; $\|\cdot\|_{(1)}$ represents another norm, defined by the formula $\|x\|_{(1)} = \|Ax\| + \|x\|$, which will be applied to the elements of $\mathbb{X}$. In connection with the operator A and its domain of definition, we will ask as a starting hypothesis that $\mathbb{X}$ be dense in $\mathbb{E}$, and that the operator A be closed; the latter is equivalent to say that $\mathbb{X}$ is a complete space in the norm $\|\cdot\|_{(1)}$.

In § 3.1 we already looked at the Navier-Stokes equations as a differential equation of the form $\mathrm{d}z/\mathrm{d}t = F(z)$ where z represented the velocity field $\boldsymbol{u}$.

In this connection, a fundamental role was played by the Stokes-Helmholtz decomposition, which allows to associate every vector field $\boldsymbol{v}$ on Ω with the part of it which is solenoidal and parallel to the boundary, which we will denote as $P\,\boldsymbol{v}$. With this notation, we can be more explicit and write the Navier-Stokes equations in the form (69) with

$$z = \boldsymbol{u}, \qquad A\,z = -\nu\,P\,\Delta\boldsymbol{u}, \qquad B(z) = -P\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \tag{70}$$

In a natural way, the supplementary conditions (14.2)–(14.4) come into play at the time of specifying both the space $\mathbb{E}$ and the subset $\mathbb{X}$ where one considers the operator A to be defined. In the spirit of § 4.2, one can start from the set $\mathcal{D}_\sigma(\Omega)$, whose elements satisfy all those conditions, and pass from there to $\mathbb{E}$ and $\mathbb{X}$ by means of certain constructions of completion. Concerning $\mathbb{E}$, it is simply a matter of completing that set in some specific norm $\|\cdot\|$. According to what has been seen in the preceding sections, it is quite natural to opt for the quadratic norm, but one can also work in other norms. Concerning the operator A and its domain of definition $\mathbb{X}$, one uses a certain closure operation which is equivalent to completing $\mathcal{D}_\sigma(\Omega)$ in the norm $\|\cdot\|_{(1)}$. In particular, this leaves us under the above starting hypotheses on A and $\mathbb{X}$. In the following we will refer to A as the **Stokes operator**.

In equations (67)–(69) the derivative $\mathrm{d}z/\mathrm{d}t$ is understood as a functional limit in the norm $\|\cdot\|$ of $\mathbb{E}$. This is not the same as the partial derivative $\partial\boldsymbol{u}/\partial t$, which properly means a numerical limit for every point $\boldsymbol{x}\in\Omega$. So, in principle the concept of solution of (67)–(69) is also an extension of the classical concept of solution of the corresponding partial differential equations.

Coming back to the general problem of semigroup theory, when $\mathbb{X}$ coincides with $\mathbb{E}$ then the linear operator A is bounded (i. e. continuous) from $\mathbb{E}$ to $\mathbb{E}$. In such a case (which is the only one that can happen in finite dimension) the solution of problem (67) can be expressed in the form

$$z(t) = \mathrm{e}^{-At}\,z_0, \tag{71}$$

where e^{-At} represents a bounded linear operator from $\mathbb{E}$ to $\mathbb{E}$ that depends on t and can be constructed from A by means of the power series of the exponential function. To a large extent, semigroup theory deals with constructing such operators e^{-At} in other situations where A is not a bounded linear operator from $\mathbb{E}$ to $\mathbb{E}$ (but only from $\mathbb{X}$ to $\mathbb{E}$). These operators must be defined at least for $t\geq 0$ and formula (71) must provide the solution of (67) at least for $z_0\in\mathbb{X}$; moreover, one asks that for every $t\geq 0$ the operator e^{-At} extends to a bounded linear operator from $\mathbb{E}$ to $\mathbb{E}$, and that for every $z_0\in\mathbb{E}$ formula (71) defines a continuous function from $t\in[0,\infty)$ to $\mathbb{E}$. For every $t\geq 0$, the operator e^{-At} goes from the initial state to the state after time t, so that one has $\mathrm{e}^{-A(t+s)} = \mathrm{e}^{-At}\,\mathrm{e}^{-As}$ $(t,s\geq 0)$. Because of this property, the family of operators e^{-At} $(t\geq 0)$ is said to form a **semigroup**. Notice that in the case where A is the Stokes operator the operators e^{-At} are nothing else than the integral operators of kernel $\Gamma(t)$ introduced in section 5, that is $\mathrm{e}^{-At}\,z_0 =$

$\Gamma(t) * \boldsymbol{u}_0$. As it will be seen below, the methods of semigroup theory allow to construct and analyse these operators not only in the case $\Omega = \mathbb{R}^n$ but also when $\Omega \subset \mathbb{R}^n$. Notice also that as soon as the solution of (67) is available in the form (71), then the solution of the non-homogeneous problem (68) is expected to be given by the formula

$$z(t) = \mathrm{e}^{-At}\, z_0 + \int_0^t \mathrm{e}^{-A(t-s)}\, f(s)\, \mathrm{d}s. \tag{72}$$

10.3.2. A situation where it is relatively easy to construct the operators e^{-At} $(t \geq 0)$ is that where $\mathbb{E}$ is a Hilbert space and the operator A is self-adjoint and positive. Under the starting hypotheses stated above, these two conditions are equivalent to saying that $\langle Ax, y\rangle = \langle x, Ay\rangle$ and $\langle Ax, x\rangle \geq 0$, where x, y are arbitrary elements of $\mathbb{X}$ and $\langle \cdot\,, \cdot \rangle$ represents the scalar product of $\mathbb{E}$. For the Stokes operator in the quadratic norm setting, these properties follow easily from the orthogonal character of the Stokes-Helmholtz decomposition and the validity of the formula $\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{v}\ \mathrm{d}V = -\int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{v}\,\mathrm{d}V$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{D}_\sigma(\Omega)$. In such a situation the operators e^{-At} $(t \geq 0)$ can be defined by means of the so-called spectral decomposition of the operator A. This is specially simple when one has a basis of $\mathbb{E}$ formed by eigenfunctions of A (in the sense of a denumerable basis in the style of Fourier series, like the one that we have already considered in § 10.1). As it was proved by David Hilbert and Erhard Schmidt (1904–07), such a basis exists whenever A has a certain property (compact resolvent) which in the case of the Stokes operator is satisfied as soon as the region Ω is bounded. Form now on we will assume this hypothesis. In such a situation, the eigenfunctions and the corresponding eigenvalues $\boldsymbol{\psi}_k$ and λ_k $(k = 1, 2, ...)$ have the following properties: (a) by definition

$$A\,\boldsymbol{\psi}_k = \lambda_k\,\boldsymbol{\psi}_k; \tag{73}$$

(b) the eigenvalues are positive real numbers and they form a non-decreasing sequence that tends to ∞; (c) suitably adjusted, the eigenfunctions are orthogonal to each other and their norm is equal to one; and (d) the following formulae establish a one-to-one correspondence between the elements x of $\mathbb{E}$ and the numerical sequences ξ_k $(k = 1, 2, ...)$ with a convergent quadratic sum $\sum_{k=1}^{\infty} \xi_k^2$:

$$x = \sum_{k=1}^{\infty} \xi_k\,\boldsymbol{\psi}_k, \qquad \xi_k = \langle x, \boldsymbol{\psi}_k\rangle, \qquad \|x\|^2 = \sum_{k=1}^{\infty} \xi_k^2. \tag{74}$$

By operating term-wise on (74.1), equality (73) entails that Ax should be given by the series

$$Ax = \sum_{k=1}^{\infty} \lambda_k\,\xi_k\,\boldsymbol{\psi}_k. \tag{75}$$

This can be viewed as saying that in the basis under consideration the coefficients of x and those of Ax are related to each other through an (infinite)

diagonal matrix. However, one cannot expect the preceding series to converge for any $x \in \mathbb{E}$ (because $\lambda_k \to \infty$ as $k \to \infty$). Even so, the closed character of operator A ensures that the convergence of this series is exactly equivalent to x being a member of $\mathbb{X}$, and that equality (75) holds as soon as these conditions are satisfied (in other words, as soon as one of the two sides of (75) is defined the other one is defined too and the equality holds).

On the other hand, equality (73) entails that the function $z(t) = \mathrm{e}^{-\lambda_k t}\,\boldsymbol{\psi}_k$ solves equation (67) with $z_0 = \boldsymbol{\psi}_k$, which leads to defining the operators e^{-At} $(t \geq 0)$ in the following way:

$$\mathrm{e}^{-At}\,x \;=\; \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_k t}\,\xi_k\,\boldsymbol{\psi}_k. \tag{76}$$

In contrast to (75), here the property $\lambda_k > 0$ ensures that the preceding series converges for every $x \in \mathbb{E}$ (with the bound $\|\mathrm{e}^{-At}x\| \leq \|x\|$). The operators e^{-At} $(t \geq 0)$ defined in this way are easily checked to satisfy the desired properties. In particular, the two following equalities are easily justified for $x \in \mathbb{X}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathrm{e}^{-At}\,x) = -\sum_{k=1}^{\infty} \lambda_k\,\mathrm{e}^{-\lambda_k t}\,\xi_k\,\boldsymbol{\psi}_k, \qquad A\,\mathrm{e}^{-At}\,x = \sum_{k=1}^{\infty} \lambda_k\,\mathrm{e}^{-\lambda_k t}\,\xi_k\,\boldsymbol{\psi}_k, \tag{77}$$

which ensures that in this case ($x \in \mathbb{X}$) the function $z(t) = \mathrm{e}^{-At}\,x$ is indeed a solution of (67.1).

The idea of using the preceding spectral method for solving the linear problem (67) associated with the Stokes operator was proposed already in 1931 by Folke Odqvist [17]. Since 1950, several authors developed this idea in a functional setting like the one above (see for instance [35a, 24a, 37]). It is interesting to notice that in that article of 1931 Odqvist suggested also another method, based upon the Laplace transform, which later on has been used systematically in semigroup theory. Here we will remark only that the method based upon the Laplace transform has a wider application than the preceding one. In particular, it allows to work in other norms instead of the quadratic one.

10.3.3 Semigroups like the one that we have just constructed have certain special properties which become crucial when looking at non-linear equations of the form (69). For instance, using the fact that the numerical function $\lambda \mathrm{e}^{-\lambda}$ is bounded independently of $\lambda \geq 0$, one immediately sees that for $t > 0$ the series in (77) is convergent for any $x \in \mathbb{E}$. Therefore, the equalities in (77) can be justified not only for $x \in \mathbb{X}$ and $t \geq 0$, but also for any $x \in \mathbb{E}$ and $t > 0$. This fact allows to say that the function $z(t) = \mathrm{e}^{-At}\,z_0$ gives the solution of the evolution problem (67) not only for $z_0 \in \mathbb{X}$, but also for every $z_0 \in \mathbb{E}$: in fact, by an argument like the ones used in § 6.2 and § 9.3, one can ensure that $z(t)$ is the only function continuous in the closed interval $0 \leq t < \infty$ and derivable in

the open interval $0 < t < \infty$ which satisfies the initial condition (67.2) and later on evolves according to the differential equation (67.1).

In the terminology of semigroup theory, we are in front of a **differentiable semigroup**. By definition, this means that, for any $x \in \mathbb{E}$, the function $t \mapsto \mathrm{e}^{-At}x$ is differentiable in the open interval $0 < t < \infty$. As it is proved in semigroup theory, this automatically implies that, for $t > 0$, the values of $\mathrm{e}^{-At}x$ belong to $\mathbb{X}$, the domain of the operator A. Not only that, in fact the semigroup property allows to derive that, for any $x \in \mathbb{E}$, the function $t \mapsto \mathrm{e}^{-At}x$ is infinitely differentiable in the open interval $0 < t < \infty$, and in this interval the values of $\mathrm{e}^{-At}x$ belong to $\bigcap_{n=1}^{\infty} \mathbb{X}^n$, where $\mathbb{X}^n$ represents the domain of the operator A^n. In the case that interests us —and generally whenever x represents a function of some variable $\boldsymbol{x}$, and A involves derivatives with respect to this variable— $\mathbb{X} = \mathbb{X}^1$ is a strict subset of $\mathbb{E} = \mathbb{X}^0$, and this entails that $\mathbb{X}^n$ is a strict subset of $\mathbb{X}^{n-1}$ for every positive integer n. More specifically, the sets $\mathbb{X}^n$ $(n = 0, 1, 2, ...)$ are formed by functions which are progressively more differentiable with respect to $\boldsymbol{x}$ (in some generalized sense). So, the differentiability of the semigroup corresponds to the regularizing properties that we were meeting already in section 5.

In the case of § 10.3.2, this regularizing character is quantified by the following bound, which follows from (77) by taking into account that $\lambda \mathrm{e}^{-\lambda} \leq \mathrm{e}^{-1}$ $(\forall \lambda \geq 0)$: For every $x \in \mathbb{E}$ and every $t > 0$ one has

$$\|A\, e^{-At}\, x\| \leq C_1\, t^{-1}\, \|x\|, \tag{78}$$

where $C_1 = \mathrm{e}^{-1}$. A bound of this form, possibly with a different value of the constant C_1, characterizes the so-called **analytic semigroups**. As it is suggested by this name, from (78) one can deduce that the function $t \mapsto \mathrm{e}^{-At}x$ is not only differentiable but also analytic (in the open interval $0 < t < \infty$, from which one can then extend it to complex values of t).

Together with the preceding bound, analytic semigroups satisfy the following generalization in terms of a parameter $\alpha > 0$:

$$\|A^{\alpha}\, \mathrm{e}^{-At}\, x\| \leq C_{\alpha}\, t^{-\alpha}\, \|x\|. \tag{79}$$

As we will see in a while, for studying the non-linear equation (69) it is especially interesting the case where α is a fractional number between 0 and 1. In the case of § 10.3.2, the above **fractional powers** of A can be defined by the following formula analogous to (75) and (76):

$$A^{\alpha}x = \sum_{k=1}^{\infty} \lambda_k^{\alpha}\, \xi_k\, \boldsymbol{\psi}_k, \tag{80}$$

and their domain of definition $\mathbb{X}^{\alpha}$ is simply characterized by the convergence of the series of the right-hand side of the preceding formula. Combined with (76), this definition immediately gives the bound (79) with $C_{\alpha} = (\alpha/\mathrm{e})^{\alpha}$. Analogously to the case $\alpha = 1$, the domain $\mathbb{X}^{\alpha}$ is a complete space in the norm $\|\cdot\|_{(\alpha)}$ defined by the formula $\|x\|_{(\alpha)} = \|A^{\alpha}x\| + \|x\|$.

10.3.4. The preceding tools allow to study equation (69) by a method of successive approximations in the style of section 5. Analogously to what we did there, the idea is using formula (72) with $f(t) = B(z(t))$, that is

$$z(t) = \mathrm{e}^{-At}\, z_0 + \int_0^t \mathrm{e}^{-A(t-s)}\, B(z(s))\, \mathrm{d}s, \tag{81}$$

and following a process of successive approximations that corresponds to putting $z = z_{m-1}$ in the right-hand side of this integral equation and $z = z_m$ in its left-hand side.

In connection with all this, the problem arises that the non-linear operator B has also, like A, a domain of definition smaller than $\mathbb{E}$, let us call it $\mathbb{Y}$. So, in order to give a meaning to (81) and the corresponding process of successive approximations, it must be ensured that z takes values in $\mathbb{Y}$. Now, this is not so difficult if one can take $\mathbb{Y} = \mathbb{X}^\alpha$ for some $\alpha < 1$: in fact, the bound (79) together with the convergence of the integral $\int_0^t (t-s)^{-\alpha}\mathrm{d}s$ ensure that the integral in the right-hand side of (81) gives an element of $\mathbb{X}^\alpha$. More specifically, in order to get a result analogous to theorem 1, it suffices to assume that B maps $\mathbb{X}^\alpha$ to $\mathbb{E}$ and satisfies a bound of the form $\|B(X)-B(Y)\| \le L\,\|X-Y\|_{(\alpha)}$ whenever the norms $\|X\|_{(\alpha)}$ and $\|Y\|_{(\alpha)}$ are bounded.

In order to check the validity of the latter condition in specific applications, one must rely on less abstract characterizations of the spaces $\mathbb{X}^\alpha$ and their norms $\|\cdot\|_{(\alpha)}$ for $0 \le \alpha \le 1$. Obtaining them for the Stokes operator is relatively easy for $\alpha = 1/2$, but the case $\alpha = 1$ is more technical. Once this case is solved, which was done in 1960–61 by Vsevolod A. Solonnikov [33] and Lamberto Cattabriga [34], then it is not so difficult to deal with the remaining values of α.

The application of these ideas to the Navier-Stokes equations was carried out around 1960 with the contributions of Selim G. Kreĭn [35], Pavel E. Sobolevskiĭ [36], and of Hiroshi Fujita together with Tosio Kato [38]. According to the final results of these works, the Navier-Stokes equations for $n = 2$ or 3 can be put into the preceding scheme (in the quadratic norm setting) by taking $\alpha > n/4$.

In general terms, the techniques described in this section allow to extend the results of the preceding sections to the case of a region $\Omega \subset \mathbb{R}^n$. However, the core of the problem is still essentially the same.

11 Uniqueness versus regularity · Local dissipation · Partial regularity of locally dissipative solutions

11.1. Let us recall the fundamental questions which are awaiting an answer. They are concerned with the evolution problem for the Navier-Stokes equations, i. e. the problem of finding a solution, or solutions, of (14) for a given initial state. In order to simplify the wording, from now on it will be understood

that we are considering *global solutions*, i. e. solutions defined for all $t \geq 0$. With this convention, together with a certain loosening of the terminology in a direction which will be explained shortly, questions 1 and 2, posed respectively in § 3.4 and § 7.1, can be reformulated in the following way (question $2'$ goes in a direction opposite to that of question 2):

Question 1. Existence and uniqueness: *Is it true that the problem has one solution only?*

Question $2'$. Existence and regularity: *Is it true that the problem has (at least) one regular solution?*

Let us explain what do we mean in each case. We will start with question $2'$ and the term **regular solution**. This term can be viewed as embracing a wide range of meanings which generally speaking are different from each other but in the present case turn out to be equivalent.

One possible meaning is a "solution without singularities". More precisely, the absence of singularities should be understood as saying that the function $\boldsymbol{u}$ is bounded and continuous on every set of the form $\Omega \times [0, t]$. In such a context, the condition of being a solution can be understood in the weak sense or in the sense of equation (19) of section 5.

Another possible meaning is a "solution with a maximal regularity", say infinitely differentiable in the open set $\Omega \times (0, \infty)$ and "as differentiable as possible" in the closed set $\overline{\Omega} \times [0, \infty)$. What one can hope for in the latter set depends on the regularity of the data $\boldsymbol{u}_0$ and Ω. In this context, the condition of being a solution can be understood in the classical sense, at least when the data are regular enough. A low regularity of $\boldsymbol{u}_0$ can prevent the solution $\boldsymbol{u}$ from being continuous in the closed set $\overline{\Omega} \times [0, \infty)$, in which case the initial condition (14.5) must be understood in some weaker sense (in particular, this is the case of the semi-regular solutions considered in section 9, where $\boldsymbol{u}(t)$ approaches $\boldsymbol{u}_0$ in the quadratic norm as $t \downarrow 0$).

In principle, the "classical solutions" are located between the "solutions without singularities" and those with a "maximal regularity". In fact, the classical interpretation of equations (14) assumes only a limited regularity.

As we have seen in section 5 and in § 10.3.3, the Navier-Stokes equations have a regularizing character that entails the equivalence between these different meanings. In other words, if $\boldsymbol{u}$ is a solution of (14) without singularities then it is "maximally regular". More generally, there are many results of the following type: if $\boldsymbol{u}$ is a solution of (14) and it is "minimally regular" in some sense, then it is "maximally regular" (the problem is that this minimal regularity is never low enough to include the weak solutions, at least in the results of this kind obtained so far).

In particular, the term "regular solution" can therefore be considered as a synonym of "classical solution". So, question $2'$ is the same that we were

considering in sections 5–7, namely whether classical solutions can be extended to all $t \geq 0$. However, now we can also look at it from another point of view: among the weak solutions, which are already known to exist for all $t \geq 0$, is there one of them (at least) which contains no singularities?

Let us consider now question 1: Is it true that the problem has one solution only? In contrast to question $2'$, here we are not specifying which kind of solution. The idea is that we are now open to any generalized meanings of the term "solution". On the other hand, it is obvious that not any thing can be admitted as a solution. In this connection, for a generalized notion of solution to be admissible it is natural to require it to satisfy the following condition: when $\boldsymbol{u}$ is regular enough, then it is a generalized solution if and only if it is a classical one. Anyway, *an affirmative answer to question 1 must specify in which sense it understands the term "solution".*

As we have seen in the preceding sections, the weak notion of solution satisfies the property of (global) existence, but the solution is not ensured to be unique. On the other hand, the property of (global) existence holds in classes of solutions more restrictive than the weak ones. In fact, we have seen it to hold in the class of globally dissipative solutions, which is contained in the class of weak solutions. In front of this, it is natural to ask whether uniqueness holds in such more restrictive classes of solutions. This line of thought will be explored a little further in a while. Before that, however, we will make a remark about the relationship between questions 1 and $2'$.

11.2. Questions 1 and $2'$ are not independent from each other. In fact, by the uniqueness part of theorem 1, it is clear that *an affirmative answer to question $2'$ automatically entails an affirmative answer to question 1 in the class of regular solutions.*

On the other hand, the converse of *this statement* is also true: if we have one and only one regular solution, we obviously have at least one such solution. However, *an affirmative answer to question 1 in a class of generalized solutions does* ***not*** *necessarily entail an affirmative answer to question $2'$.* In general, it might happen that every initial state determined a unique solution defined for all $t \geq 0$, but this solution were not regular. As Michael Struwe points out in [64], in fact such a situation occurs in certain partial differential equations which are not so different from the Navier-Stokes equations (like the equation $\partial \boldsymbol{u}/\partial t = \Delta \boldsymbol{u} + |\nabla \boldsymbol{u}|^2 \boldsymbol{u}$, where $\boldsymbol{x}$ varies over a disc and $\boldsymbol{u}$ takes values in a sphere).

Therefore, the two questions posed above are not equivalent to each other. From a philosophical point of view, the most important one is certainly question 1, that is, the question of determinism. However, the "millennium problem" as specified by the *Clay Mathematics Institute* is not question 1 but question $2'$ [78].

11.3. In connection with question 1, in 1969 Ladyzhenskaya explored the possible non-uniqueness of weak solutions by means of an example which contains some undesirable features but is still quite interesting [29]. The example in question is based upon an idea of Golovkin (1964) and certain constructions akin to Leray's self-similar solutions (§ 7.2). The main difference from the latter is that here the singularities do not arise as $t \uparrow T$ but as $t \downarrow 0$. In other words, the example is not about the formation of a singularity, but about its future evolution. On the other hand, the singularity is associated with an infinite value of the energy, which makes it less interesting. Nevertheless, Ladyzhenskaya adapts her construction to a time-varying domain of the form $\{\,(\boldsymbol{x},t) \mid t > 0,\ \boldsymbol{x}/\sqrt{t} \in \Omega_1\,\}$ —where the spatial region becomes contracted to a point as $t \downarrow 0$— and she argues that the two functions constructed can be considered as two weak solutions of the same boundary value problem on this time-varying domain.

11.4. As we have seen in § 8.6 and § 10.1, the weak solutions obtained by Leray and Hopf satisfy the so-called energy *inequality* instead of the corresponding *equality* satisfied by the regular solutions. However, while the latter equality is a consequence of the very fact of being a regular solution, the energy inequality is not automatically ensured to hold for any weak solution. In front if this, Leray opted for explicitly including that inequality in his definition of a "turbulent" solution. Let us recall that we refer to such solutions as "globally dissipative", which emphasizes that they satisfy that inequality.

According to the energy equality, when the time increases from t_0 to t, the kinetic energy of a regular solution decreases exactly in the amount $\int_{t_0}^{t}\int_\Omega \nu\,|\nabla \boldsymbol{u}|^2\,\mathrm{d}V\mathrm{d}s$. In contrast, the energy inequality allows the kinetic energy to decrease by a larger amount. Therefore, requiring a weak solution to satisfy the energy inequality can be put in the following way: if the singularities result in a deviation from the energy equality, this deviation should be in the direction of having an *additional decrease* of energy.

Such a restriction is quite in line with the second principle of thermodynamics, which in the present context can be interpreted in terms of the "dissipation" of "macroscopic" kinetic energy into "microscopic" energy. Now, the second principle is not just a matter of what happens on the whole of Ω, but it places restrictions on what happens in every part of the fluid. This strongly suggests to consider a *local version of the energy inequality*.

Symbolically, this local condition of additional dissipation corresponds to scalarly multiplying the differential equation (14.1) by $\boldsymbol{u}$ and then replacing the equality sign by an inequality one, *whithout any integration* with respect to $\boldsymbol{x}$ nor t:

$$\partial\left(\tfrac{1}{2}|\boldsymbol{u}|^2\right)/\partial t \;-\; \nu\,\boldsymbol{u}\cdot\Delta\boldsymbol{u} \;+\; (\boldsymbol{u}\cdot\nabla)\left(p+\tfrac{1}{2}|\boldsymbol{u}|^2\right) \;\leq\; 0. \tag{82}$$

Somehow, this relation should be satisfied with an equality sign at every point where $\boldsymbol{u}$ is regular and with a strict inequality at any singular point.

In an attempt to be more accurate, we could look at such a condition from the point of view of the integral equation that keeps account of the amount of energy contained in a part ω. This corresponds to integrating the preceding relation on a space-time region of the form $\omega \times [t_0, t]$, which gives the following result:

$$\int_\omega \tfrac{1}{2}|\boldsymbol{u}(t)|^2 \,\mathrm{d}V \,+\, \int_{t_0}^{t}\int_\omega \nu\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}V\mathrm{d}s \;\;\leq$$
$$\int_\omega \tfrac{1}{2}|\boldsymbol{u}(t_0)|^2 \,\mathrm{d}V \,+\, \int_{t_0}^{t}\int_{\partial\omega} \Big(\nu\,\boldsymbol{u}\cdot\nabla_\perp \boldsymbol{u} + \big(p+\tfrac{1}{2}|\boldsymbol{u}|^2\big)\,u_\perp\Big)\,\mathrm{d}S\,\mathrm{d}s.$$
$$\forall t_0 \geq 0, \quad \forall t \geq t_0, \quad \forall \omega \in \mathcal{P}(\Omega). \tag{83}$$

For $\omega = \Omega$, the last integral vanishes because of the non-slip condition and so one recovers the global energy inequality that has been considered until now. In the general case, it must be noticed that the integral of $(p+\frac{1}{2}|\boldsymbol{u}|^2)u_\perp$ which appears in the last term might be not well defined for an arbitrary weak solution.

Alternatively, we can look at the question from a point of view akin to the integral equations which appear in the definition of a weak solution (section 7). Since the relation that we want to translate into a weak form is not an equality but an inequality, it is natural to restrict ourselves to *positive test functions* on the space-time region $\Omega\times(0,\infty)$. In the following, the set of such test functions will be denoted by $\mathcal{D}_+(\Omega\times(0,\infty))$. As in section 7, we will multiply relation (82) by an arbitrary element of this set, after which we will integrate on $\Omega\times(0,t)$, and finally we will apply the formula of integration by parts so as to transfer the derivatives to the test function. By so doing, one arrives at the following formulation:

$$\int_\Omega \tfrac{1}{2}|\boldsymbol{u}(t)|^2\,\phi(t)\,\mathrm{d}V \,+\, \int_0^t\int_\Omega \nu\,|\nabla \boldsymbol{u}|^2\,\phi\,\mathrm{d}V\mathrm{d}s \;\;\leq$$
$$\int_0^t\int_\Omega \Big(\tfrac{1}{2}|\boldsymbol{u}|^2\;(\partial\phi/\partial t + \nu\Delta\phi) + \big(p+\tfrac{1}{2}|\boldsymbol{u}|^2\big)\,\boldsymbol{u}\cdot\nabla\phi\Big)\,\mathrm{d}V\mathrm{d}s.$$
$$\forall t > 0, \quad \forall \phi \in \mathcal{D}_+(\Omega\times(0,\infty)). \tag{84}$$

Although it is not obvious at all, it turns out that the definition of a weak solution entails certain bounds on the space-time integrals of $|\boldsymbol{u}|^3$ and $|p\,\boldsymbol{u}|$ which give a proper meaning to the different terms of (84). In consonance with our previous terminology, the weak solutions which satisfy this condition will be called **locally dissipative solutions**.

Let us mention also that there exists an additional and quite interesting reformulation of the preceding condition which is due to Jean Duchon and Raoul Robert [50].

11.5. In fact, condition (84) was introduced in 1977 by Vladimir Scheffer [40], who remarked that its being fulfilled implied certain properties of partial

regularity for $\boldsymbol{u}$. These properties say essentially that $\boldsymbol{u}$ is regular except on a set of low dimension. Besides, Scheffer checked that Leray's proof of the existence of weak solutions (section 8) can be adapted so that the solutions obtained satisfy the local dissipation condition (84). In 1982 these results were improved by Luis Caffarelli, Robert Kohn and Louis Nirenberg [42] (who refer to such solutions as "suitable weak solutions"), and afterwards they have been refined by Fang-Hua Lin [48] and by Ol'ga A. Ladyzhenskaya and Grigoriĭ A. Serëgin [49]. Altogether these works allow to state the two following theorems:

Theorem 7 *Let $\Omega \subseteq \mathbb{R}^3$. For any initial state $\boldsymbol{u}_0$ whose square is integrable and whose divergence is zero in the weak sense, problem* (14) *has at least one locally dissipative solution $\boldsymbol{u}(t)$ which is defined for all $t \geq 0$.*

Theorem 8 *Let $\Omega \subseteq \mathbb{R}^3$. Let $\boldsymbol{u}$ be a locally dissipative solution of* (14) *in the interval $[0, \infty)$. Then $\boldsymbol{u}$ is regular in an open subset of $\Omega \times (0, \infty)$ whose complement is a set of dimension less than or equal to* 1 *in the sense of Haussdorff.*

11.6. As a matter of fact, the proof of this last theorem does not make use of the hypothesis that $\boldsymbol{u}$ solves (14) in the weak sense, but it is based only on the inequality (84) and the boundedness of certain integrals associated to it. In this connection, towards 1985 Scheffer showed that one can construct functions $\boldsymbol{u}$ that satisfy all of these premises but have a singular set of fractional dimension arbitrarily near to 1 [45]. This means that if a result stronger than theorem 8 is ever obtained, its proof will have to rely on the full hypothesis that $\boldsymbol{u}$ does solve (14) in the weak sense.

In fact, the funtion $\boldsymbol{u}$ constructed by Scheffer in [45] is not a solution of problem (14), but it solves a more general one where an external force $\boldsymbol{f}$ is added to equation (14.1), like it was considered in equation (9). On the other hand, $\boldsymbol{u}$ and $\boldsymbol{f}$ are constructed in such a way that the inequality $\boldsymbol{f} \cdot \boldsymbol{u} \leq 0$ holds (everywhere), i. e. $\boldsymbol{f}$ contributes to an additional dissipation of energy. At first sight, this should act against the formation of singularities. However, it must be remarked that in the presence of the external force $\boldsymbol{f}$ the condition of local dissipation in the spirit of the second principle of thermodynamics is not given by (82)–(84), but the right-hand side of (82) must be changed into $\boldsymbol{f} \cdot \boldsymbol{u}$, which results in a more restrictive condition (because of the sign of that product). However, Scheffer's construction satisfies only (82). It would seem that this margin plays an essential role in order to make it possible that $\boldsymbol{u}$ develops singularities.

Let us remark also that Scheffer's example uses certain ideas of self-similarity that remind of Leray's attempt in the same direction (§ 7.2). In the case of Scheffer, this self-similarity gives rise to a "fractal" character that plays an essential role in order to obtain a positive fractional dimension of the singular set.

According to Scheffer, one could make $\boldsymbol{f}$ infinitely differentiable in $\Omega\times[0,T)$, where T represents the first singular time. However, it seems unavoidable that $\boldsymbol{f}$ has some kind of singularity at time T. So, the singularities of $\boldsymbol{u}$ are somehow a consequence of those of $\boldsymbol{f}$.

11.7. Especially after the preceding example, theorem 8 can be seen as a frustated attempt to give a positive answer to question $2'$ (existence and regularity). On the other hand, independently of their possible regularity, the locally dissipative solutions are also quite interesting in connection with question 1 (existence and uniqueness). Since the existence part is already established by theorem 7, the question reduces to the following one:

Question 4 *Is it true that the evolution problem for the Navier-Stokes equations has no more than one locally dissipative solution?*

In support of an affirmative answer to this question, one can adduce that, for non-viscous compressible fluids, there are some cases where an "entropy condition" imposed at the singularities has indeed the virtue of determining a unique solution of the evolution problem.

12 Epilogue: After all, celestial mechanics is not perfect either

Because of space limitations, the present account has left out many other interesting contributions about the question of determinism of the Navier-Stokes equations. For more information, the reader is directed to other monographs, like [51–54]. From a more historical and human point of view, we strongly recommend also the scientific biographies of Carl Oseen, Jean Leray and Ol'ga Layzhenskaya [55–66].

In spite of the earnest efforts of other authors, one can still say that the state of the question has not changed in an essential way since the works of Jean Leray in 1933–34. According to him, in that time Henri Lebesgue already recognized that the problem was really a difficult one: "Do not devote too much time to such a refractory question. Do something else!". This quotation comes from [22], a brief retrospective vision of Jean Leray published in 1994 (which is also his last article). As it is pointed out in there, in the last decades of the twentieth century one has arrived at a similar state of affairs in two other cases of fluid mechanics, namely, the Navier-Stokes equations for a viscous compressible fluid in isentropic regime, and the Botzmann equation of kinetic gas theory. Similarly to the case of a viscous incompressible fluid, in those two other cases one has also been able to prove, on the one hand, the local existence and uniqueness of a classical solution, and on the other hand, the global existence of some kind of generalized solutions; however, as yet one has not been able to ensure both the global existence and uniqueness at the same time.

So, in its present state fluid mechanics is quite far from being able to substantiate the possibility of making long-term predictions. As we have said in the introduction, the problem could be more severe than the "butterfly effect" invoked by the meteorologists: if future researches confirmed that a single initial state can give rise to a multiplicity of solutions, then we would be in front of a sort of "butterfly effect with no butterfly".

Certainly, the picture is quite different from the one offered by celestial mechanics, where eclipses can be predicted thousands of years in advance. However, the truth is that even *celestial mechanics is not completely deterministic* in the sense that we are asking of Navier-Stokes equations: Even in the ideal case of point bodies one cannot completely discard the possibility of collisions. When a collision involves only two bodies, one can show that there is just one "reasonable" way to extend the solution past the collision. In the terminology of celestial mechanics, one says that such singularities are "regularizable"; in the terminology that we have been using in this article, we would say that there is a generalized solution that goes across the singularity without any branching. However, one has shown also that this statement is no longer true for most triple collisions (see for instance [71]). In this case there are often several different ways to continue the motion.

So, perhaps fluid mechanics is not so different from celestial mechanics in the question of determinism. Anyway, the difference between them is that celestial mechanics knows quite well its limitations in that connection, whereas fluid mechanics still lacks such a knowledge.

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We use the following symbols: $\equiv$ means a subsequent publication of the same work; $\gg$ means a translation; finally, $>$ introduces the translation of the title or other bibliographical data. In some references, especially the oldest ones, we have distinguished between the year of publication and that when the memoir was read before a scholar society or when the manuscript was received by the publisher; in that case, the year that appears next to the author's name is the year of reading or receipt, whereas the year of publication is indicated after the title of the corresponding book or series.

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LAS ESCUELAS HISPANO-FRANCESAS JACQUES-LOUIS LIONS DE SIMULACIÓN NUMÉRICA EN FíSICA E INGENIERíA (1984-2006)

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Resumen

El objetivo de esta memoria es sintetizar las orientaciones y realizaciones de las doce primeras Escuelas Hispano-Francesas celebradas en España, con frecuencia bianual, entre 1984 y 2006. Estas escuelas se han denominado *"Escuelas Jacques-Louis Lions"* por iniciativa de SeMA desde 2004 para honrar la memoria de Jacques-Louis Lions quien, con sus excepcionales cualidades científicas y humanas, se implicó enormemente en la puesta en marcha de numerosas colaboraciones fructíferas entre la mayor parte de los grupos de análisis y simulación numérica de nuestros dos países. Tras una breve sinopsis de los principales aspectos del Análisis y de la Simulación Numérica en Física e Ingeniería, daremos una reseña detallada de las doce primeras Escuelas.

1 Principales aspectos del Análisis y la Simulación Numérica en Física e Ingeniería

Los procesos de diseño, de producción, de análisis de comportamientos, de financiación, ligados a problemas concretos de ingeniería de origen mecánico, físico, químico, médico... se realizan hoy día esencialmente por vía numérica, en el marco de lo que se denomina *"Ingeniería Virtual"*. Ésta se ha desarrollado considerablemente en estrecho paralelismo con el auge de los útiles informáticos (materiales y lógicos). A la modelización de los problemas unidisciplinares de base siguió la de los problemas multidisciplinares acoplados y el estudio de los problemas inversos. Indicamos en las Figuras 1, 2 y 3 los esqueletos de las diferentes etapas, remitiendo a las publicaciones generales y especializadas sobre los diferentes temas para más detalles. El objeto de estas Escuelas es esencialmente enfocar el proyector sobre las diversas técnicas subyacentes a fin de ayudar y motivar a los investigadores que se enfrentan a ellas.

*con la colaboración de Carlos Parés Madroñal y Carlos Vázquez Cendón para la traducción al castellano y la presentación.

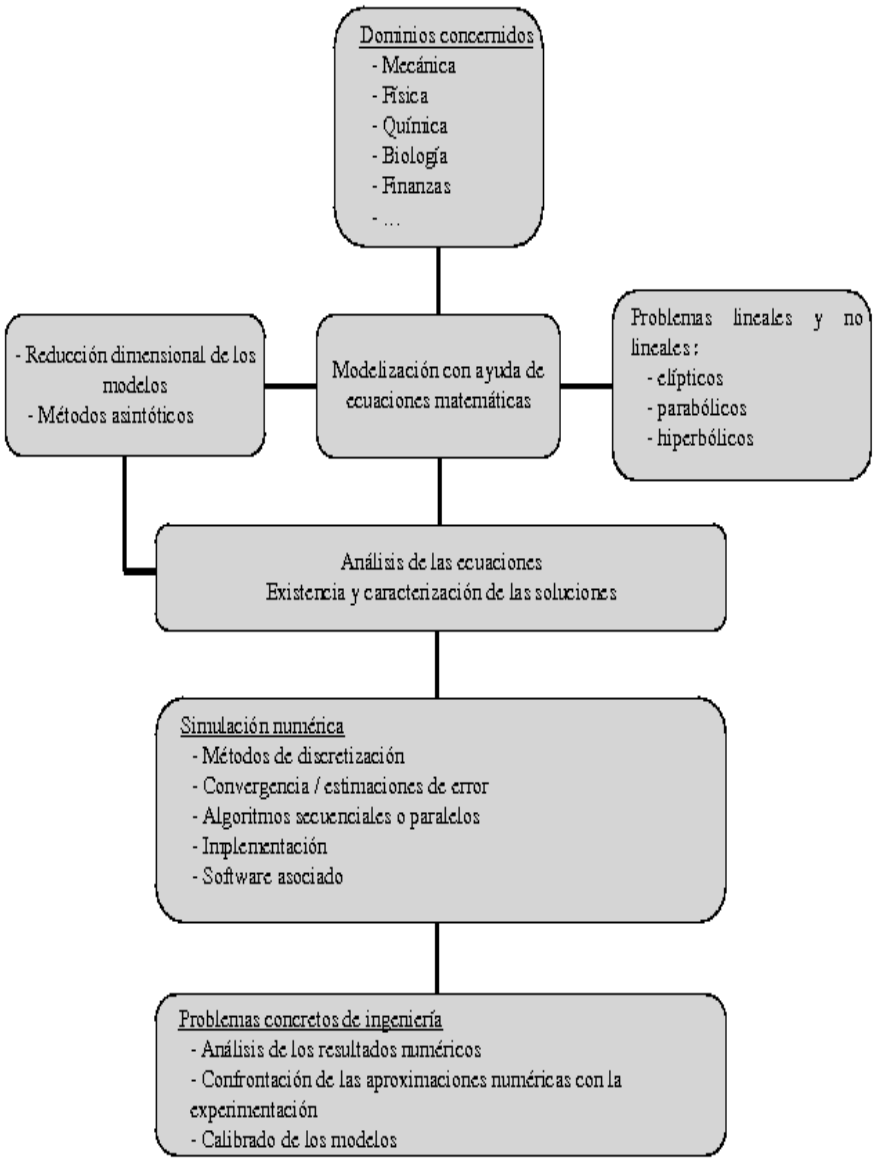


Figura 1: Problemas unidisciplinares directos

2 El contexto general de las colaboraciones hispano-francesas

La creación y la continuidad de las Escuelas de Otoño Hispano-Francesas desde hace más de 20 años se inscriben de lleno en el contexto de la muy

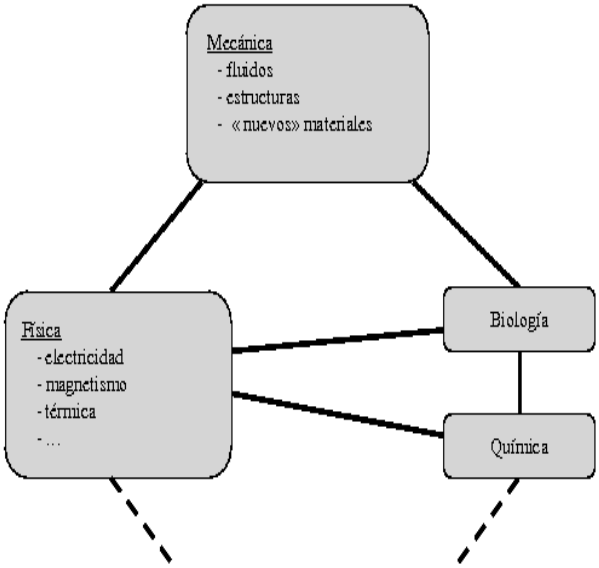


Figura 2: Problemas acoplados

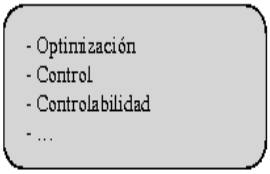


Figura 3: Problemas inversos

importante colaboración desarrollada desde hará pronto 40 años entre nuestros dos países en el dominio del Análisis Numérico y, más precisamente como lo indica el título de las Escuelas, en la Simulación Numérica en Física e Ingeniería. Por tanto, antes de abordar el contenido de las doce primeras Escuelas, recordaremos los principales aspectos de esta larga y muy rica colaboración que ha asociado científica y amistosamente a la mayor parte de los mejores especialistas del Análisis y de la Simulación Numérica de nuestros dos países.

La génesis de esta colaboración se remonta a 1963, cuando Alberto Dou de Madrid sugiere a uno de sus discípulos, Antonio Valle, que realice el doctorado bajo la dirección de Jacques-Louis Lions, que acababa de incorporarse a la Facultad de Ciencias de París. Las cualidades científicas y humanas de J.L. Lions y el incondicional apoyo brindado a A. Valle sellaron una amistad y una estima mutua que se amplificaron en el transcurso del tiempo y de las sucesivas

colaboraciones cada vez más fructíferas.

Inspirándose en su propia experiencia y con el objetivo de formar un núcleo duro de científicos españoles especialistas en Análisis y Métodos Numéricos, A. Valle propició a su vez estancias en Francia de algunos de sus mejores alumnos de las Universidades de Santiago de Compostela, de Sevilla a continuación y, finalmente, de Málaga. La mayor parte de ellos obtuvieron el DEA de Análisis Numérico de la Universidad Pierre y Marie Curie (París VI) y prolongaron posteriormente su estancia a fin de realizar la tesis doctoral en el INRIA (Institut Nacional de Recherche en Informatique et Automátique) o también, en algunos casos, en el propio Laboratorio de Análisis Numérico de la citada Universidad o en la Escuela Politécnica. Estos intercambios, inicialmente informales, recibieron rápidamente el precioso apoyo de las universidades españolas involucradas, así como el de los Servicios Científicos de las Embajadas de España en París y de Francia en Madrid, a través de la puesta en marcha de *Acciones Integradas* bilaterales (financiación de viajes y estancias de los responsables franceses y españoles de los intercambios, financiación de becas de estudios de tercer ciclo), que asociaban universidades españolas (Santiago de Compostela, Sevilla, Universidad Politécnica de Madrid, Málaga...) y, del lado francés, el INRIA, piedra angular de la colaboración, así como el Laboratorio de Análisis Numérico de la Universidad París VI, la Escuela Politécnica...

Añadamos que estas numerosas y fructructíferas colaboraciones en los dominios del Análisis y de la Simulación Numérica se extendieron enseguida a otros sectores, particularmente en lo que se refiere al INRIA, a la informática, a la visión por ordenador y al tratamiento de imágenes, tanto en el marco de los proyectos bilaterales PICASSO como en el de la participación común en Redes o Proyectos de Investigación europeos multilaterales.

3 Puesta en marcha de las Escuelas de Otoño

Volvamos al principio de los años 80. Las Acciones Integradas bilaterales habían permitido institucionalizar los intercambios que aumentaban sin cesar especialmente entre (la lista no es exhaustiva):

- las Universidades de Santiago de Compostela, Sevilla, Politécnica de Madrid, Málaga
- el IRIA (convertido en el INRIA a finales de 1979), la Universidad París VI, la Escuela Politécnica

Estas colaboraciones comportaban la acogida de doctorandos españoles en los equipos franceses y estancias de trabajo de corta duración de los responsables españoles y franceses.

Mientras tanto, J.L. Lions fue nombrado Presidente Director General del INRIA y se propuso dar un fuerte impulso complementario a esta colaboración a la que A. Valle y él mismo se sentían muy vinculados. En este contexto y en mi calidad de responsable en el INRIA de las actividades de Formación para la Investigación, J.L. Lions y R. Glowinski me pidieron que viajara a Madrid

los días 6 y 7 de Junio de 1983 para asistir a una reunión científica en la que participarían un gran número de científicos españoles especialistas en Análisis y Métodos Numéricos. En paralelo a esta reunión, se abordaría una iniciativa para reforzar los intercambios franco-españoles: *"ya verás sobre el terreno como están las cosas"*, me dijeron. Siguiendo estas instrucciones generales, viajé a Madrid donde fui acogido calurosamente en el aeropuerto por F. Michavila, Director del Departamento de Análisis Numérico de la Escuela de Minas de Madrid (Universidad Politécnica de Madrid) y por H. Carsalade, Consejero Científico de la Embajada de Francia en Madrid. De allí, nos dirigimos inmediatamente hacia Alcalá de Henares, donde la reunión acababa de comenzar. Y, efectivamente, tras una jornada y media de intercambios científicos y algunas consumiciones nocturnas en la muy bella plaza de Alcalá de Henares, empezaron a dibujarse los contornos de una primera Escuela de Otoño Hispano-Francesa sobre Simulación Numérica en Mecánica y Física. Fue A. Bermúdez de Castro, Profesor en la Universidad de Santiago de Compostela y primer doctorando español del INRIA, el encargado de organizar este primer encuentro.

4 Objetivos de las Escuelas

Los objetivos generales de las Escuelas se pueden resumir en los siguiente cuatro puntos:

- Proporcionar información a los interesados en las Matemáticas Aplicadas, y, más precisamente, en las nuevas posibilidades ofrecidas por la Simulación Numérica que son objeto de investigación y desarrollo tanto en España y Francia.
- Servir de punto de encuentro entre investigadores, profesores, ingenieros industriales y jóvenes titulados de los dos países
- Presentar aplicaciones efectivas de la Simulación Numérica en empresas industriales españolas y francesas
- Analizar las contribuciones futuras que la Simulación Numérica puede aportar en los diversos sectores industriales de nuestros dos países.

5 Las Escuelas sucesivas

Una vez fijados el contexto de la creación de esta serie de Escuelas y sus objetivos, recordamos sucintamente los principales hitos y las evoluciones que se han ido sucediendo. Para cada una de las ediciones, indicaremos sistemáticamente:

- el contexto,
- los profesores, los conferenciantes y los programas,
- los participantes,

- los trípticos.

Conviene señalar que todas estas Escuelas se han beneficiado de un apoyo constante y decisivo de las universidades españolas organizadoras, del INRIA y de los Servicios Científicos de la Embajada de Francia en Madrid.

5.1 I Escuela: Santiago de Compostela, del 22-10 al 28-10 de 1984

La elección de Santiago de Compostela no fue fruto del azar. Tras haberse doctorado en París, A. Valle organizó allí un primer grupo de Análisis Numérico antes de trasladarse a Sevilla, donde impulsaría la creación de un segundo equipo. El grupo de Santiago de Compostela quedaría en adelante bajo la responsabilidad de A. Bermúdez de Castro.

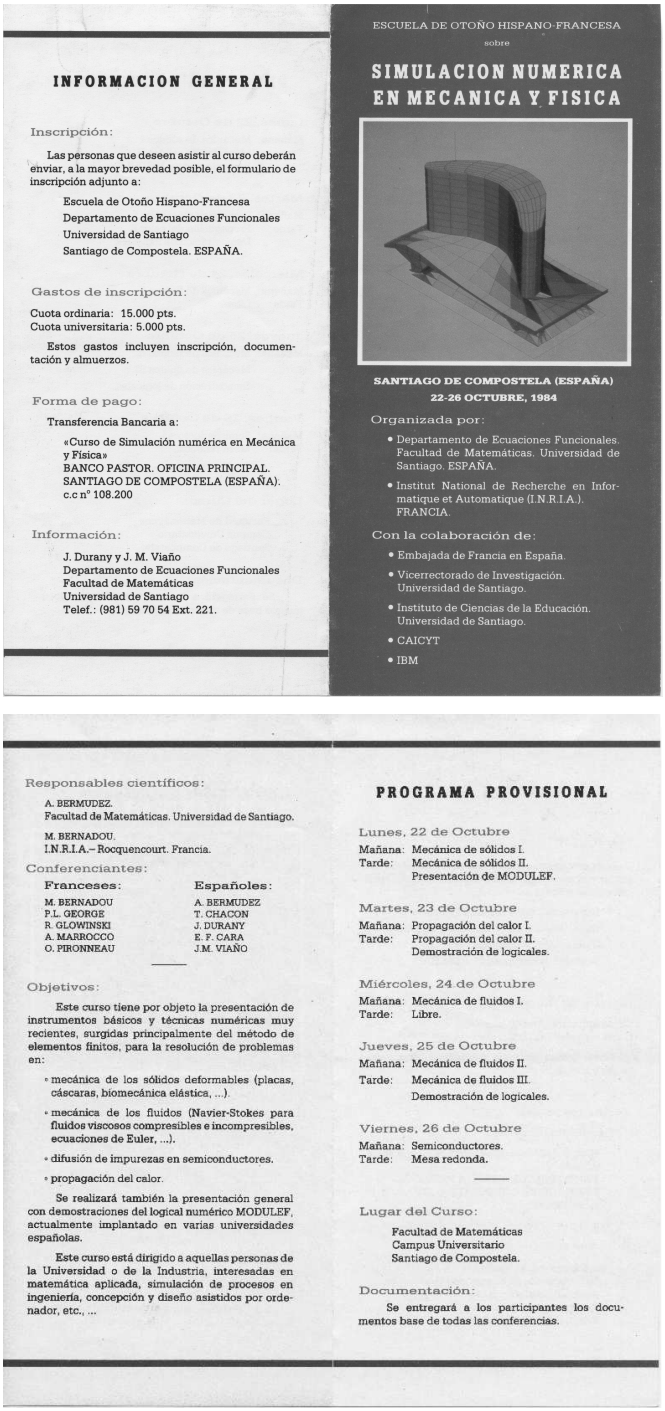
Esta Escuela, que no es aún calificada de primera (nada aseguraba entonces que fuera haber otras...) es ya titulada *"de Otoño"*, ya que tuvo lugar a finales de Octubre. Habrá algunas distorsiones en el calendario más adelante: el resto de las Escuelas tendrá lugar los años pares hacia el final del verano, por la general en la tercera semana de Septiembre. La última semana del verano ofrece la ventaja de situarse en España entre la segunda sesión de exámenes y la vuelta a clase de los estudiantes universitarios.

La Escuela se celebró en los modernos locales de la Universidad de Santiago de Compostela, situados no demasiado lejos del notable centro de la ciudad, que fue declarado por la UNESCO poco después, y con toda justicia, Patrimonio Cultural de la Humanidad. Es inútil precisar que sus riquezas históricas, religiosas y monumentales ocuparon completamente la tarde libre del miércoles... ¡ con mayor razón (al menos para la parte francesa) si se tiene en cuenta que, en España, la tarde empieza hacia las 15h30-16h, y no tras el mediodía como en Francia!

5.1.1 Programa

En esta primera Escuela, los cinco cursos trataron sobre temas *"clásicos"* de métodos numéricos.

- *Propagación del calor*, A. Bermúdez de Castro y J. Durany (Univ. Santiago de Compostela): Modelización. Marco funcional. Aproximación numérica
- *Mecánica del sólido*, M. Bernadou (INRIA) y J.M. Viaño (Univ. Santiago de Compostela): Placas y láminas delgadas. Biomecánica ósea
- *Mecánica de fluidos*, T. Chacón (Univ. de Sevilla), R. Glowinski (Univ. París VI e INRIA) y O. Pironneau (Univ. París XIII e INRIA): Ecuaciones de Navier-Stokes para fluidos viscosos incompresibles. Ecuaciones de Navier-Stokes para fluidos viscosos compresibles. Ecuaciones de Euler
- *Métodos numéricos para la difusión de impurezas en un semiconductor*, E. Fernández Cara (Univ. de Sevilla) y A. Marrocco (INRIA)



INFORMACION GENERAL

Inscripción:

Las personas que deseen asistir al curso deberán enviar, a la mayor brevedad posible, el formulario de inscripción adjunto a:

Escuela de Otoño Hispano-Francesa
Departamento de Ecuaciones Funcionales
Universidad de Santiago
Santiago de Compostela. ESPAÑA.

Gastos de inscripción:

Cuota ordinaria: 15.000 pts.
Cuota universitaria: 5.000 pts.

Estos gastos incluyen inscripción, documentación y almuerzos.

Forma de pago:

Transferencia Bancaria a:

«Curso de Simulación numérica en Mecánica y Física»
BANCO PASTOR. OFICINA PRINCIPAL.
SANTIAGO DE COMPOSTELA (ESPAÑA).
c.c nº 108.200

Información:

J. Durany y J. M. Viaño
Departamento de Ecuaciones Funcionales
Facultad de Matemáticas
Universidad de Santiago
Telef.: (981) 59 70 54 Ext. 221.

ESCUELA DE OTOÑO HISPANO-FRANCESA
sobre

SIMULACION NUMERICA EN MECANICA Y FISICA

SANTIAGO DE COMPOSTELA (ESPAÑA)
22-26 OCTUBRE, 1984

Organizada por:

- Departamento de Ecuaciones Funcionales. Facultad de Matemáticas. Universidad de Santiago. ESPAÑA.
- Institut National de Recherche en Informatique et Automatique (I.N.R.I.A.). FRANCIA.

Con la colaboración de:

- Embajada de Francia en España.
- Vicerrectorado de Investigación. Universidad de Santiago.
- Instituto de Ciencias de la Educación. Universidad de Santiago.
- CAICYT
- IBM

Responsables científicos:

A. BERMUDEZ.
Facultad de Matemáticas. Universidad de Santiago.

M. BERNADOU.
I.N.R.I.A.– Rocquencourt. Francia.

Conferenciantes:

Franceses:	Españoles:
M. BERNADOU	A. BERMUDEZ
P.L. GEORGE	T. CHACON
R. GLOWINSKI	J. DURANY
A. MARROCCO	E. F. CARA
O. PIRONNEAU	J.M. VIAÑO

Objetivos:

Este curso tiene por objeto la presentación de instrumentos básicos y técnicas numéricas muy recientes, surgidas principalmente del método de elementos finitos, para la resolución de problemas en:

- mecánica de los sólidos deformables (placas, cáscaras, biomecánica elástica, ...).
- mecánica de los fluidos (Navier-Stokes para fluidos viscosos compresibles e incompresibles, ecuaciones de Euler, ...).
- difusión de impurezas en semiconductores.
- propagación del calor.

Se realizará también la presentación general con demostraciones del logical numérico MODULEF, actualmente implantado en varias universidades españolas.

Este curso está dirigido a aquellas personas de la Universidad o de la Industria, interesadas en matemática aplicada, simulación de procesos en ingeniería, concepción y diseño asistidos por ordenador, etc., ...

PROGRAMA PROVISIONAL

Lunes, 22 de Octubre
Mañana: Mecánica de sólidos I.
Tarde: Mecánica de sólidos II.
Presentación de MODULEF.

Martes, 23 de Octubre
Mañana: Propagación del calor I.
Tarde: Propagación del calor II.
Demostración de logicales.

Miércoles, 24 de Octubre
Mañana: Mecánica de fluidos I.
Tarde: Libre.

Jueves, 25 de Octubre
Mañana: Mecánica de fluidos II.
Tarde: Mecánica de fluidos III.
Demostración de logicales.

Viernes, 26 de Octubre
Mañana: Semiconductores.
Tarde: Mesa redonda.

Lugar del Curso:

Facultad de Matemáticas
Campus Universitario
Santiago de Compostela.

Documentación:

Se entregará a los participantes los documentos base de todas las conferencias.

Figura 4: Cartel de la EHF1984

- *Software de investigación Modulef*, P.L. George (INRIA): Presentación general. Algunos ejemplos de problemas resueltos con Modulef. Demostraciones de uso

5.1.2 Participantes

Esta Escuela reunió a 93 participantes (incluyendo a los conferenciantes) que procedían esencialmente de universidades y centros de investigación españoles muy repartidos en el territorio del país. Más precisamente:

- *España*: Universidades Autónoma de Barcelona (1); de Cádiz (2, de los cuales 1 de Algeciras); Cantabria (4); Córdoba (2); Autónoma (2) y Politécnica (8) de Madrid; Oviedo (15); País Vasco (2); Politécnica de Cataluña (1); Politécnica de Valencia (3); UNED (1); Santiago de Compostela (32, de los cuales 6 de Vigo, 5 de La Coruña, 1 de Orense, 1 del Ferrol); Sevilla (2); Valladolid (4, de los cuales 2 de Burgos); Pontificia (1); San Sebastián (2); Zaragoza (2); Telefónica (1), INT Aerospacial (1), ENIEPSA (1) de Madrid.
- *Francia*: INRIA Rocquencourt (4); Univ. París VI (1)
- *Portugal*: IST de Lisboa (1)

5.1.3 Conclusión

El objetivo de reunir durante una semana a los principales investigadores y utilizadores españoles en métodos numéricos, jóvenes y consagrados, fue plenamente alcanzado. La demanda de colaboración entre los dos países fue fuertemente percibida, así como la de formación complementaria en Francia. Evidentemente, esta demanda estaba relacionada en primer lugar con los métodos de simulación numérica, aunque también, en un sentido más amplio, con la formación de informáticos de alto nivel y de futuros responsables universitarios, por lo que el INRIA fue especialmente solicitado.

5.2 II Escuela: Castillo de Bil-Bil, Benalmádena-Costa, Málaga, del 22-9 al 4-10 de 1986

Tras el éxito cosechado por la primera Escuela tan bien organizada por A. Bermúdez de Castro, J. Durany y J.M. Viaño en la ciudad histórica de Santiago de Compostela, se confirmó la celebración de una segunda Escuela, esta vez en el sur de España. A. Valle acababa de dejar Sevilla para volver a su ciudad natal, Málaga, donde impulsó la creación, tras Santiago de Compostela y Sevilla, de un tercer grupo de Análisis Numérico. En la celebración de esta Escuela vio la ocasión de dar a conocer a su joven equipo de especialistas en Numérico, tanto a la comunidad española como a algunos responsables franceses. Esta forma de proceder, que reiteró bastantes veces y bajo distintas formas, contribuyó fuertemente a dar a conocer al grupo de Málaga, poco numeroso pero de gran calidad.

INFORMACION GENERAL

Gastos de inscripción:

Cuota ordinaria: 50.000 ptas.
Cuota universitaria: 15.000 ptas.
(sólo hasta final de agosto)

Forma de pago:

Transferencia bancaria a la siguiente cuenta.
Nº 100.610-05 "Universidad de Malaga. Contratos, cursos y convenios"
Banco de Granada, Urbana Nº 2,
Paseo de Reding 19, 29016 Malaga
Remitir fotocopia del resguardo o aviso de la transferencia, a:
Prof. A. Valle
(Escuela Hispano-Francesa de Simulación Numérica)
Facultad de Ciencias
29080 Málaga
Para cualquier consulta, dirigirse a la misma dirección.
Teléfono (952) 28 13 00, ext. 334.

Situacion:

La sede de la Escuela se encuentra en la costa, a unos 20 Km. de Málaga (12 del aeropuerto) en dirección oeste.
Los alojamientos que se recomendaron, están en su inmediata proximidad.
Se aconseja a los participantes en la Escuela, su llegada el Domingo 28 de Septiembre por la tarde.

Presidentes:

A. DOU, de la Real Academia de Ciencias
A. BENSOUSSAN, Presidente del I.N.R.I.A.

Responsables:

A. VALLE, Universidad de Málaga.
M. BERNADOU, I.N.R.I.A.

Comité Científico:

A. BERMUDEZ DE CASTRO, Univ. de Santiago.
J. BLUM, I.N.R.I.A.
J. CEA, Université de Nice.
E. FERNANDEZ CARA, Univ. de Sevilla.
R. GLOWINSKI, I.N.R.I.A. / Univ. de Houston.
F. MICHAVILA, E.T.S.I. de Minas, U.P.M.
C. MORENO, Universidad Autónoma de Madrid.
O. Pironneau, I.N.R.I.A. / Univ. de París VI.

Profesores:

Españoles	*Franceses*
C. CONDE E.T.S.I. Minas, U.P.M.	M. BERNADOU I.N.R.I.A.
T. CHACON Univ. de Sevilla	J.CEA Univ. de Niza
J. DURANY Univ. de Santiago	PH. DESTUYNDER Ecole Centrale des Arts et Manufactures París.
F. MICHAVILA E.T.S.I. Minas, U.P.M.	P. LE TALLEC Lab. Central des Ponts et Chausseés, París
J.M. VIAÑO Univ. de Santiago	J. PERIAUX Avions Marcel Dassault-Bréguet Aviation, St. Cloud.

Objetivos:

Siguiendo la pauta establecida por la anterior Escuela de la misma denominación (Santiago de Compostela, octubre 1984), se trata de presentar instrumental matemático básico y técnicas numéricas muy recientes relacionadas con el método de elementos finitos, para la resolución de problemas en:

- Mecánica de Fluidos (Navier-Stokes para fluidos viscosos compresibles e incompresibles, modelización de la turbulencia etc.)
- Elasto-visco-plasticidad
- "Cáscaras"
- Mecánica de la Fractura
- Optimización.

Se desea asimismo, realizar una presentación general del logical numérico MODULEF, implantado en varias universidades españolas.

La Escuela está dirigida a aquellas personas de la Universidad y/o de la Industria, interesadas en la Matemática Aplicada, Simulación de Procesos en Ingeniería, Concepción y diseño asistidos por ordenador, etc.

PROGRAMA PROVISIONAL

Lunes, 29 de Septiembre:

9,15 h.: Apertura
9,30 h.: Instrumental previo I (1 ½h.)
11,15 h.: Instrumental previo II (1 ½h.)
16,15 h.: Instrumental previo III (1½ h.)
Estas tres sesiones serán impartidas conjuntamente por los Profesores C. Conde y J. Durany.
18,00 h.: Elasto-visco-plasticidad (E-V-P) I (Prof. Viaño, 1½ h.)

Martes 30 de Septiembre:

9,30 h.: E-V-P. II (Prof. Le Tallec, 1 ½h.)
11,15 h.: Mecánica de la Fractura (Prof. Michavila, 1 ½ h.)
16,15 h.: E-V-P III (Prof. Viaño, 1 h.)
17,30 h.: En paralelo (1½ h.):
Turbulencia (Prof. Chacón);
Fractura (Prof. Michavila);

Miércoles 1 de Octubre:

Probable desplazamiento a Sevilla
11,30-13,30 Demostraciones MODULEF en el Centro de Cálculo de dicha Universidad.
Tarde Libre en Sevilla
18,00h.: Regreso a la sede de la Escuela.

Jueves 2 de Octubre:

9,30 h.: Optimización (Prof. Cea, 1 h.)
10,45 h.: Mecánica de Fluídos (Prof. Periaux, 2 h.)
Tarde: En Paralelo
16,15 h.: Turbulencia (Prof. Chacón)
Fractura (Prof. Michavila) (1 h.)
17,30 h.: En paralelo
Mecánica de Fluídos (Prof. Periaux)
Cáscaras (Prof. Bernadou) (2 h.)

Viernes 3 de Octubre:

9,30 h.: En paralelo
E-V-P IV (Prof. Le Tallec)
Fractura (Prof. Destuynder) (1 ½h.)
11,15 h.: Optimización (Prof. Cea, 1½ h.)
16, 15 h.: En paralelo:
E-V-P. V (Prof. Viaño, 1 ½h.)
18,00 h.: En paralelo
Turbulencia (Prof. Chacón)
Cáscaras (Prof. Bernadou) (1½ h.)

Sabado, 4 de Octubre:

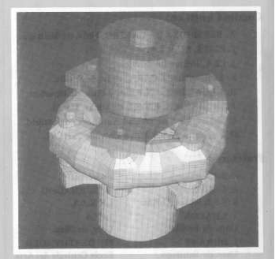
9,15 h.: Optimización (Prof. Cea, 1½ h.)
11,00 h.: En paralelo:
E-V-P VI (Prof. Le Tallec)
Fractura (Prof. Destuynder) (1 h.)
12,15 h.: Mesa redonda y Clausura de la Escuela.

Observaciones

El Prof. O. Pironneau, podría participar en la Escuela, teniendo a su cargo alguna de las exposiciones del bloque Mecánica de Fluidos-Turbulencia.

II ESCUELA DE OTOÑO HISPANO-FRANCESA
sobre

SIMULACION NUMERICA EN MECANICA Y FISICA



CASTILLO DE BIL-BIL,
BENALMADENA-COSTA (MALAGA)
29 de septiembre a 4 de octubre, 1986

Con la colaboración de:

- C.A.I.C.Y.T.
- Servicio de Ciencia y Tecnología de la Embajada de Francia.
- Dirección General de Universidades e Investigación de la Junta de Andalucía.
- Asociación Hispano-Francesa de Cooperación Técnica y Científica.
- Universidad de Málaga / Vicerrectorado de Investigación.
- Centro de Cálculo de la Universidad de Sevilla.
- Compañía Sevillana de Electricidad.
- I.B.M. - España.
- Caja de Ahorros Provincial de Málaga.
- Caja de Ahorros de Ronda.
- Caja de Ahorros de Antequera.
- Ayuntamiento de Benalmadena (Málaga).

Organizada por:

- Grupos españoles de Investigación en "Métodos numéricos aplicados a la Ciencia y a la Técnica".
- Institut de Recherche en Informatique et Automatique (I.N.R.I.A.), FRANCIA

Figura 5: Cartel de la EHF1986

En lo que se refiere a la ubicación de la Escuela, A. Valle reservó el seductor Castillo de Bil-Bil que se alza en la playa como un castillo de arena. Fue su deseo además asociar a sus antiguos colegas de Sevilla en la celebración de esta Escuela: así, el conjunto de los participantes se desplazó a Sevilla para asistir a una presentación de Modulef seguida de un almuerzo y una excursión al centro de la ciudad, incluida la ascensión de la Giralda.

5.2.1 Programa

El programa fue muy rico, ya que se establecieron sesiones en paralelo y la Escuela se prolongó hasta el sábado por la mañana. En total, casi 35 horas de intervenciones científicas, un record absoluto aún imbatido:

- *Las herramientas de base de los métodos numéricos*, C. Conde (Escuela de Minas de Madrid) y J. Durany (Univ. de Santiago de Compostela)
- *Elasto-visco-plasticidad*, P. Le Tallec (LCPC París) y J.M. Viaño (Univ. de Santiago de Compostela)
- *Turbulencia*, T. Chacón (Univ. de Sevilla)
- *Fractura*, P. Destuynder (Escuela Central de París) y F. Michavila (Escuela de Minas de Madrid)
- *Modulef*, E. Fernández Cara y R. Echevarría (Univ. de Sevilla)
- *Optimización*, J. Céa (Univ. de Niza)
- *Mecánica de Fluidos*, J. Périaux (Avions Marcel Dassault Bréguet Aviation, París) y O. Pironneau (Univ. París VI)
- *Láminas*, M. Bernadou (INRIA).

5.2.2 Participantes

La Escuela contó con 80 participantes, incluyendo los conferenciantes, procedentes en su mayoría de universidades y centros de investigación españoles con un excelente reparto geográfico y un buen equilibrio entre matemáticos, físicos/químicos e ingenieros. Los conferenciantes franceses se expresaron en francés, salvo J. Céa, cuyo bilingüismo francés/español fue muy apreciado, y yo mismo, que intenté hablar en español: ¡el sonoro *"somos partidos"* con el que arrancó mi curso no dejó ninguna duda sobre mi dominio del español!

5.3 III Escuela: Madrid, del 26-9 al 30-9 de 1988

Esta Escuela fue organizada por F. Michavila que era entonces Director de la Escuela de Minas de la Universidad Politécnica de Madrid, en colaboración con los profesores C. Conde, M. López y A. Ruiz. Tuvo lugar en el anfiteatro de la Fundación Gómez-Pardo situado en las proximidades inmediatas de la Escuela

de Minas y, por consiguiente, muy cerca del centro histórico de Madrid. Esta proximidad fue evidentemente aprovechada durante la media jornada libre para visitar dicho centro histórico (Plaza Mayor, Palacio Real) y sobre todo el Prado, que figura entre los principales Museos del Mundo, con los pintores del Siglo de Oro (s. XVII): el Greco, Velázquez, Zurbarán, Murillo así como Ribera y el más importante pintor español del siglo XVIII, Goya.

5.3.1 Programa

Esta Escuela permitió analizar la situación de algunos de los temas objeto de colaboración hispano-francesa:

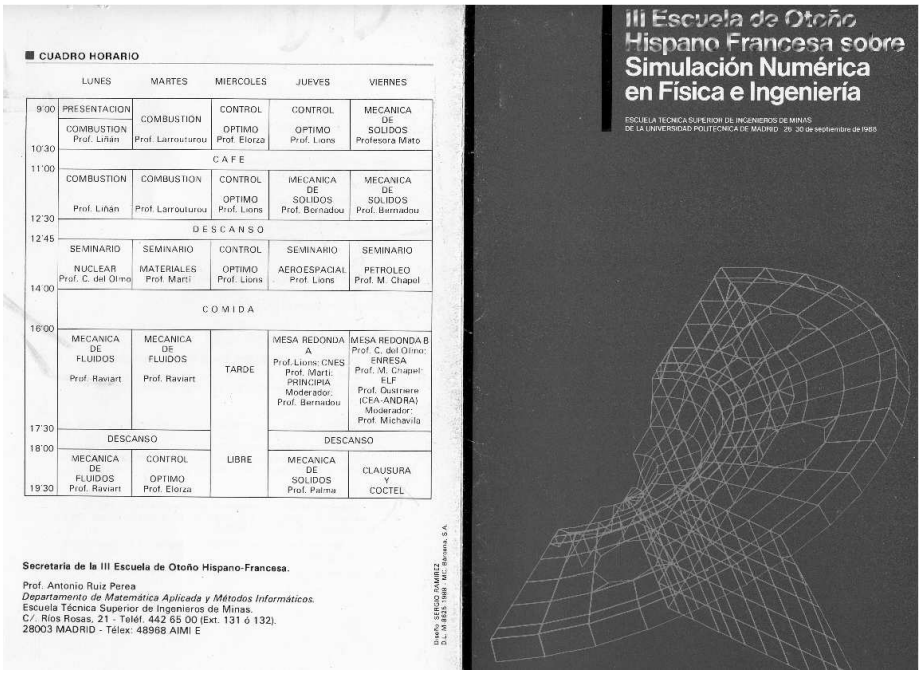
- *Nociones generales de control*, F.J. Elorza (Escuela de Minas de Madrid)
- *Uniones de placas, confluencias de láminas*, M. Bernadou (INRIA)
- *Optimización de forma de láminas*, F. Palma (Univ. de Málaga)
- *Aproximación de láminas mediante métodos DKT*, P. Mato Eiroa (Univ. Santiago de Compostela)

así como de algunos temas más recientes que debían constituir vías de investigación prometedoras para el futuro:

- *Controlabilidad exacta*, J.L. Lions (Collège de France y CNES)
- *Problemas hiperbólicos no lineales*, P.A. Raviart (Escuela Politécnica de París)
- *Combustión*, B. Larrouturou (INRIA) y A. Liñán (ETSI Aeronáuticos de Madrid).

Mas allá de los cursos, ya clásicos en la forma, los organizadores desearon acentuar la presentación de métodos de simulación numérica para el diseño, la puesta a punto y la optimización de nuevos productos industriales. Algunas colaboraciones entre analistas numéricos e industriales comenzaban a establecerse en España y era importante llamar la atención de los nuevos titulados y de los ingenieros de los gabinetes de estudio españoles sobre estas nuevas posibilidades que experimentaban ya un auge considerable en los países anglosajones, en Alemania y en Francia. Para cubrir este objetivo, complementando los cursos tradicionales, se organizaron cursos en el marco de esta *apertura industrial*:

- Seminarios *"industriales"*: Estos seminarios, presentados por los responsables de cuatro polos industriales diferentes, permitieron definir e ilustrar algunos problemas resueltos (o susceptibles de serlo) gracias a las nuevas contribuciones de la simulación numérica:
  - *Seminario Nuclear*, C. del Olmo (ENRESA)
  - *Seminario Aerospacial*, J.L. Lions (CNES)



■ CUADRO HORARIO

	LUNES	MARTES	MIERCOLES	JUEVES	VIERNES
9'00	PRESENTACION	COMBUSTION	CONTROL	CONTROL	MECANICA DE SOLIDOS
10'30	COMBUSTION Prof. Liñán	Prof. Larrouturou	OPTIMO Prof. Elorza	OPTIMO Prof. Lions	Profesora Mato
11'00	CAFE				
12'30	COMBUSTION Prof. Liñán	COMBUSTION Prof. Larrouturou	CONTROL OPTIMO Prof. Lions	MECANICA DE SOLIDOS Prof. Bernadou	MECANICA DE SOLIDOS Prof. Bernadou
12'45	DESCANSO				
14'00	SEMINARIO NUCLEAR Prof. C. del Olmo	SEMINARIO MATERIALES Prof. Martí	CONTROL OPTIMO Prof. Lions	SEMINARIO AEROESPACIAL Prof. Lions	SEMINARIO PETROLEO Prof. M. Chapel
16'00	COMIDA				
17'30	MECANICA DE FLUIDOS Prof. Raviart	MECANICA DE FLUIDOS Prof. Raviart	TARDE	MESA REDONDA A Prof. Lions: CNES Prof. Martí: PRINCIPIA Moderador: Prof. Bernadou	MESA REDONDA B Prof. C. del Olmo: ENRESA Prof. M. Chapel: ELF Prof. Dustniere (CEA-ANDRA) Moderador: Prof. Michavila
18'00	DESCANSO			DESCANSO	
19'30	MECANICA DE FLUIDOS Prof. Raviart	CONTROL OPTIMO Prof. Elorza	LIBRE	MECANICA DE SOLIDOS Prof. Palma	CLAUSURA Y COCTEL

Secretaría de la III Escuela de Otoño Hispano-Francesa.

Prof. Antonio Ruiz Perea
Departamento de Matemática Aplicada y Métodos Informáticos.
Escuela Técnica Superior de Ingenieros de Minas.
C/. Ríos Rosas, 21 - Teléf. 442 65 00 (Ext. 131 ó 132).
28003 MADRID - Télex: 48968 AIMI E

Diseño: SERGIO RAMIREZ
D.L.: M.4425-1988 - Mec. Barcelona, S.A.

III Escuela de Otoño Hispano Francesa sobre Simulación Numérica en Física e Ingeniería

ESCUELA TECNICA SUPERIOR DE INGENIEROS DE MINAS
DE LA UNIVERSIDAD POLITECNICA DE MADRID 26-30 de septiembre de 1988

■ **ORGANIZADA POR**

- *Grupos españoles de investigación en métodos numéricos de:*
  - E.T.S.I. Minas de la Universidad Politécnica de Madrid.
  - -- Facultad de Ciencias de la Universidad de Málaga.
  - — Facultad de Matemáticas de la Universidad de Santiago.
  - — Facultad de Matemáticas de la Universidad de Sevilla.
- *Institut National de Recherche en Informatique et Automatique (INRIA) FRANCIA.*

■ **PRESIDIDA POR**

J.L. Lions.
Presidente du Centre National d'Etudes Spatiales (C.N.E.S.)

R. Portaencasa.
Rector de la Universidad Politécnica de Madrid (U.P.M.)

■ **RESPONSABLES**

M. Bernadou *(INRIA)*

F. Michavila *(E.T.S.I.M.M.)*

■ **COMITE CIENTIFICO**

A. Bermudez
Universidad de Santiago

R. Glowinski
Universidad de Houston

C. Conde
E.T.S.I. de Minas, U.P.M.

B. Larrouturou
INRIA

E. Fernández Cara
Universidad de Sevilla

O. Pironneau
INRIA / Univ. Paris VI

A. Valle
Universidad de Málaga

P.A. Raviart
Ecole Polytechnique

■ **COMITE ORGANIZADOR**

F. Michavila, *E.T.S.I. Minas - Universidad Politécnica de Madrid*
M. Bernadou, *INRIA.*
C. Conde, *E.T.S.I. Minas - Universidad Politécnica de Madrid*
M. López Quero, *Universidad Politécnica de Madrid*
A. Ruiz, *E.T.S.I. Minas - Universidad Politécnica de Madrid*

■ **OBJETIVOS**

a) Iniciar a las personas con interés por la Matemática Aplicada y en concreto por la Simulación Numérica, en líneas de investigación que se desarrollan en España y Francia.

b) Servir de punto de encuentro entre Investigadores, Profesores, especialistas de la industria y jóvenes titulados en ambos países.

c) Mostrar usos actuales de la Simulación Numérica en la Industria y en la Empresa española y francesa.

d) Analizar el futuro que la Simulación Numérica tendrá en ciertos sectores de la Industria de ambos países.

■ **CON LA COLABORACION DE**

Universidad Politécnica de Madrid.
Embajada de Francia en España.
Asociación Hispano Francesa de Cooperación Tecnica y Científica.
I.N.R.I.A.
Colegio de Ingenieros de Minas.
Instituto Geológico y Minero de España (IGME).
Centre national d'Etudes Spatiales (CNES).
Empresa Nacional de Residuos Radiactivos (ENRESA).
ELF.
Principia España.
Fundación Benéfico-docente GOMEZ PARDO.
REPSOL.
Construcciones Aeronáuticas (CASA).

■ **INFORMACION GENERAL**

Cuota de inscripción: 10.000 ptas (hasta el 15 de septiembre).

Esta cuota da derecho a la asistencia, documentación y comida de mediodía, durante las jornadas que dure esta Escuela.

Becas:

Se ha previsto otorgar una serie de becas, que cubran los gastos de alojamiento y transporte para aquellas personas que residan fuera de Madrid. Los interesados deberán dirigir su petición a la Secretaría de la III Escuela de Otoño Hispano-Francesa, antes del 15 de junio. El Comité Organizador resolverá según las peticiones habidas antes del 15 de julio.

Forma de pago:

— Transferencia bancaria a nombre de Fundación General U.P.M. E.T.S. Ingenieros de Minas a la c/c. 0-002346 del Banco de Bilbao. Avda. Reina Victoria, 66. 28003 MADRID.

– Talón Nominativo adjunto a nombre de Fundación General U.P.M. -E.T.S.I. de Minas.

Las inscripciones se aceptarán por riguroso orden de llegada.

Figura 6: Cartel de la EHF1988

  - *Seminario Materiales*, J. Martí (PRINCIPIA)
  - *Seminario Petróleo*, P. Samier (ELF-SNEA(P))

- Mesas redondas: Dos mesas redondas fueron destinadas a aportar otros enfoques sobre estos temas industriales de actualidad y a abrir la discusión con los participantes:

  - *Mesa redonda sobre Nuevos Materiales en Aerospacial*, con la participación de G. Duvaut (ONERA), J.L. Lions (CNES), J. Martí (PRINCIPIA), M. Simon (CASA), M. Bernadou (Moderador) ;
  - *Mesa redonda sobre Energía*, con la participación de P. Oustrière (CEA-ANDRA), L. Pérez (REPSOL Exploración), P. Samier (ELF), A. Uliberri (ENRESA), F. Michavila (Moderador).

Así, esta Escuela puso especialmente el acento en las repercusiones industriales efectivas de la Simulación Numérica. Introdujo también la novedad de Seminarios/Conferencias de una hora acompañando a los cursos, a fin de ilustrar algunas aplicaciones importantes.

5.3.2 Participantes

La presencia de J.L. Lions, las aperturas, la posición central de Madrid y la excelente organización explican sin ninguna duda el número de participantes: 145 incluyendo profesores y conferenciantes. Más precisamente, los centros de origen de los participantes fueron:

- *España* : Universidades de Cantabria (2); Castilla-La Mancha (4, de los cuales 1 de Albacete y 3 de Almadén); Córdoba (3); Granada (2); Las Palmas (1); Málaga (6); Murcia (2); Oviedo (8); País Vasco (6); Politécnica de Cataluña (1); Politécnica de Valencia (2); Salamanca (1); Santiago de Compostela (16, de los cuales 1 del Ferrol, 2 de La Coruña y 2 de Vigo); Sevilla (8); Valencia (6); Valladolid (2); Zaragoza (3); Instituto de Mijas (Málaga) (1); INCAPSA (1); Murcia (1), y finalmente de Madrid, procedentes de Univ. Politécnica (38) ; UNED (5) ; CASA (3) ; CIEMAT (2) ; ENRESA (5) ; ICAI (1) ; EGME (1) ; INYPSA (1) ; REPSOL (4) ; SITECSA (2) ; 1 otros.

- *Francia*: CEA-Andra (1) ; CNES París (1) ; ELF Pau (1) ; INRIA Rocquencourt (1) y Niza (1) ; Escuela Politécnica, Palaiseau (2) ; ONERA, Châtillon (1)

5.4 IV Escuela: Santiago de Compostela, del 24-9 al 28-09 de 1990

Después de las tres primeras Escuelas organizadas por los tres principales polos españoles de colaboración con Francia, esto es, las Universidades de Santiago de Compostela, de Málaga y Sevilla, y la Escuela de Minas de la Universidad Politécnica de Madrid, la experiencia fue juzgada como muy satisfactoria y se decidió efectuar un nuevo ciclo de tres Escuelas que serían nuevamente

organizadas por cada uno de los tres polos sucesivamente. Esta cuarta Escuela fue así organizada en Santiago de Compostela por A. Bermúdez de Castro, P. Quintela y J.M. Viaño. Se retomó la fórmula mixta de cursos acompañados de conferencias, de seminarios de aplicaciones y de una mesa redonda y fueron casi una centena los participantes que asistieron a esta nueva edición.

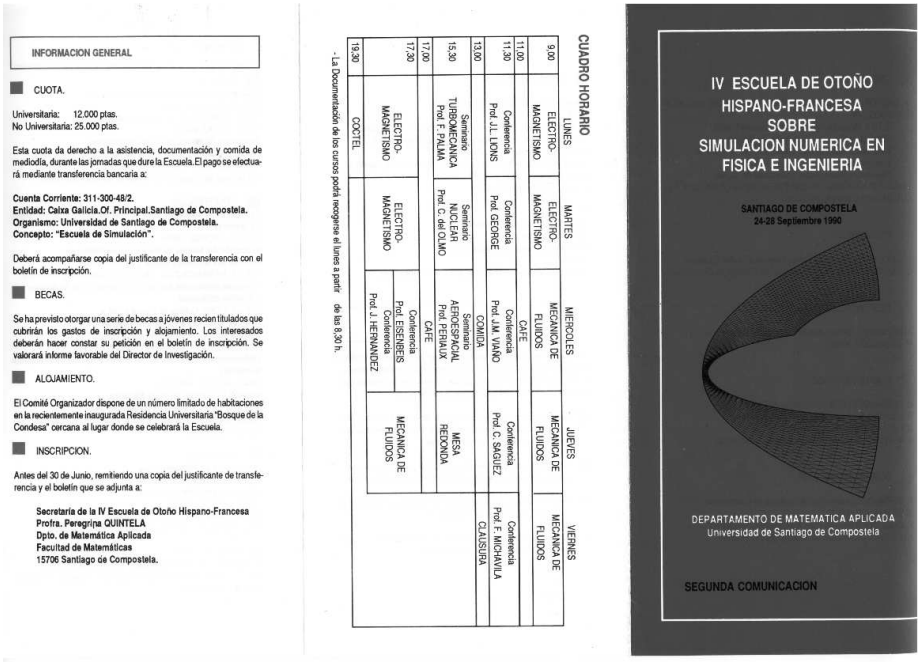
5.4.1 Programa

En los cursos se abordaron dos temas en plena expansión:

- *Métodos numéricos en electromagnetismo*, A. Bermúdez de Castro (Santiago de Compostela) y A. Bossavit (Electricité de France)
- *Algunos métodos numéricos para los diferentes problemas de la mecánica de fluido*, J.A. Désidéri (Univ. de Niza), L. Fezoui (INRIA) y E. Fernández Cara (Univ. de Sevilla).

Junto a las 16 horas de curso, nueve seminarios, de hora y media cada uno, permitieron presentar de forma sintetizada o bien algunos resultados recientes en otros ejes de investigación, o bien algunas aplicaciones de estos métodos numéricos en el medio industrial:

- Conferencias: Las conferencias permitieron analizar la situación de
  - *Las metodologías utilizadas para realizar mallados*, P.L. George (INRIA)
  - *Los modelos asintóticos de barras elásticas*, J.M. Viaño (Univ. Santiago de Compostela)
  - *Los métodos numéricos para el diseño de dispositivos optoeléctricos*, F.J. Mustieles (Telefónica I+D)
  - *Algunos avances recientes en la modelización de singularidades mediante el método de elementos finitos*, F. Michavila (Escuela de Minas de Madrid)
  - *La evolución de superordenadores para la simulación numérica*, Ch. Eisenbeis (INRIA)
- Seminarios industriales: El objetivo de estos seminarios era presentar algunos ejemplos de uso en el medio industrial de resultados recientes de investigación en métodos numéricos.
  - *Simulación numérica de problemas de láminas delgadas y algunas aplicaciones a la optimización de forma*, F. Palma (Univ. de Málaga)
  - *Problemáticas numéricas en la industria nuclear*, C. del Olmo (ENRESA)
  - *Simulación de flujos en diseño aerospacial*, B. Stoufflet (Avions Marcel Dassault Bréguet Aviation)



INFORMACION GENERAL

CUOTA.

Universitaria: 12.000 ptas.
No Universitaria: 25.000 ptas.

Esta cuota da derecho a la asistencia, documentación y comida de mediodía, durante las jornadas que dure la Escuela. El pago se efectuará mediante transferencia bancaria a:

Cuenta Corriente: 311-300-48/2.
Entidad: Caixa Galicia.Of. Principal.Santiago de Compostela.
Organismo: Universidad de Santiago de Compostela.
Concepto: "Escuela de Simulación".

Deberá acompañarse copia del justificante de la transferencia con el boletín de inscripción.

BECAS.

Se ha previsto otorgar una serie de becas a jóvenes recien titulados que cubrirán los gastos de inscripción y alojamiento. Los interesados deberán hacer constar su petición en el boletín de inscripción. Se valorará informe favorable del Director de Investigación.

ALOJAMIENTO.

El Comité Organizador dispone de un número limitado de habitaciones en la recientemente inaugurada Residencia Universitaria "Bosque de la Condesa" cercana al lugar donde se celebrará la Escuela.

INSCRIPCION.

Antes del 30 de Junio, remitiendo una copia del justificante de transferencia y el boletín que se adjunta a:

Secretaría de la IV Escuela de Otoño Hispano-Francesa
Profra. Peregrina QUINTELA
Dpto. de Matemática Aplicada
Facultad de Matemáticas
15706 Santiago de Compostela.

CUADRO HORARIO

	LUNES	MARTES	MIERCOLES	JUEVES	VIERNES
9.00	ELECTRO-MAGNETISMO	ELECTRO-MAGNETISMO	MECANICA DE FLUIDOS	MECANICA DE FLUIDOS	MECANICA DE FLUIDOS
11.00	Conferencia Prof. J.L. LIONS	Conferencia Prof. GEORGE	CAFE Conferencia Prof. J.M. VIAÑO	Conferencia Prof. C. SAGUEZ	Conferencia Prof. F. MICHAVILA
13.00	COMIDA	COMIDA	COMIDA	COMIDA	CLAUSURA
15.30	Seminario TURBOMECANICA Prof. F. PALMA	Seminario NUCLEAR Prof. C. del OLMO	Seminario AEROESPACIAL Prof. PERIAUX	MESA REDONDA	
17.00	CAFE	CAFE			
17.30	ELECTRO-MAGNETISMO	ELECTRO-MAGNETISMO Conferencia Prof. ESDIBERS Conferencia Prof. J. HERNANDEZ		MECANICA DE FLUIDOS	
19.30	COCTEL				

La Documentación de los cursos podrá recogerse el lunes a partir de las 6.30 h.

IV ESCUELA DE OTOÑO
HISPANO-FRANCESA
SOBRE
SIMULACION NUMERICA EN
FISICA E INGENIERIA

SANTIAGO DE COMPOSTELA
24-28 Septiembre 1990

DEPARTAMENTO DE MATEMATICA APLICADA
Universidad de Santiago de Compostela

SEGUNDA COMUNICACION

ORGANIZACION

• GRUPOS ESPAÑOLES DE INVESTIGACION EN METODOS NUMERICOS DE:
E.T.S.I. Minas. Universidad Politécnica de Madrid.
Facultad de Ciencias. Universidad de Málaga.
Facultad de Matemáticas. Universidad de Santiago de Compostela.
Facultad de Matemáticas. Universidad de Sevilla.
• INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE (INRIA). FRANCIA.

PRESIDENCIA

J. L. LIONS (Presidente del Centre National d'Etudes Spatiales)
C. PAJARES (Rector de la Universidad de Santiago de Compostela)

RESPONSABLES

A. BERMUDEZ (Universidad de Santiago de Compostela)
M. BERNADOU (INRIA)

COMITE CIENTIFICO

E. FERNANDEZ CARA (Universidad de Sevilla)
F. MICHAVILA (Universidad Politécnica de Madrid)
A.VALLE (Universidad de Málaga)
J. M. VIAÑO (Universidad de Santiago de Compostela)

COMITE ORGANIZADOR

A. BERMUDEZ (Universidad de Santiago de Compostela)
M. BERNADOU (INRIA)
P. QUINTELA (Universidad de Santiago de Compostela)
J.M. VIAÑO (Universidad de Santiago de Compostela)

PATROCINADORES

Universidad de Santiago de Compostela.
INRIA.
Dir. Xeral Ensinanzas Univ. e Política Científica. Xunta de Galicia.
Dirección General de Investigación Científica y Técnica. M.E.C.
Embajada de Francia en España.
Asociación Hispano-Fancesa de Cooperación Técnica y Científica.
Empresa Nacional de Electricidad.
Convex Supercomputer, S.A.E.
Data General.
IBM.
Centre National d'Etudes Spatiales.
Electricitié de France.
Avions Marcel Dassault. Breguet Aviation.
Simulog.
(La lista definitiva se incluirá en la próxima comunicación).

OBJETIVOS

Continuación de otras celebradas en Santiago de Compostela (1984), Málaga-Sevilla (1986) y Madrid (1988), esta Escuela pretende presentar métodos y técnicas en la simulación numérica de fenómenos que tienen interés en la Física y la Ingeniería con objeto de:

- Iniciar a las personas con interés por la Matemática Aplicada y en concreto por la Simulación Numérica, en líneas de investigación que se desarrollan en España y Francia.
- Servir de punto de encuentro entre investigadores, profesores, especialistas de la industria y jóvenes titulados de ambos países.
- Mostrar usos actuales de la Simulación Numérica en la Industria y en la Empresa española y francesa.
- Analizar el futuro que la Simulación Numérica tendrá en ciertos sectores de la industria de ambos países.

ACTIVIDADES

CURSOS.
Simulación Numérica en Mecánica de Fluidos.
Prof. J.A. DESIDERI. INRIA
Prof. E. FERNANDEZ CARA. Universidad de Sevilla.
Prof. L. FEZOUI. INRIA
Simulación Numérica en Electromagnetismo.
Prof. A. BERMUDEZ. Universidad de Santiago de Compostela.
Prof. A. BOSSAVIT. Electricité de France.
(Cada curso será impartido en cuatro sesiones de dos horas).

CONFERENCIAS.
Generation de Maillages pour la Mise en Ouevre de Méthodes d'Eléments Finis.
Prof. P.L. GEORGE. INRIA.
Aplicación del Método de los Elementos Finitos en Dispositivos Autoelectrónicos y de Microondas.
Prof. J.F. HERNANDEZ GIL. Telefónica Investigación y Desarrollo.
Quelques Remarques sur les Raccordements de Problèmes d'échelles différentes.
Prof. J.L. LIONS. Collège de France y CNES.
Avances Recientes en la Modelización de Singularidades mediante Elementos Finitos.
Prof. F. MICHAVILA. Universidad Politécnica de Madrid.
Environnement Utilisateur et Fiabilité de Logiciels de Simulation Numérique.
Prof. C. SAGUEZ. SIMULOG.
Modelos Asintóticos en Vigas Elásticas.
Prof. J.M. VIAÑO. Universidad de Santiago de Compostela.
Impact des Supercalculateurs sur la Simulation Numérique.
Prof. C. EISENBEIS.

SEMINARIOS.
Aplicaciones en Turbomecánica.
Prof. F. PALMA. Universidad de Málaga.
Aplicaciones en Energía Nuclear.
Prof. C. del OLMO. ENRESA.
Aplicaciones en la Industria Aeroespacial.
Prof. J. PERIAUX. Avions Marcel Dassault-Breguet Aviation.

MESA REDONDA.
Se organizará una mesa redonda sobre el presente y el futuro de la Simulación Numérica en la Industria, con presencia de especialistas de ambos países.

Figura 7: Cartel de la EHF1990

– *Entorno de utilizador y fiabilidad de software de simulación numérica*, C. Saguez (SIMULOG).

Dos volúmenes que reunían la casi totalidad de los textos de las intervenciones fueron entregados a los participantes en el momento de la apertura de la Escuela. En estos volúmenes figura el soporte escrito de la intervención que J.L. Lions, que se encontraba en Moscú por cuestiones relacionadas con asuntos espaciales, no pudo efectuar: *Algunos comentarios sobre el ensamblado de problemas de escalas diferentes.*

5.4.2 Participantes

A pesar de la situación geográfica excéntrica de Santiago de Compostela, esta Escuela reunió 81 participantes procedente en casi su totalidad del sector universitario: Universidades de Córdoba (3), Las Palmas (2), Málaga (3), Murcia (1), Oviedo (3), País Vasco (1), Politécnica de Madrid (13), Politécnica de Valencia (5), Santiago de Compostela (25, de los cuales 1 de La Coruña y 4 de Vigo), Salamanca (2), Sevilla (12), Valladolid (7). La audiencia estaba compuesta esencialmente por jóvenes investigadores así como por algunos profesores en ejercicio. A estos 81 participantes hay que añadir los 16 conferenciantes, lo que da un total de 97 asistentes.

5.4.3 Perspectivas

Esta colaboración franco-española en Análisis Numérico y Cálculo Científico es ya muy importante. Entre los desarrollos recientes, conviene destacar:

- la estancia de Matthieu Hallard en Santiago de Compostela: modelización de la combustión de carbón (abril a junio de 1990)
- la obtención de un programa SCIENCE de la Comunidad Europea que asociaba al INRIA con la Universidad de Santiago de Compostela y la Universidad Técnica de Lisboa, en el tema de *Modelización de uniones de estructuras*
- la obtención de un contrato ERASMUS entre las Universidades París VI (con la implicación del INRIA), de Sevilla, Málaga, Santiago de Compostela y la Universidad Politécnica de Madrid
- la ida a París VI de tres estudiantes españoles (1 de Santiago de Compostela, 1 de Sevilla y 1 de Málaga) para la preparación del DEA de Análisis Numérico. Tras la obtención de este DEA, dichos estudiantes continuarían su estancia en Francia a fin de preparar sus tesis de doctorado en los Laboratorios asociados a la citada formación doctoral (entre ellos el INRIA)

Estos cuatro ejemplos no sólo ponen de manifiesto la vitalidad de la colaboración, sino que muestran la evolución de la misma hacia un régimen de intercambios a partes iguales.

5.5 V Escuela: Castellón de la Plana, del 28-9 al 2-10 de 1992

Es la Universidad Jaume I de Castellón de la Plana, en pleno desarrollo bajo la acción enérgica de su nuevo Rector, F. Michavila ayudado por C. Conde, quien nos acoge. De hecho, la Escuela tuvo lugar en el balneario próximo a Benicassim *Villa Elisa*, perteneciente a la ciudad de Castellón y situado frente al mar, lo que hace recordar al seductor Castillo de Bil-Bil que albergó la Escuela de1986 organizada por la Universidad de Málaga. Si a esto se añade que todos los participantes fueron alojados en el Hotel Voramar, muy próximo y también situado frente al mar, se comprende que estos emplazamientos dieron un carácter distendido a esta Escuela que fue muy apreciado por todos. Sin olvidar una excursión memorable por la costa, seguida de una excelente comida y, para terminar, la degustación de una *Queimada* del Maestro J.M. Viaño en la playa.

5.5.1 Programa

Se eligieron tres temas de cursos y cada uno de ellos fue abordado por tres profesores con enfoques muy complementarios. Estos temas fueron:

- *Modelización de estructuras delgadas (placas, láminas), de sus uniones y de sus aproximaciones por elementos finitos*, P.G. Ciarlet (Univ. París VI), F. Palma (Univ. de Málaga), P. Quintela (Univ. Santiago de Compostela)

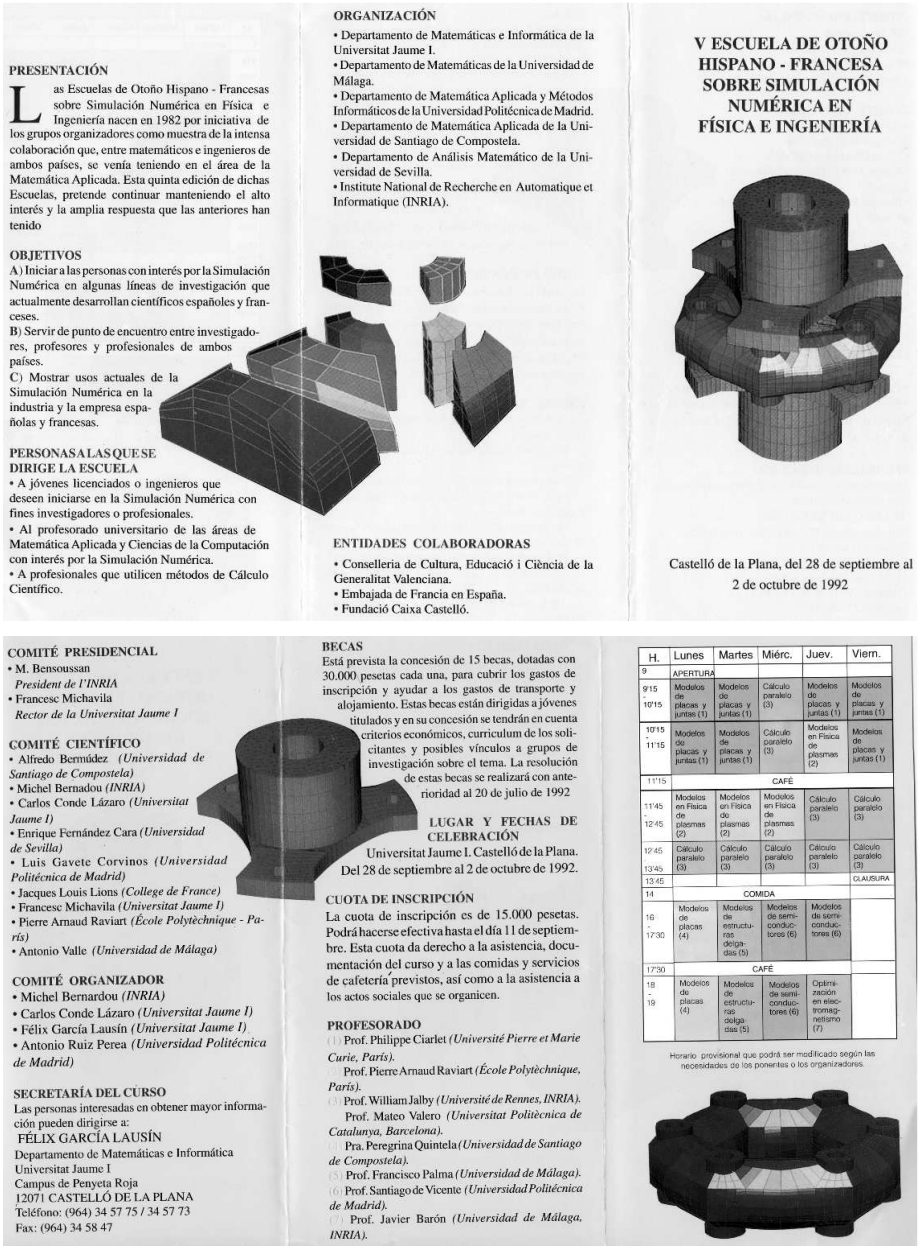
- *Modelización de problemas de la física: plasmas*, P.A. Raviart (Escuela Politécnica), *semiconductores*, S. de Vicente (Univ. Politécnica de Madrid), *y optimización en electromagnetismo*, J. Barón (Univ. de Málaga)

- *Calculo paralelo*, W. Jalby (Univ. de Rennes), M. Valero (Univ. Politécnica de Cataluña), M. Vidrascu (INRIA)

Cuatro volúmenes reunieron los apuntes de los cursos. Algunas notas complementarias fueron fotocopiadas y distribuidas sobre la marcha.

5.5.2 Participantes

La Escuela reunió a 86 participantes incluyendo a los profesores y conferenciantes, cuya mayor parte procedía del sector universitario. Estos últimos procedían de las Universidades de Barcelona (1), Complutense de Madrid (1), Córdoba (3), La Coruña (1), Granada (1), Jaume I (18), Málaga (6), Murcia (1), Oviedo (6), País Vasco (3), Politécnica de Madrid (10), Santiago de Compostela (10), Sevilla (6), Valencia (7) y Zaragoza (1), lo que pone de manifiesto la diversidad y la gran representatividad geográfica de la audiencia. El nivel de los participantes fue muy variado, yendo desde jóvenes estudiantes a profesores e investigadores curtidos.



PRESENTACIÓN

Las Escuelas de Otoño Hispano - Francesas sobre Simulación Numérica en Física e Ingeniería nacen en 1982 por iniciativa de los grupos organizadores como muestra de la intensa colaboración que, entre matemáticos e ingenieros de ambos países, se venía teniendo en el área de la Matemática Aplicada. Esta quinta edición de dichas Escuelas, pretende continuar manteniendo el alto interés y la amplia respuesta que las anteriores han tenido

OBJETIVOS

A) Iniciar a las personas con interés por la Simulación Numérica en algunas líneas de investigación que actualmente desarrollan científicos españoles y franceses.

B) Servir de punto de encuentro entre investigadores, profesores y profesionales de ambos países.

C) Mostrar usos actuales de la Simulación Numérica en la industria y la empresa españolas y francesas.

PERSONAS A LAS QUE SE DIRIGE LA ESCUELA

- A jóvenes licenciados o ingenieros que deseen iniciarse en la Simulación Numérica con fines investigadores o profesionales.
- Al profesorado universitario de las áreas de Matemática Aplicada y Ciencias de la Computación con interés por la Simulación Numérica.
- A profesionales que utilicen métodos de Cálculo Científico.

ORGANIZACIÓN

- Departamento de Matemáticas e Informática de la Universitat Jaume I.
- Departamento de Matemáticas de la Universidad de Málaga.
- Departamento de Matemática Aplicada y Métodos Informáticos de la Universidad Politécnica de Madrid.
- Departamento de Matemática Aplicada de la Universidad de Santiago de Compostela.
- Departamento de Análisis Matemático de la Universidad de Sevilla.
- Institute National de Recherche en Automatique et Informatique (INRIA).

ENTIDADES COLABORADORAS

- Conselleria de Cultura, Educació i Ciència de la Generalitat Valenciana.
- Embajada de Francia en España.
- Fundació Caixa Castelló.

V ESCUELA DE OTOÑO HISPANO - FRANCESA SOBRE SIMULACIÓN NUMÉRICA EN FÍSICA E INGENIERÍA

Castelló de la Plana, del 28 de septiembre al 2 de octubre de 1992

COMITÉ PRESIDENCIAL

- M. Bensoussan
  President de l'INRIA
- Francesc Michavila
  Rector de la Universitat Jaume I

COMITÉ CIENTÍFICO

- Alfredo Bermúdez *(Universidad de Santiago de Compostela)*
- Michel Bernadou *(INRIA)*
- Carlos Conde Lázaro *(Universitat Jaume I)*
- Enrique Fernández Cara *(Universidad de Sevilla)*
- Luis Gavete Corvinos *(Universidad Politécnica de Madrid)*
- Jacques Louis Lions *(College de France)*
- Francesc Michavila *(Universitat Jaume I)*
- Pierre Arnaud Raviart *(École Polytèchnique - París)*
- Antonio Valle *(Universidad de Málaga)*

COMITÉ ORGANIZADOR

- Michel Bernardou *(INRIA)*
- Carlos Conde Lázaro *(Universitat Jaume I)*
- Félix García Lausín *(Universitat Jaume I)*
- Antonio Ruiz Perea *(Universidad Politécnica de Madrid)*

SECRETARÍA DEL CURSO

Las personas interesadas en obtener mayor información pueden dirigirse a:

FÉLIX GARCÍA LAUSÍN
Departamento de Matemáticas e Informática
Universitat Jaume I
Campus de Penyeta Roja
12071 CASTELLÓ DE LA PLANA
Teléfono: (964) 34 57 75 / 34 57 73
Fax: (964) 34 58 47

BECAS

Está prevista la concesión de 15 becas, dotadas con 30.000 pesetas cada una, para cubrir los gastos de inscripción y ayudar a los gastos de transporte y alojamiento. Estas becas están dirigidas a jóvenes titulados y en su concesión se tendrán en cuenta criterios económicos, curriculum de los solicitantes y posibles vínculos a grupos de investigación sobre el tema. La resolución de estas becas se realizará con anterioridad al 20 de julio de 1992

LUGAR Y FECHAS DE CELEBRACIÓN

Universitat Jaume I. Castelló de la Plana. Del 28 de septiembre al 2 de octubre de 1992.

CUOTA DE INSCRIPCIÓN

La cuota de inscripción es de 15.000 pesetas. Podrá hacerse efectiva hasta el día 11 de septiembre. Esta cuota da derecho a la asistencia, documentación del curso y a las comidas y servicios de cafetería previstos, así como a la asistencia a los actos sociales que se organicen.

PROFESORADO

(1) Prof. Philippe Ciarlet *(Université Pierre et Marie Curie, París).*
(2) Prof. Pierre Arnaud Raviart *(École Polytèchnique, París).*
(3) Prof. William Jalby *(Université de Rennes, INRIA).*
Prof. Mateo Valero *(Universitat Politècnica de Catalunya, Barcelona).*
(4) Pra. Peregrina Quintela *(Universidad de Santiago de Compostela).*
(5) Prof. Francisco Palma *(Universidad de Málaga).*
(6) Prof. Santiago de Vicente *(Universidad Politécnica de Madrid).*
(7) Prof. Javier Barón *(Universidad de Málaga, INRIA).*

H.	Lunes	Martes	Miérc.	Juev.	Viern.
9	APERTURA				
9'15 - 10'15	Modelos de placas y juntas (1)	Modelos de placas y juntas (1)	Cálculo paralelo (3)	Modelos de placas y juntas (1)	Modelos de placas y juntas (1)
10'15 - 11'15	Modelos de placas y juntas (1)	Modelos de placas y juntas (1)	Cálculo paralelo (3)	Modelos en Física de plasmas (2)	Modelos de placas y juntas (1)
11'15	CAFÉ				
11'45 - 12'45	Modelos en Física de plasmas (2)	Modelos en Física de plasmas (2)	Modelos en Física de plasmas (2)	Cálculo paralelo (3)	Cálculo paralelo (3)
12'45 - 13'45	Cálculo paralelo (3)	Cálculo paralelo (3)	Cálculo paralelo (3)	Cálculo paralelo (3)	Cálculo paralelo (3)
13'45					CLAUSURA
14	COMIDA				
16 - 17'30	Modelos de placas (4)	Modelos de estructuras delgadas (5)	Modelos de semiconductores (6)	Modelos de semiconductores (6)	
17'30	CAFÉ				
18 - 19	Modelos de placas (4)	Modelos de estructuras delgadas (5)	Modelos de semiconductores (6)	Optimización en electromagnetismo (7)	

Horario provisional que podrá ser modificado según las necesidades de los ponentes o los organizadores.

Figura 8: Cartel de la EHF1992

5.5.3 Otras colaboraciones en curso

Esta colaboración franco-española en análisis numérico y cálculo científico es cada vez más importante. Sus desarrollos recientes se orientan en dos direcciones:

- Acogida en París de un flujo regular de estudiantes españoles que vienen a prepara el DEA de Análisis Numérico y, posteriormente, la tesis (2 a 3 estudiantes por año).
- Colaboraciones bilaterales cada vez más equilibradas que se traducen en particular en:
  - visitas de corta duración (2 a 4 semanas) de españoles a Francia y de franceses a España. Estas visitas permiten finalizar trabajos efectuados en común y publicados como artículos conjuntos
  - obtención de un programa SCIENCE que reune a la Universidad París VI (Responsable y director científico: P.G. Ciarlet), INRIA (coordinador: M. Bernadou), Universidad de Santiago de Compostela (coordinador: J.M. Viaño) y la Universidad Técnica de Lisboa (coordinador: L. Trabucho). El tema de este programa es la *Modelización de uniones de estructuras* y su duración es de 3 años (1991 a 1993).

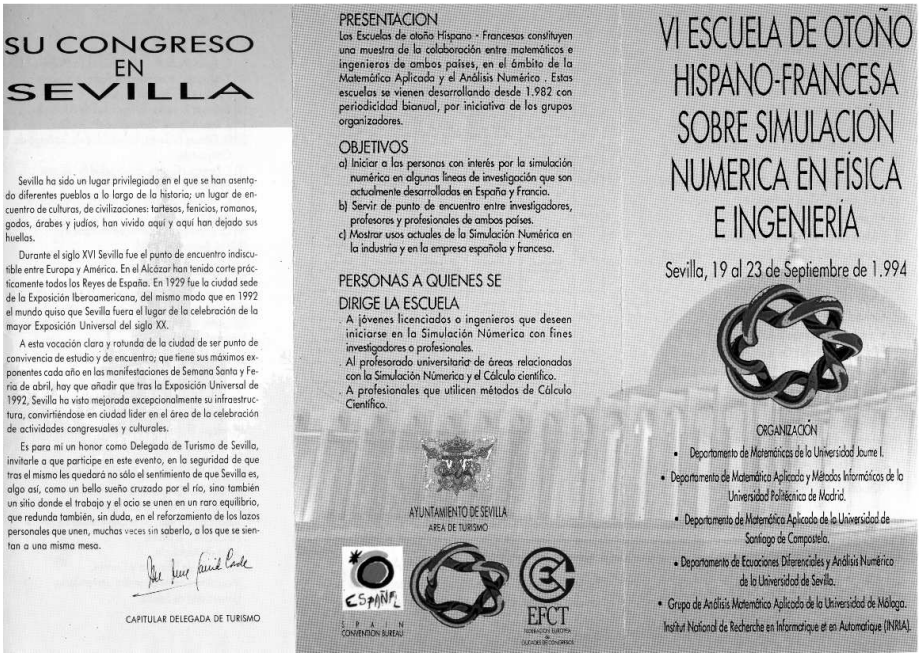
5.6 VI Escuela: Sevilla, del 19-9 al 23-9 de 1994

La Escuela vuelve a Andalucía, esta vez a Sevilla, donde tuvo lugar dos años antes la emblemática y muy conseguida Exposición Universal. Los organizadores son T. Chacón, E. Fernández Cara, D. Franco, F. Guillén y F. Ortegón. Junto a un programa científico de primer plano, consiguieron restituir el ambiente festivo de la Expo: visita al Ayuntamiento de Sevilla, conferencia general de J.L. Lions en la magnífica Sala de Fiestas del Ayuntamiento, visita de la Catedral y del Alcázar, declaradas por la UNESCO patrimonio mundial de la humanidad, seguida de una cena y de una muy calurosa velada de Sevillanas. Además, el alojamiento de los participantes en la residencia Hernando Colón, próxima a las salas de curso, dieron un carácter residencial a esta Escuela, que fue muy apreciado.

5.6.1 Programa

Se impartieron cuatro cursos de cinco horas:

- *Control óptimo de sistemas distribuidos*, E. Casas (Univ. de Cantabria)
- *Introducción al método de elementos finitos en mecánica de fluidos*, C. Parés (Univ. de Málaga)
- *Controlabilidad exacta y aproximada para sistemas con parámetros distribuidos*, J.L. Lions (Collège de France)



COMITÉ CIENTÍFICO

- Alfredo Bermúdez de Castro (Universidad de Santiago de Compostela).
- Michel Bernadou (INRIA)
- Tomás Chacón Rebollo (Universidad de Sevilla)
- Carlos Conde Lázaro (Universidad Politécnica de Madrid)
- Enrique Fernández Cara (Universidad de Sevilla)
- J. L. Lions (Collège de France)
- Francisco Michavila Pitarch (Universidad Jaume I de Castellón)
- Antonio Valle Sánchez (Universidad de Málaga)

COMITÉ ORGANIZADOR

- Michel Bernadou (INRIA)
- Tomás Chacón Rebollo (Universidad de Sevilla)
- Enrique Fernández Cara (Universidad de Sevilla)
- Daniel Franco Coronil (Universidad de Sevilla)
- Francisco Guillén González (Universidad de Sevilla)
- Francisco Ortegón Gallego (Universidad de Sevilla)

SECRETARÍA DEL CURSO

• FRANCISCO GUILLEN GONZALEZ
Departamento de Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla - Facultad de Matemáticas
c/ Tarfia, s/n - 41012 SEVILLA
Teléfono (95)455.79.99/455.79.89
Fax (95) 455.28.98
e-mail : guillen @ cica.es

HORARIO PROVISIONAL

(Sujeto a modificaciones si las necesidades de organización lo requieren)

	LUNES	MARTES	MIÉRCOLES	JUEVES	VIERNES
8h30'	APERTURA				
9h00'	MEF en Mecánica de Fluidos (1)	MEF en Mecánica de Fluidos (1)	MEF en Mecánica de Fluidos (1)	MEF en Mecánica de Fluidos (1)	MEF en Mecánica de Fluidos (1)
10h00'	Cálculo de Flujos por Métodos Espectrales (2)	Cálculo de Flujos por Métodos Espectrales (2)	Cálculo de Flujos por Métodos Espectrales (2)	Cálculo de Flujos por Métodos Espectrales (2)	Cálculo de Flujos por Métodos Espectrales (2)
11h00'	CAFÉ				
11h30'	Controlabilidad de Sistemas Distribuidos (3)	Controlabilidad de las Ecuaciones de Navier-Stokes (4)	Controlabilidad de Sistemas Distribuidos (3)	Controlabilidad de Sistemas Distribuidos (3)	Controlabilidad de Sistemas Distribuidos (3)
12h30'	Control Óptimo de Sistemas Distribuidos (5)	Control Óptimo de Sistemas Distribuidos (5)	Control Óptimo de Sistemas Distribuidos (5)	Control Óptimo de Sistemas Distribuidos (5)	Control Óptimo de Sistemas Distribuidos (5)
13h30'					CLAUSURA
14h00'	ALMUERZO				
16h00'	Guías de Ondas Electromagnéticas (6)	Matemáticas y Medio Ambiente (7)	Teoría Matemática de la Plasticidad (8)	MODULEF (10)	
17h00'	Adaptabilidad y Métodos Multimalla (9)	ACTOS SOCIALES	Simulación Numérica de la Turbulencia (9)	Balance	Asamblea SEMA

ACTOS SOCIALES

Está prevista una visita al Ayuntamiento de Sevilla y a los Reales Alcázares, seguida de la cena de la Escuela.

ASAMBLEA DEL SEMA

La Asamblea del SEMA tendrá lugar con ocasión de la celebración de la Escuela, pero no forma parte de la misma.

PROFESORADO

(1) C. Parés Madroñal, Universidad de Málaga.
(2) Y. Maday, Universidad de Paris VI
(3) J.L. Lions, Collège de France.
(4) E. Casas Renteria, Universidad de Santander.
(5) J. Real Anguas, Universidad de Sevilla.
(6) D. Gómez Pedreira, Universidad de Santiago de Compostela.
(7) L. Ferragut Canals, Universidad Politécnica de Madrid.
(8) C. Moreno González, Universidad Politécnica de Madrid
(9) O. Pironneau, Universidad de París VI
(10) M. Bernadou, INRIA

BECAS

Está prevista la concesión de cierto número de becas dirigidas a jóvenes licenciados. Su importe cubrirá los gastos de inscripción y, en su caso, los gastos de transporte y alojamiento.

INSCRIPCIÓN

La cuota de inscripción es de 20.000 pesetas. Esta cuota da derecho a asistencia y documentación del curso, comidas y asistencia a los actos sociales que se organicen. El plazo de inscripción será del 1 Junio al 9 de Septiembre de 1.994.

ENTIDADES COLABORADORAS

. La Cruz del Campo, S.A.
. Embajada de Francia en España.
. Fundación Cámara de la Universidad de Sevilla.
. Fundación " El Monte ".
. Junta de Andalucía.
. Ministerio de Educación y Ciencia.
. Vicerrectorado de Extensión Universitaria, Universidad de Sevilla.

Figura 9: Cartel de la EHF1994

- *Resultados fundamentales sobre métodos espectrales*, Y. Maday (Univ. París VI)

Los dos primeros cursos tenían el carácter de introducción a métodos muy utilizados en cálculo científico desde hacía muchos años y que aún hoy son objeto de numerosas investigaciones. Estos cursos estaban destinados a jóvenes investigadores. Los dos últimos cursos presentaban teorías recientes que son también objeto de numerosas investigaciones actualmente y que, en aquel momento, empezaban a suscitar numerosas aplicaciones. El contenido y la forma de los cuatro cursos fueron extremadamente apreciados. El formato de 5 horas y la presentación *de pizarra* parece óptima... ¡las excelentes pedagógicas de los cuatro profesores tuvieron ciertamente mucho que ver con esta apreciación!

Siete conferencias completaron este ciclo de cursos y permitieron ilustrar algunas otras facetas del cálculo científico:

- *Matemáticas y medio ambiente* (conferencia general en el Ayuntamiento de Sevilla), J.L. Lions (Collège de France)
- *Estimación de error a posteriori para problemas elípticos mediante un método de dualidad*, L. Ferragut (Univ. Politécnica de Madrid)
- *Algunas cuestiones relativas a la controlabilidad aproximada de las ecuaciones de Navier-Stokes*, J. Real (Univ. de Sevilla)
- *Guías de ondas electromagnéticas*, D. Gómez (Univ. Santiago de Compostela)
- *Teoría matemática de la plasticidad*, C. Moreno (Univ. Politécnica de Madrid)
- *Observaciones sobre la modelización numérica de la turbulencia*, O. Pironneau (Univ. París VI)
- *Análisis de la evolución de MODULEF, seguido de algunas observaciones sobre las láminas piezoeléctricas*, M. Bernadou (INRIA)

En el conjunto de los cursos y conferencias, los idiomas de trabajo fueron el español y el francés. Dos volúmenes agrupaban la totalidad de los apuntes de los cursos y conferencias (casi 350 páginas cada uno).

5.6.2 Participantes

Esta Escuela reunió a 135 participantes, incluyendo profesores y conferenciantes, procedentes en su mayor parte del sector universitario. Estos últimos procedían de las Universidades de Barcelona (1), Cádiz (5), Cantabria (2), Córdoba (3), La Coruña (1), Granada (2), la Laguna (1), Las Palmas (4), Madrid Autónoma (2), Madrid Complutense (7), Málaga (6), Oviedo (8), País Vasco (1), Politécnica de Madrid (10), Salamanca (3), Santiago de Compostela (13), Sevilla (21), Vigo (5), junto a una treintena de participantes de los que no nos consta la afiliación,

lo que pone de manifiesto una vez más la diversidad y la gran representatividad geográfica de los participantes. Nuevamente, el nivel fue muy variado, yendo desde jóvenes estudiantes de doctorado a profesores e investigadores veteranos.

5.6.3 Otras colaboraciones en curso

Además de las colaboraciones señalada en el resumen de la Escuela de 1992, colaboraciones éstas que se continúan y amplifican, conviene mencionar la obtención y financiación de una nueve red europea HCM *Mathematical Modeling and Analysis of Thin Shell Problems*, que arrancó en enero de 1995 con una duración de 3 años. Este programa reune al INRIA (P.G. Ciarlet, Director Científico y M. Bernadou), París VI (H. Le Dret), Málaga (A. Valle y F. Palma), Santiago de Compostela (J.M. Viaño), Lisboa (L. Trabucho), Coimbra (I. Figueiredo), Pavía (F. Brezzi), Stuttgart (K. Kirchgassner), Bochum (W. Kratzig).

5.7 VII Escuela: Oviedo, del 24-9 al 28-9 de 1996

Tras los dos ciclos de tres Escuelas organizadas por los polos fundadores: Santiago de Compostela, Málaga/Sevilla, Madrid/Castellón, se sugirió que, a fin de ampliar el público de la Escuela, ésta fuera organizada por otros centros españoles. Tras Sevilla, es pues la Universidad de Oviedo, y más concretamente J. Valdés, O. Menéndez, S. de Vicente, S. Meddahi y P. Pérez quienes nos acogen. Esta ciudad, de pasado muy rico y dotada de muy bellos monumentos salpicados por magníficos espacios verdes, ha logrado reconvertirse tras el cierre de sus minas. Durante la media jornada libre, los participantes pudieron visitar una de dichas minas abierta al turismo en la actualidad: ¡una experiencia excelente! Otra experiencia inolvidable: la visita a las sidrerías con degustación incluida.

5.7.1 Programa

Esta séptima Escuela contó con cuatro cursos de cinco horas cada uno:

- *Métodos de volúmenes finitos*, R. Eymard (LCPC, París)
- *Modelos matemáticos y numéricos en propagación de ondas*, P. Joly (Univ. París-Dauphine e INRIA)
- *Resolución efectiva de las ecuaciones de Navier-Stokes mediante métodos de elementos finitos*, R. Echevarría (Univ. de Sevilla)
- *Simulación y resolución numérica de problemas de frontera libre*, J. Durany, P. Quintela, C. Vázquez (Univ. de Vigo, Santiago de Compostela y La Coruña)

Todos los profesores destacaron los desarrollos recientes de sus disciplinas, sobre la base de una presentación clara y precisa de los resultados de base, lo que permitió a los más jóvenes participantes adquirir una panorámica general

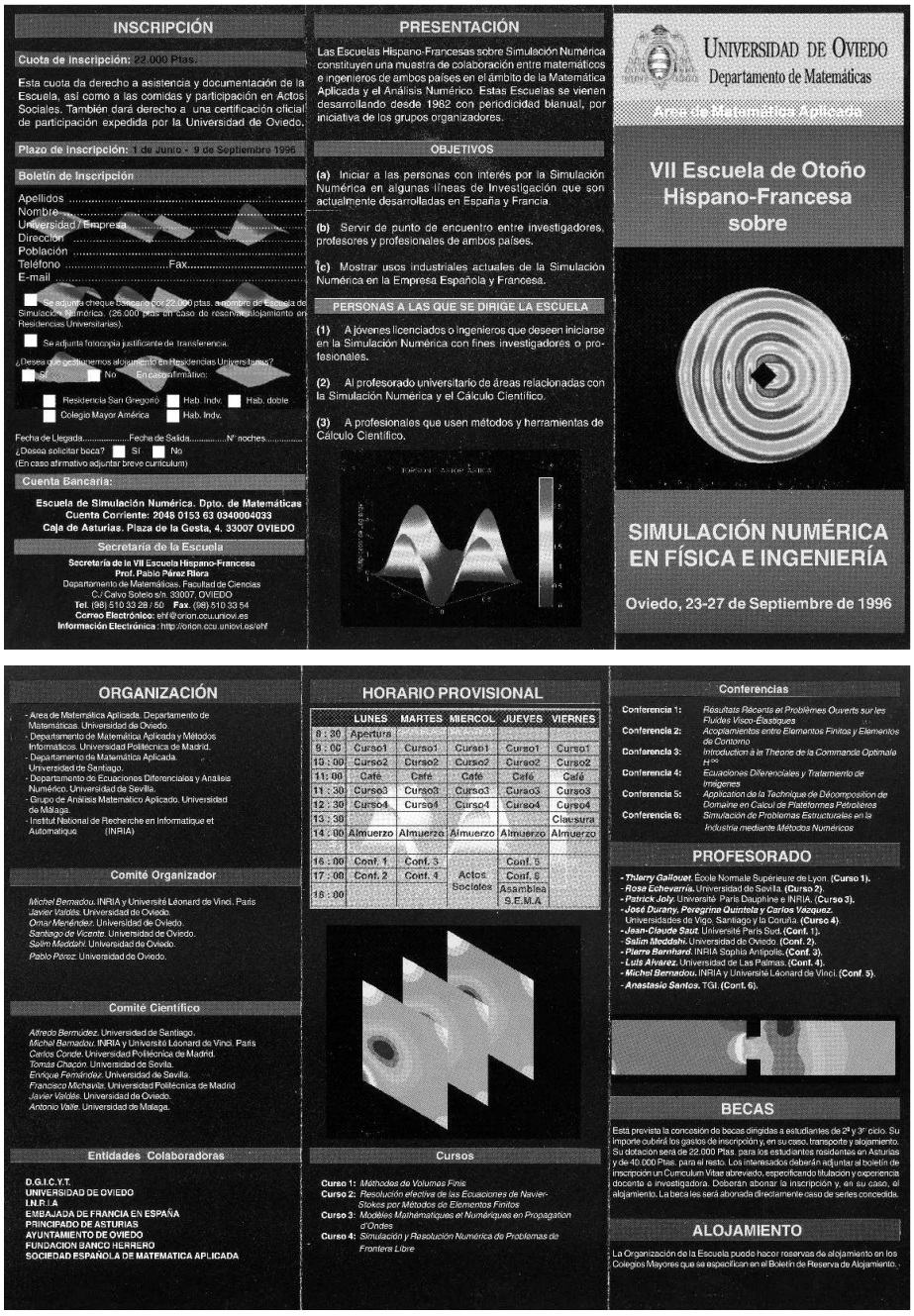


Figura 10: Cartel de la EHF1996

de cada uno de los temas. En los dos primeros cursos, los profesores insistieron además en la puesta en práctica numérica efectiva de los métodos.

Seis conferencias completaron este ciclo de conferencias y permitieron ilustrar otras tantas facetas del cálculo científico, que se pueden clasificar en académicas:

- *Introducción a la teoría del control óptimo H^∞*, P. Bernhard (INRIA Sophia-Antípolis)
- *Resultados recientes y problemas abiertos sobre los fluidos visco-elásticos*, J.C. Saut (Univ. París-Sur)
- *Acoplamientos entre elementos finitos y elementos de contorno*, S. Meddahi (Univ. de Oviedo)
- *Ecuaciones diferenciales y tratamiento de imágenes*, L. Álvarez (Univ. de Las Palmas)

e industriales:

- *Aplicación de la técnica de descomposición de dominio al cálculo de plataformas petrolíferas*, M. Bernadou (Univ. Leonardo da Vinci e INRIA)
- *Simulación de problemas estructurales en la industria mediante métodos numéricos*, A. Santos (TGI SA).

Tres volúmenes agruparon la totalidad de las notas de los cursos y conferencias (casi 920 páginas), lo que fue muy apreciado por los participantes.

5.7.2 Participantes

Esta escuela reunió a 113 participantes, incluyendo profesorado. Nuevamente, la mayor parte procedían del sector universitario: Cádiz (3), Cantabria (1), Castilla-La Mancha (1), Complutense de Madrid (1), Córdoba (3), La Coruña (2), Granada (1), Las Palmas (1), León (1), Málaga (4), Oviedo (26), Politécnica de Madrid (11), Salamanca (8), Santiago de Compostela (8), Sevilla (18), UNED (1), Valladolid (6), Vigo (7) y Zaragoza (4), además de 5 franceses y una portuguesa. De nuevo, se puso de manifiesto la diversidad y representatividad geográfica de la audiencia cuyo nivel, también en este caso, iba desde jóvenes estudiantes de doctorado a investigadores curtidos.

5.8 VIII Escuela: Córdoba, del 21-9 al 25-9 de 1998

Vuelta al sur para descubrir otra faceta de Andalucía: Córdoba, cuyo centro histórico, encrucijada de varias culturas, fue declarado por la UNESCO Patrimonio Mundial de la Humanidad. Como era de esperar, los organizadores J.L. Cruz, M. Marín, M.C. Calzada, J.R. Galo y J.A. Herencia incluyeron visitas al centro en el programa « social » de la Escuela; incluso añadieron un concierto de cuerdas, un *Perol* en la sierra de Córdoba y la cena de gala. Todo ello sin descuidar el programa científico, que fue del mejor nivel y muy completo.

Esta Escuela tuvo lugar en el Colegios Mayor Ntra. Sra. De la Asunción de Córdoba, lo que garantizó la unidad del emplazamiento (alojamiento, restauración y sala de conferencias) en un entorno confortable muy apreciado por los participantes.

5.8.1 Programa

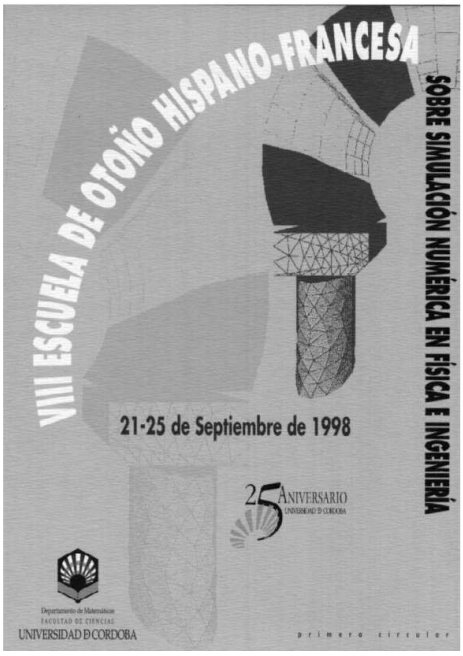
Esta octava Escuela contó con cuatro cursos de cuatro horas cada uno:

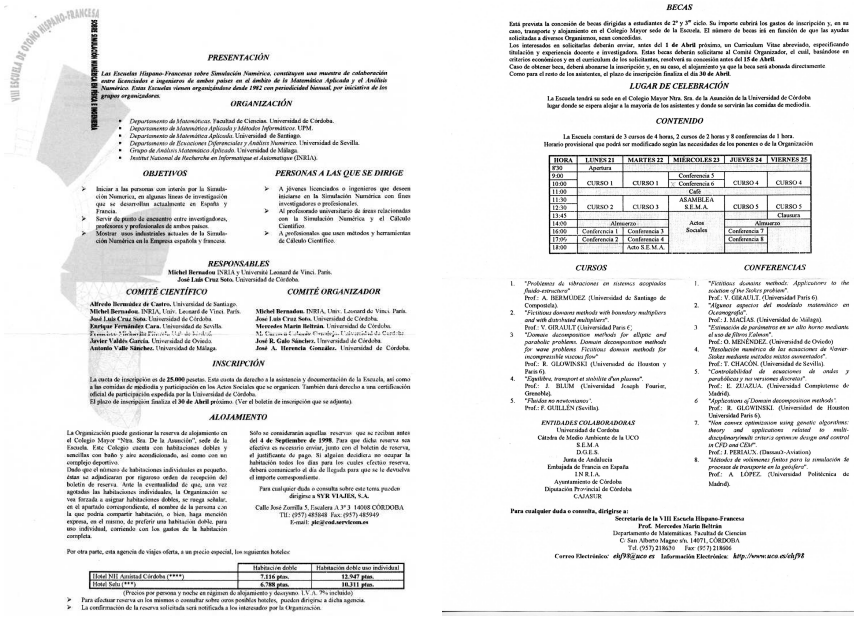
- *Problemas de vibraciones en sistemas acoplados fluido-estructura*, A. Bermúdez de Castro (Univ. de Santiago de Compostela)
- *Métodos de dominios ficticios con multiplicadores en la frontera y multiplicadores distribuidos; aplicaciones*, V. Girault y R. Glowinski (Univ. de París VI y Houston)
- *Equilibrio, transporte y estabilidad de un plasma*, J. Blum (Univ. de Grenoble)
- *Fluidos no newtonianos*, F. Guillén (Univ. de Sevilla)

En cada uno de los cursos, los profesores se propusieron presentar el origen físico o mecánico de los problemas considerados, su modelización en términos de ecuaciones en derivadas parciales, su formulación matemática, su aproximación numérica y algunas aplicaciones concretas. Además, el segundo curso presentó las bases teóricas y los últimos avances del método de los dominios ficticios que permite, para problemas formulados en dominios de geometría compleja, aproximar las soluciones utilizando una malla estructurada de un dominio regular de mayor tamaño (rectángulo, paralelepípedo...) combinada con una malla grosera de la frontera del dominio inicial.

Diez conferencias completaron este ciclo de cursos e ilustraron otras aplicaciones recientes de la simulación numérica:

- *Aplicación del método de los dominios ficticios a la resolución de problemas de Stokes*, V. Girault (Univ. París VI)
- *Estudio del fenómeno de corrientes oceánicas « El Niño » mediante un modelo acoplado híbrido océano/atmósfera*, J. Macías (Univ. de Málaga)
- *Modelización asintótica de lagos*, J. Simon (Univ. de Clermont-Ferrand)
- *Aplicación de los filtros de Kalman a la estimación de parámetros en un Alto Horno*, O. Menéndez (Univ. de Oviedo)
- *Resolución numérica de las ecuaciones de Navier-Stokes mediante métodos mixtos aumentados*, T. Chacón (Univ. de Sevilla)
- *Controlabilidad de ecuaciones de ondas y de ecuaciones parabólicas y sus versiones discretas*, E. Zuazua (Univ. Complutense de Madrid)





VIII ESCUELA DE OTOÑO HISPANO-FRANCESA

PRESENTACIÓN

Las Escuelas Hispano-Francesas sobre Simulación Numérica, constituyen una muestra de colaboración entre licenciados e ingenieros de ambos países en el ámbito de la Matemática Aplicada y el Análisis Numérico. Estas Escuelas vienen organizándose desde 1982 con periodicidad bianual, por iniciativa de los grupos organizadores.

ORGANIZACIÓN

- *Departamento de Matemáticas.* Facultad de Ciencias. Universidad de Córdoba.
- *Departamento de Matemática Aplicada y Métodos Informáticos.* UPM.
- *Departamento de Matemática Aplicada.* Universidad de Santiago.
- *Departamento de Ecuaciones Diferenciales y Análisis Numerico.* Universidad de Sevilla.
- *Grupo de Análisis Matemático Aplicado.* Universidad de Málaga.
- *Institut National de Recherche en Informatique et Automatique* (INRIA).

OBJETIVOS

- Iniciar a las personas con interés por la Simulación Numerica, en algunas líneas de investigación que se desarrollan actualmente en España y Francia.
- Servir de punto de encuentro entre investigadores, profesores y profesionales de ambos países.
- Mostrar unas industrias actuales de la Simulación Numérica en la Empresa española y francesa.

PERSONAS A LAS QUE SE DIRIGE

- A jóvenes licenciados o ingenieros que deseen iniciarse en la Simulación Numérica con fines investigadores o profesionales.
- Al profesorado universitario de áreas relacionadas con la Simulación Numérica y el Cálculo Científico
- A profesionales que usen métodos y herramientas de Cálculo Científico.

RESPONSABLES

Michel Bernadou INRIA y Université Leonard de Vinci. París.
José Luis Cruz Soto. Universidad de Córdoba.

COMITÉ CIENTÍFICO

Alfredo Bermúdez de Castro. Universidad de Santiago.
Michel Bernadou. INRIA, Univ. Leonard de Vinci. París.
José Luis Cruz Soto. Universidad de Córdoba.
Enrique Fernández Cara. Universidad de Sevilla.
Francisco Ochoa de Echaguen. Univ. de Málaga.
Javier Valdés García. Universidad de Oviedo.
Antonio Valle Sánchez. Universidad de Málaga.

COMITÉ ORGANIZADOR

Michel Bernadou. INRIA, Univ. Leonard de Vinci. París.
José Luis Cruz Soto. Universidad de Córdoba.
Mercedes Marín Beltrán. Universidad de Córdoba.
M. Encina Conde Fernández. Universidad de Córdoba.
José R. Galo Sánchez. Universidad de Córdoba.
José A. Herencia González. Universidad de Córdoba.

INSCRIPCIÓN

La cuota de inscripción es de **25.000** pesetas. Esta cuota da derecho a la asistencia y documentación de la Escuela, así como a las comidas de mediodía y participación en los Actos Sociales que se organicen. También dará derecho a una certificación oficial de participación expedida por la Universidad de Córdoba.
El plazo de inscripción finaliza el **30 de Abril próximo**. (Ver el boletín de inscripción que se adjunta).

ALOJAMIENTO

La Organización puede gestionar la reserva de alojamiento en el Colegio Mayor "Ntra. Sra. De la Asunción", sede de la Escuela. Este Colegio cuenta con habitaciones dobles y sencillas con baño y aire acondicionado, así como con un complejo deportivo.
Dado que el número de habitaciones individuales es pequeño, éstas se adjudicarán por riguroso orden de recepción del boletín de reserva. Ante la eventualidad de que, aún voz agotadas las habitaciones individuales, la Organización se vea forzada a asignar habitaciones dobles, se ruega señalar, en el apartado correspondiente, el nombre de la persona con la que podría compartir habitación, o bien, haga mención expresa, en el mismo, de preferir una habitación doble para uso individual, corriendo con los gastos de la habitación completa.

Sólo se considerarán aquellas reservas que se reciban antes del **1 de Septiembre de 1998**. Para que dicha reserva sea efectiva es necesario enviar, junto con el boletín de reserva, el justificante de pago. Si alguien decidiera no ocupar la habitación todos los días para los cuales efectúo reserva, deberá comunicarlo al día de llegada para que se le devuelva el importe correspondiente.

Para cualquier duda o consulta sobre este tema pueden dirigirse a **NYR VIAJES, S.A.**

Calle José Zorrilla 5, Escalera A 3º 3 14008 CORDOBA
TE: (957) 485868 Fax: (957) 485949
E-mail: **plc@cod.servicom.es**

Por otra parte, esta agencia de viajes oferta, a un precio especial, los siguientes hoteles:

	Habitación doble	Habitación doble uso individual
Hotel NH Amistad Córdoba (****)	**7.116 ptas.**	**12.947 ptas.**
Hotel Sabi (***)	**6.708 ptas.**	**10.311 ptas.**

(Precios por persona y noche en régimen de alojamiento y desayuno I.V.A. 7% incluido)

- Para efectuar reserva en los mismos o consultar sobre otros posibles hoteles, pueden dirigirse a dicha agencia.
- La confirmación de la reserva solicitada será notificada a los interesados por la Organización.

BECAS

Está prevista la concesión de becas dirigidas a estudiantes de 2º y 3er ciclo. Su importe cubrirá los gastos de inscripción y, en su caso, transporte y alojamiento en el Colegio Mayor sede de la Escuela. El número de becas irá en función de que las ayudas solicitadas a diversos Organismos, sean concedidas.
Los interesados en solicitarlas deberán enviar, antes del **1 de Abril** próximo, un Curriculum Vitae abreviado, especificando titulación y experiencia docente e investigadora. Estas becas deberán solicitarse al Comité Organizador, el cuál, basándose en criterios económicos y en el curriculum de los solicitantes, resolverá su concesión antes del **15 de Abril**.
Caso de obtener beca, deberá abonarse la inscripción y, en su caso, el alojamiento ya que la beca será abonada directamente
Como para el resto de los asistentes, el plazo de inscripción finaliza el día **30 de Abril**.

LUGAR DE CELEBRACIÓN

La Escuela tendrá su sede en el Colegio Mayor Ntra. Sra. de la Asunción de la Universidad de Córdoba
lugar donde se espera alojar a la mayoría de los asistentes y donde se servirán las comidas de mediodía.

CONTENIDO

La Escuela constará de 3 cursos de 4 horas, 2 cursos de 2 horas y 8 conferencias de 1 hora.
Horario provisional que podrá ser modificado según las necesidades de los ponentes o de la Organización

HORA	LUNES 21	MARTES 22	MIÉRCOLES 23	JUEVES 24	VIERNES 25
9:30	Apertura				
9:00	CURSO 1	CURSO 1	Conferencia 5	CURSO 4	CURSO 4
10:00			Conferencia 6		
11:00			Café		
11:30	CURSO 2	CURSO 3	ASAMBLEA S.E.M.A.	CURSO 5	CURSO 5
12:30					Clausura
13:45					
14:00	Almuerzo		Actos Sociales	Almuerzo	
16:00	Conferencia 1	Conferencia 3		Conferencia 7	
17:00	Conferencia 2	Conferencia 4		Conferencia 8	
18:00		Acto S.E.M.A.			

CURSOS

1. *"Problemas de vibraciones en sistemas acoplados fluido-estructura"*
   Prof.: A. BERMÚDEZ. (Universidad de Santiago de Compostela).
2. *"Ficticious domains methods with boundary multipliers and with distributed multipliers"*
   Prof.: V. GIRAULT (Universidad Paris 6).
3. *"Domain decomposition methods for elliptic and parabolic problems. Domain decomposition methods for wave problems. Ficticious domain methods for incompressible viscous flow"*
   Prof.: R. GLOWINSKI (Universidad de Houston y Paris 6).
4. *"Equilibre, transport et stabilité d'un plasma".*
   Prof.: J. BLUM (Universidad Joseph Fourier, Grenoble).
5. *"Fluides no newtoniens".*
   Prof.: F. GUILLEN (Sevilla).

CONFERENCIAS

1. *"Ficticious domains methods: Applicatons to the solution of two Stokes problem".*
   Prof.: V. GIRAULT (Universidad Paris 6).
2. *"Algunos aspectos del modelado matemático en Oceanografía".*
   Prof.: J. MACÍAS. (Universidad de Málaga).
3. *"Estimación de parámetros en un alto horno mediante el uso de filtros Fubini".*
   Prof.: O. MENÉNDEZ. (Universidad de Oviedo)
4. *"Resolución numérica de las ecuaciones de Navier-Stokes mediante métodos mixtos convertidos".*
   Prof.: T. CHACÓN. (Universidad de Sevilla).
5. *"Controlabilidad de ecuaciones de ondas y parabólicas y sus versiones discretas".*
   Prof.: E. ZUAZUA. (Universidad Compiutense de Madrid).
6. *"Applications of Domain decomposition methods".*
   Prof.: R. GLOWINSKI. (Universidad de Houston Universidad Paris 6).
7. *"Non convex optimization using genetic algorithms: theory and applications related to multi-disciplinary/multi criteria optimum design and control in CFD and CEM".*
   Prof.: J. PERIAUX. (Dassault-Aviation)
8. *"Métodos de ventanas finitas para la simulación de procesos de transporte en la geosfera".*
   Prof.: A. LOPEZ. (Universidad Politécnica de Madrid).

ENTIDADES COLABORADORAS

Universidad de Córdoba
Cátedra de Medio Ambiente de la UCO
S.E.M.A
D.G.E.S.
Junta de Andalucía
Embajada de Francia en España
I.N.R.I.A.
Ayuntamiento de Córdoba
Diputación Provincial de Córdoba
CAJASUR

Para cualquier duda o consulta, dirigirse a:

Secretaría de la VIII Escuela Hispano-Francesa
Prof. Mercedes María Beltrán
Departamento de Matemáticas. Facultad de Ciencias
C. San Alberto Magno s/n. 14071, CORDOBA
Tel. (957) 218630 Fax: (957) 218606
Correo Electrónico: ***eh98@uco.es*** **Información Electrónica:** ***http://www.uco.es/eh98***

Figura 11: Cartel de la EHF1998

- *Aplicación del método de los dominios ficticios a problemas de sedimentación*, R. Glowinski (Univ. de París VI y Houston)

- *Optimización no convexa mediante algoritmos genéticos: teoría y aplicaciones problemas de optimización de forma y de control multidisciplinar/multicriterios en CFD (Computational Fluids Dynamics) y CEM (Computational ElectroMagnétism)*, J. Périaux (Avions Marcel Dassault)

- *Métodos de volúmenes finitos para la simulación de procesos de transporte en la geosfera*, A. López (Univ. Politécnica de Madrid)

- *El problema 16 de Hilbert: estado de la cuestión*, J. Chavarriga (Univ. de Lérida)

Dos volúmenes agruparon la totalidad de los apuntes de cursos y conferencias (casi 550 páginas) lo que fue muy apreciado por los asistentes y muy útil para aquellos que no dominaban completamente los dos idiomas: francés y español.

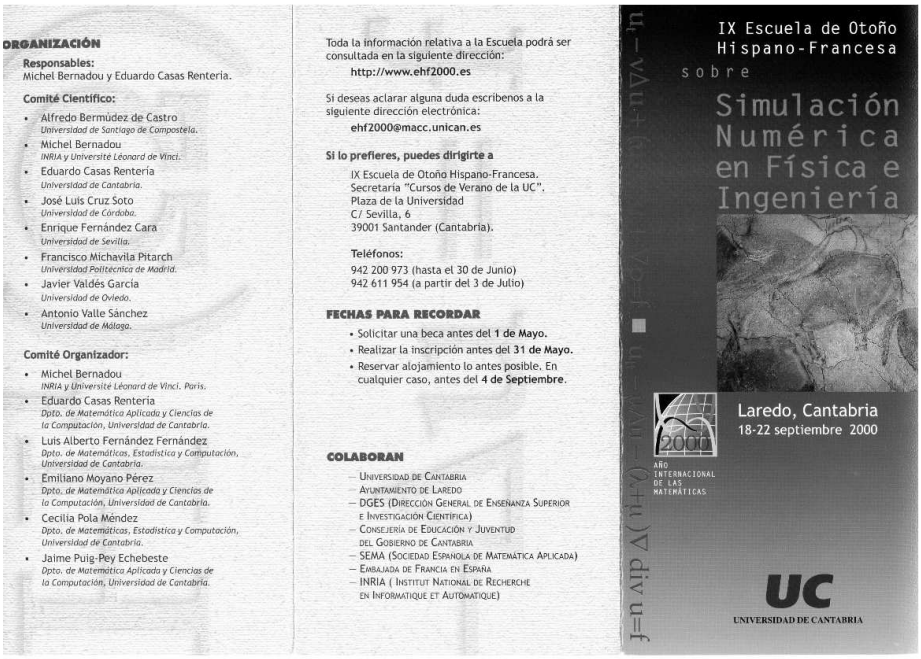
5.8.2 Participantes

Esta Escuela reunió 128 participantes, incluyendo profesores y conferenciantes, la mayor parte procedente del sector universitario. Estos últimos procedían de las Universidades de Burgos (2), Cádiz (9), Cantabria (2), Castilla-La Mancha (2), Córdoba (14), La Coruña (4), Lérida (1), Madrid (1 de la Autónoma, 5 de la Complutense, 6 de la Politécnica), Málaga (8), Murcia (1), Oviedo (11), Santiago de Compostela (12), Sevilla (28), Valencia (1), Valladolid (2), Vigo (5), Zaragoza (3) del lado español, y del lado francés: Clermont-Ferrand (1), Grenoble (1), París (4), Pau (2), Perpiñán (1), y de Lisboa (2) del lado portugués. Esta distribución ha confirmado la ampliación del público de las Escuelas en España y también, en menor medida, a Francia y Portugal.

El nivel científico del auditorio iba desde jóvenes estudiantes de doctorado a investigadores muy experimentados de estatura internacional en el caso de una veintena de entre ellos.

5.9 IX Escuela: Laredo, Cantabria, del 18-9 al 22-9 de 2000

Rumbo al norte a la ciudad de Laredo, que combina una inmensa playa de fina arena con un centro histórico reputado. Los organizadores de la Escuela, E. Casas, L.A. Fernández, E. Moyano, C. Pola y J. Puig-Pey de la Universidad de Cantabria nos acogen en la sede de los Cursos de Verano de la Universidad de Cantabria a unos 50 km al este de Santander. Además del completo programa científico que recordamos a continuación, visita al Parque Natural de Cabárceno, demostración de vuelo libre de aves de presa (muy impresionante) y cena de gala en el Centro Náutico de Laredo.



ORGANIZACIÓN

Responsables:
Michel Bernadou y Eduardo Casas Renteria.

Comité Científico:

- Alfredo Bermúdez de Castro
  Universidad de Santiago de Compostela.
- Michel Bernadou
  INRIA y Université Léonard de Vinci.
- Eduardo Casas Renteria
  Universidad de Cantabria.
- José Luis Cruz Soto
  Universidad de Córdoba.
- Enrique Fernández Cara
  Universidad de Sevilla.
- Francisco Michavila Pitarch
  Universidad Politécnica de Madrid.
- Javier Valdés García
  Universidad de Oviedo.
- Antonio Valle Sánchez
  Universidad de Málaga.

Comité Organizador:

- Michel Bernadou
  INRIA y Université Léonard de Vinci. Paris.
- Eduardo Casas Renteria
  Dpto. de Matemática Aplicada y Ciencias de la Computación, Universidad de Cantabria.
- Luis Alberto Fernández Fernández
  Dpto. de Matemáticas, Estadística y Computación, Universidad de Cantabria.
- Emiliano Moyano Pérez
  Dpto. de Matemática Aplicada y Ciencias de la Computación, Universidad de Cantabria.
- Cecilia Pola Méndez
  Dpto. de Matemáticas, Estadística y Computación, Universidad de Cantabria.
- Jaime Puig-Pey Echebeste
  Dpto. de Matemática Aplicada y Ciencias de la Computación, Universidad de Cantabria.

Toda la información relativa a la Escuela podrá ser consultada en la siguiente dirección:

http://www.ehf2000.es

Si deseas aclarar alguna duda escríbenos a la siguiente dirección electrónica:

ehf2000@macc.unican.es

Si lo prefieres, puedes dirigirte a

IX Escuela de Otoño Hispano-Francesa.
Secretaria "Cursos de Verano de la UC",
Plaza de la Universidad
C/ Sevilla, 6
39001 Santander (Cantabria).

Teléfonos:

942 200 973 (hasta el 30 de Junio)
942 611 954 (a partir del 3 de Julio)

FECHAS PARA RECORDAR

- Solicitar una beca antes del **1 de Mayo.**
- Realizar la inscripción antes del **31 de Mayo.**
- Reservar alojamiento lo antes posible. En cualquier caso, antes del **4 de Septiembre.**

COLABORAN

— Universidad de Cantabria
— Ayuntamiento de Laredo
— DGES (Dirección General de Enseñanza Superior e Investigación Científica)
— Consejería de Educación y Juventud del Gobierno de Cantabria
— SEMA (Sociedad Española de Matemática Aplicada)
— Embajada de Francia en España
— INRIA (Institut National de Recherche en Informatique et Automatique)

IX Escuela de Otoño
Hispano-Francesa
sobre
Simulación Numérica en Física e Ingeniería

Laredo, Cantabria
18-22 septiembre 2000

2000 AÑO INTERNACIONAL DE LAS MATEMÁTICAS

UC
UNIVERSIDAD DE CANTABRIA

PRESENTACIÓN

Las Escuelas Hispano-Francesas sobre Simulación Numérica constituyen una muestra de colaboración entre matemáticos e ingenieros de ambos países en el ámbito de la Matemática Aplicada y el Análisis Numérico.

Estas Escuelas se vienen desarrollando desde 1982 con periodicidad bianual, por iniciativa de los grupos organizadores.

OBJETIVOS

a. Iniciar a las personas con interés por la Simulación Numérica en algunas líneas de Investigación que se desarrollan actualmente en España y Francia.

b. Servir de punto de encuentro entre investigadores, profesores y profesionales de ambos países.

c. Mostrar usos industriales actuales de la Simulación Numérica en la Empresa Española y Francesa.

PERSONAS A LAS QUE SE DIRIGE LA ESCUELA

1. A jóvenes licenciados o ingenieros que deseen iniciarse en la Simulación Numérica con fines investigadores o profesionales.

2. Al profesorado universitario de áreas relacionadas con la Simulación Numérica y el Cálculo Científico.

3. A profesionales que usen métodos y herramientas de Cálculo Científico.

ACTIVIDADES A DESARROLLAR

Cuatro Cursos Monográficos de 5 horas

- **Métodos de Elementos Finitos en Problemas de Mecánica de Fluidos.**
  Ramón Codina Rovira
  (Universidad Politécnica de Cataluña).
- **Les Méthodes Ondelettes: de l'Analyse à la Simulation.**
  Albert Cohen
  (Université Pierre et Marie Curie).
- **Modelisation Numérique de Structures en Grandes Deformations.**
  Patrick Le Tallec *(École Polytechnique).*
- **Métodos de Elementos de Contorno.**
  F. Javier Sayas González
  (Universidad de Zaragoza).

Seis Conferencias de 45 minutos impartidas por

- José Luis Cruz Soto *(Universidad de Córdoba).*
- Luis Ferragut Canals *(Universidad de Salamanca).*
- Roland Glowinski *(University of Houston).*
- Claude Le Bris *(École Nationale des Ponts et Chaussées).*
- Salim Meddahi *(Universidad de Oviedo).*
- Carlos Parés Madroñal *(Universidad de Málaga).*

INSCRIPCIÓN

Cuota de Inscripción: 27000 ptas.
Esta cuota dará derecho a la asistencia y documentación de la Escuela, así como a las comidas del mediodía y participación en las actividades sociales. También dará derecho a una certificación oficial de participación.

Deberá ingresarse por transferencia bancaria o en efectivo en la cuenta corriente Nº 2066 0000 17 2001 89353 de Caja Cantabria antes del 31 de Mayo de 2000.

VIAJES

IBERIA, como transportista aéreo oficial de los Cursos de Verano de la Universidad de Cantabria, ofrece una reducción del 30% de su tarifa normal (clase turista) en los vuelos con destino a Santander o Bilbao. Asimismo, ALSA ofrece una reducción del 50% del importe de los billetes de autobús con destino a Laredo. En ambos casos, para poder acceder a estas reducciones será necesario acreditar la inscripción en la Escuela.

ALOJAMIENTO

La Organización de la Escuela ha concertado unas condiciones especiales para los participantes que deseen alojarse en el Hotel Cosmopol (Laredo). Los precios (incluyendo desayuno) son los siguientes:

Habitación individual ..... 6.950 ptas. +IVA por noche
Habitación doble ..... 4.950 ptas. +IVA por persona y noche

Debido a las limitaciones de plazas, se recomienda efectuar la reserva lo antes posible. En cualquier caso, las reservas se atenderán por riguroso orden de llegada y sólo se considerarán aquellas hechas antes del 4 de septiembre de 2000.

BECAS

Está prevista la concesión de algunas becas dirigidas a estudiantes de segundo y tercer ciclo. Su importe cubrirá los gastos de inscripción. Los interesados deberán solicitarlo antes del día 1 de Mayo de 2000 mediante escrito dirigido a la Secretaría, acompañando curriculum vitae abreviado.

Figura 12: Cartel de la EHF2000

5.9.1 Programa

Se impartieron cuatro cursos de cinco horas cada uno:

- *Métodos de elementos finitos para problemas de mecánica de fluidos*, R. Codina (Univ. Politécnica de Cataluña), curso panorámico sobre uno de los temas « clásicos » pero siempre de gran actualidad de la Simulación Numérica

- *Métodos wavelets: del análisis a la simulación*, A. Cohen (Univ. París VI), curso prospectivo sobre una de las nuevas disciplinas que ha experimentado un mayor desarrollo en el transcurso de la última década. Este disciplina presenta importantes aplicaciones tanto en simulación numérica como en tratamiento de la señal y de la imagen

- *Modelización numérica de estructuras en grandes deformaciones*, P. Le Tallec (Escuela Politécnica e INRIA), curso panorámico sobre la modelización y la aproximación numérica de los principales problemas no lineales que aparecen en mecánica del sólido: elasticidad no lineal, postflameado, termoviscoelasticidad, contacto, comportamiento no lineal de polímeros y de caucho sintético, aplicaciones en biomecánica. En la parte numérica, se discutieron las contribuciones de los métodos de elementos finitos mixtos, estimaciones de error a posteriori y técnicas de descomposición de dominio

- *Métodos de elementos de contorno*, F.J. Sayas (Univ. de Zaragoza), curso introductorio a la técnica consistente en reformular un problema de contorno en la forma de una ecuación integral que, posteriormente, es aproximada numéricamente. Se presentaron algunas aplicaciones a problemas de potencial interior/exterior, a las ecuaciones de Helmholtz, a la elasticidad y a otros problemas menos clásicos

Cinco conferencias completaron este ciclo de cursos y permitieron abordar algunos resultados recientes de Cálculo Científico:

- *Simulación numérica y visualización de los datos y resultados para la propagación de un incendio forestal en la cuenca del Ebro*, L. Ferragut (Univ. de Salamanca)

- *Modelización numérica en química cuántica*, C. Le Bris (CERMICS, ENPC París)

- *Cálculo paralelo para la resolución de ecuaciones en derivadas parciales*, J.L. Cruz (Univ. de Córdoba)

- *Acoplamientos elementos finitos/elementos de contorno para la resolución de problemas de Stokes en el plano*, S. Meddahi (Univ. de Oviedo)

- *Modelización numérica de problemas medioambientales en el Estrecho de Gibraltar*, C. Parés (Univ. de Málaga)

Es conveniente destacar, tanto en lo que se refiere a los cursos como a las conferencias, la excelente cualidades científicas y pedagógicas del conjunto de conferenciantes.

Un volumen que reunía los apuntes de los cursos y las conferencias (de aproximadamente 400 páginas) facilitó el trabajo de los asistentes.

5.9.2 Participantes

Esta Escuela reunió a 81 participantes, incluyendo profesores y conferenciantes, procedentes en su mayor parte del medio universitario (Universidades de Cantabria (8), Córdoba (3), Málaga (5), Oviedo (10), Salamanca (7), Santiago de Compostela (11), Sevilla (6), Valencia (2), Valladolid (2) Vigo (4), Zaragoza (3), 5 de las universidades de Madrid, 5 de las universidades catalanas y 9 de otras universidades) y de nivel científico doctoral, postdoctoral o investigador experimentado. La representatividad geográfica en España se puso una vez más de manifiesto; es de destacar que una buena proporción de los investigadores experimentados que participaron mantienen colaboraciones regulares con Francia.

La participación en esta Escuela experimentó un ligero retroceso comparada con las precedentes, lo que se debió esencialmente al gran número de Coloquios y Congresos de Matemáticas Aplicadas celebrados en España durante este año 2000 declarado Año Mundial de las Matemáticas.

5.10 X Escuela: Jaca, del 23-9 al 27-9 de 2002

La organización de esta décima Escuela, a cargo de nuestros colegas de Zaragoza F. Lisbona , F.J. Sayas, R. Celorrio, C. Clavero y F. Gaspar en la ciudad de Jaca, nos conduce por primera vez al corazón de los Pirineos españoles. La candidatura de esta villa turística a la organización de los Juegos Olímpicos de invierno de 2010 atestigua la calidad de su entorno montañoso, que pudimos apreciar durante la media jornada de excursión. Además, la propia ciudad cuenta con una gran riqueza arquitectónica, especialmente la Catedral de estilo románico, el Castillo de San Pedro y el Ayuntamiento. La Escuela tuvo lugar en la Residencia Universitaria de Jaca, con alojamiento y comida en el mismo lugar, lo que favoreció las conversaciones y discusiones científicas.

5.10.1 Programa

Se impartieron cuatro cursos de cinco horas cada uno:

- *Modelos numéricos para los fluidos hidrodinámicos*, T. Chacón (Univ. de Sevilla), presentación de la problemática de distintos tipos de modelos matemáticos, de su análisis matemático y de la aproximación de sus soluciones, así como de algunos ejemplos de aplicaciones concretas

- *Problemas de evolución parabólicos*, M. Crouzeix (Univ. de Rennes I), presentación de una síntesis de los métodos de análisis matemáticos y de

Figura 13: Cartel de la EHF2002

aproximación numérica actualmente disponibles para abordar de forma óptima la simulación numérica de los problemas

- *Esquemas numéricos de alto orden para leyes de conservación hiperbólica con buenas propiedades de captura de ondas de choque*, R. Donat (Univ. de Valencia), presentación de una amplia síntesis de métodos matemáticos y numéricos recientes para modelizar fenómenos de tipo propagación de ondas, discontinuidades de contacto, choques, rarefacciones...

- *Algunos nuevos resultados y desafíos actuales en el análisis mediante elementos finitos de problemas de láminas*, D. Chapelle (INRIA, Rocquencourt), síntesis de los resultados matemáticos y numéricos, enfatizando la combinación de la calidad de los modelos matemáticos con la eficacia y la fiabilidad de la aproximación numérica

Cinco conferencias de una hora cada una permitieron abordar cinco temas más específicos:

- *Avances recientes en el modelado numérico de la propagación del oleaje y su interacción con estructuras costeras*, I. Losada (Univ. de Cantabria)

- *Métodos iterativos para la resolución de grandes sistemas lineales con matrices vacías*, G. Meurant (CEA Bruyères-le-Châtel)

- *Métodos numéricos para problemas de valorización de opciones*, C. Moreno (Univ. Politécnica de Madrid)

- *Aplicación del método de volúmenes finitos y de esquemas de tipo Godunov a las ecuaciones de aguas someras*, E. Vázquez (Univ. Santiago de Compostela)

- *Aproximación mediante elementos finitos de problemas coercivos no elípticos*, J.M. Thomas (Univ. de Pau)

Además se celebró una sesión especial sobre algunas aplicaciones de herramientas matemáticas y métodos numéricos en la bioingeniería. En efecto, la extensión de los métodos de simulación numérica a los seres vivos ha tropezado durante mucho tiempo con la complejidad de los modelos (los medios concernidos son muy inhomogéneos, anisótropos y requieren modelos fuertemente acoplados) y con el hecho de conducir a problemas numéricos de una dimensión demasiado grande. Los enormes progresos realizados tanto en modelización de fenómenos acoplados como en la combinación *algoritmia numérica/prestaciones informáticas y cálculo paralelo* permite abordar en la actualidad de manera fiable algunos dominios importantes de la bioingeniería. Esta sesión permitió abordar tres de ellos:

- *Tratamiento de la señal biomédica en el diagnóstico y monitorización de patologías*, P. Carminal (Univ. Politécnica de Cataluña)

- *Adaptación en mecánica ósea*, S. Ramtani (Univ. París Val de Marne)

- *Algunas aplicaciones del método de elementos finitos a la modelización de tejidos vivos y al diseño de implantes*, M. Doblaré (Univ. de Zaragoza).

Un volumen recopiló los apuntes de los cursos, a los que se añadieron dos tiradas aparte (las transparencias de la conferencia de J.M. Thomas y los resúmenes de la sesión especial de bioingeniería).

5.10.2 Participantes

Esta Escuela reunió 78 participantes, incluyendo profesores y conferenciantes, procedentes en su mayor parte del sector universitario español. Más precisamente, estos últimos procedían de las Universidades de Alicante (1), Barcelona (1), Cantabria (3), Cartagena (1), Córdoba (3), La Coruña (6), Navarra (2), Oviedo (2), Salamanca (4), Santiago de Compostela (8, de los cuales 1 de Lugo), Sevilla (7), Valencia (8), Valladolid (1), Vigo (1 de Pontevedra), Zaragoza (13, 1 de Huesca), 5 de las Universidades de Madrid y, del lado francés, del CEA (1), del INRIA (1), de las Universidades de Pau (2), Rennes (1) y del Polo Universitario Leonardo da Vinci (1). Este reparto atestigua la diversidad y la gran representatividad geográfica de la audiencia. El número de participantes sigue siendo elevado, a pesar de las dificultades de acceso de la localización y de las fechas algo tardías de la Escuela, muy próximas al inicio del curso universitario.

El nivel de los participantes fue muy variado, yendo de doctorandos a profesores e investigadores con muy distinto nivel de experiencia. Un tercio de entre ellos aproximadamente participa personalmente en colaboraciones con equipos franceses (París VI, INRIA, Escuela Politécnica, Escuela Central, Universidades de Pau y de Rennes...).

5.10.3 Otras colaboraciones en curso

Numerosas colaboraciones están en marcha, especialmente codirecciones de tesis, participación en proyectos y contratos conjuntos o en Redes Europeas: el proyecto *Smart Systems* acaba de ser aprobado por la Comunidad Europea y se desarrollará en el periodo 2003/2006. El INRIA, el ESIEE, el Polo Universitario Leonardo da Vinci son miembros del lado francés y la Universidad de Santiago de Compostela del lado español.

5.11 XI Escuela: Cádiz, del 20-9 al 24-9 de 2004

A propuesta de A. Valle de la Universidad de Málaga y por iniciativa de SeMA, la Escuela pasó a denominarse « Escuela Jacques-Louis Lions Hispano-Francesa sobre Simulación Numérica en Física e ingeniería ». De esta manera, SeMA deseaba honrar la memoria de Jacques-Louis Lions, fallecido en 2001, quien, con sus excepcionales cualidades científica y humana, se implicó en gran medida en el establecimiento de numerosas y fructíferas colaboraciones entre la mayor parte de los grupos de simulación numérica españoles y franceses.

Información general

Las Escuelas Hispano-Francesas sobre Simulación Numérica constituyen una muestra de la colaboración entre investigadores de España y Francia, en el ámbito del análisis numérico y los modelos matemáticos de las ciencias experimentales. Se celebran con carácter bianual desde 1984 y se orienta fundamentalmente a

- la formación de nuevos investigadores;
- iniciar a profesionales y estudiantes en algunas líneas de investigación relacionadas con la simulación numérica;
- servir de punto de encuentro entre investigadores, profesores y profesionales de ambos países; y
- mostrar algunos usos actuales de la simulación numérica en la industria y en la empresa.

Destinatarios

- Estudiantes de últimos cursos, doctorandos y jóvenes licenciados o ingenieros que deseen iniciarse en la simulación numérica con fines investigadores o profesionales.
- Profesores universitarios de áreas relacionadas con la simulación numérica y la computación científica.
- Profesionales y técnicos que utilicen métodos de computación científica.

In memoriam

En la actual edición, el consejo ejecutivo de la SEMA ha decidido un cambio en la denominación de la escuela en homenaje al profesor Jacques-Louis Lions, fallecido en el año 2001, valedor e impulsor y participante de este tipo de encuentros, y vinculado especialmente con la mayoría de grupos de simulación numérica españoles y franceses.

Entidades colaboradoras

- Vicerrectorado de Investigación y Desarrollo (UCA).
- Ministerio de Ciencia y Tecnología.
- Consejería de Educación y Ciencia (Junta de Andalucía).
- AIRBUS España, S.L.
- Dragados Off-Shore, S.A.
- Embajada de Francia en España.
- Excma. Diputación Provincial de Cádiz.
- Sociedad Española de Matemática Aplicada (SEMA).
- INRIA

Inscripción

La información precisa sobre las diversas modalidades de inscripción se encuentra en

http://www.uca.es/ehf2004

Cuotas de inscripción

Socio SEMA	hasta el 31 de mayo de 2004	220 €
	a partir del 1 de junio de 2004	250 €
Ordinaria	hasta el 31 de mayo de 2004	250 €
	a partir del 1 de junio de 2004	280 €

La cuota de inscripción da derecho a la asistencia a los cursos y conferencias, a la documentación de la Escuela, a los almuerzos y a participar en los actos sociales.

Alojamiento

La información precisa sobre la reserva de alojamiento se encuentra en

http://www.uca.es/ehf2004

Becas

La organización tiene prevista la concesión de un cierto número de becas dirigidas a estudiantes en las modalidades de inscripción (220 €) y de inscripción y alojamiento (370 €). Las solicitudes, junto con el curriculum vitae y fotocopia del carnet de estudiante en vigor, deberán ser remitidas a la secretaría de la Escuela antes del 15 de mayo de 2004. Más información en

http://www.uca.es/ehf2004

Sesiones de pósteres

A partir de la presente edición, los alumnos tienen la posibilidad de presentar trabajos en pósteres en las tres sesiones programadas.

UCA
Universidad de Cádiz

XI Escuela Jacques-Louis Lions
Hispano-Francesa
sobre
Simulación Numérica en Física e Ingeniería

Cádiz, 20–24 de septiembre de 2004

Responsables de organización

Michel Bernadou
Pole Universitaire Léonard de Vinci. Paris

Francisco Ortegón Gallego
Universidad de Cádiz

Organización

Grupo de investigación
Análisis teórico y numérico de modelos de las ciencias experimentales (FQM-315)
Departamento de Matemáticas de la Universidad de Cádiz

Comité científico

- Michel Bernadou (P. U. Léonard de Vinci, Paris).
- Eduardo Casas Renteria (Universidad de Cantabria).
- Tomás Chacón Rebollo (Universidad de Sevilla).
- José Luis Cruz Soto (Universidad de Córdoba).
- Francisco Lisbona Cortés (Universidad de Zaragoza).
- Francisco Michavila Pitarch (U. Politécnica de Madrid).
- Francisco Ortegón Gallego (Universidad de Cádiz).
- Javier Valdés García (Universidad de Oviedo).
- Antonio Valle Sánchez (Universidad de Málaga).
- Juan Viaño Rey (U. de Santiago de Compostela).

Comité organizador

- Michel Bernadou (P. U. Léonard de Vinci, Paris).
- Francisco Ortegón Gallego (Universidad de Cádiz).
- José Manuel Díaz Moreno (Universidad de Cádiz).
- Rafael Rodríguez Galván (Universidad de Cádiz).
- María Victoria Redondo Nebles (Universidad de Cádiz).
- Concepción García Vázquez (Universidad de Cádiz).
- Mª Teresa González Montesinos (Universidad de Cádiz).

Secretaría de la Escuela

José Manuel Díaz Moreno
CASEM
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Información completa sobre la Escuela

http://www.uca.es/ehf2004

Cursos

- *Modelos matemáticos en los procesos industriales.*
  Alfredo Bermúdez de Castro y López-Varela.
  Universidad de Santiago de Compostela.
- *Problèmes mathématiques et numériques posés par la securité des déchets nucléaires.*
  Olivier Pironneau.
  Université Pierre et Marie Curie. Paris 6.
- *Génération de maillages tridimensionnels.*
  Paul-Louis George.
  Institut National de Recherche en Informatique et Automatique.
- *Modelos matemáticos difusivos para la descripción del clima global.*
  Jesús Ildefonso Díaz Díaz.
  Universidad Complutense de Madrid.

Conferencias

- *Cálculo paralelo para la resolución numérica de las ecuaciones de Navier-Stokes.*
  Enrique Fernández Cara.
  Universidad de Sevilla.
- *Finite volume schemes and Lax theorem.*
  Bruno Després.
  Université Pierre et Marie Curie, Paris 6.
- *Resolución numérica de las ecuaciones de aguas someras bicapa mediante esquemas de volúmenes finitos.*
  Manuel Jesús Castro Díaz.
  Universidad de Málaga.
- *Schémas numériques à deux grilles pour les équations de Navier-Stokes.*
  Vivette Girault.
  Université Pierre et Marie Curie, Paris 6.
- *Algunas ecuaciones en derivadas parciales para el procesamiento de imágenes.*
  José Manuel Mazón Ruiz.
  Universidad de Valencia.
- *Simulación de procesos. Digital Factory.*
  Antonio Montosa Jurado.
  AIRBUS España, S.L.

Programa provisional

Domingo, 19/09
De 18:00 a 20:00 Entrega de documentación

Lunes, 20/09
8:30 Entrega de documentación
9:00 Acto inaugural
9:30 Curso
11:00 Sesión de pósteres / Pausa (café)
12:00 Conferencia
13:00 Conferencia
14:00 Pausa (comida)
16:00 Curso
17:30 Curso

Martes, 21/09
9:00 Curso
10:30 Conferencia
11:30 Sesión de pósteres / Pausa (café)
12:30 Curso
14:00 Pausa (comida)
16:00 Curso
17:30 Conferencia
18:30 Pausa (café)
19:00 Conferencia premio Joven Investigador
19:20 Conferencia premio Divulgación Matemática

Miércoles, 22/09
9:00 Curso
10:30 Curso
12:00 Sesión de pósteres / Pausa (café)
12:30 Curso
14:00 Pausa (comida)
16:00 Curso
17:30 Conferencia
18:30 Pausa (café)
19:00 Asamblea SEMA
21:00 Acto social: cena

Jueves, 23/09
Actos sociales.
Se prevé una visita a bodega y asistencia al espectáculo de la Real Escuela Ecuestre de Jerez de la Frontera.

Viernes, 24/09
9:00 Curso
10:30 Conferencia
11:30 Pausa (café)
12:00 Curso
13:30 Acto de clausura

La Asamblea de la SEMA tendrá lugar con ocasión de la celebración de la Escuela, pero no forma parte de la misma.

Figura 14: Cartel de la EHF2004

Esta undécima Escuela fue organizada en Cádiz por F. Ortegón y sus colaboradores J.M. Díaz, R. Rodríguez, M.V. Redondo, C. García y M.T. González.

Junto a la parte científica de la Escuela, una excursión a Jerez de la Frontera nos permitió visitar una bodega, asistir al espectáculo de la célebre Real Escuela Ecuestre y visitar el Museo Taurino; además, la cena de gala en el Parador Atlántico nos permitió degustar las especialidades locales.

El formato general de esta Escuela fue análogo a la de las cuatro o cinco precedentes: cuatro cursos de unas cuatro horas cada uno permitieron presentar de forma detallada los retos y los resultados más significativos de cuatro disciplinas de actualidad en simulación numérica y seis conferencias permitieron analizar sucintamente el progreso de otros tantos temas de actualidad.

5.11.1 Programa

Se impartieron los cuatro cursos siguientes:

- *Modelos matemáticos en los procesos industriales*, A. Bermúdez de Castro (Univ. de Santiago de Compostela), el curso se apoyó en dos ejemplos concretos: el primero, la combustión en una central térmica y el segundo, la reducción del ruido en recintos cerrados mediante técnicas *anti-ruido*
- *Generación de mallados tridimensionales*, P.L. George (INRIA Rocquencourt), presentación de los principales métodos de generación de mallados tridimensionales que permiten responder de manera fiable y eficaz a las necesidades industriales, particularmente en términos de rapidez y de tamaño de los mallados tridimensionales generados
- *Actuar a favor del medio ambiente: reflexiones generales y análisis matemático de dos problemas concretos*, J.I. Díaz (Univ. Complutense de Madrid y Real Academia de Ciencias), consecuencias de la revolución industrial en la evolución del clima e ilustración mediante algunos ejemplos que conducen a estudios de controlabilidad
- *Problemas matemáticos y numéricos planteados por la seguridad de los residuos nucleares*, O. Pironneau (Univ. París VI y Académie des Sciences), presentación de herramientas matemáticas apropiadas: desarrollos multi-escala, los métodos de descomposición de dominios y de homogeneización, así como los métodos numéricos correspondientes

Los cursos se completaron con las conferencias:

- *Esquemas numéricos con dos mallas para las ecuaciones de Navier-Stokes*, V. Girault (Univ. París VI)
- *Cálculo paralelo para la resolución numérica de las ecuaciones de Navier-Stokes*, E. Fernández Cara (Univ. de Sevilla)
- *Resolución numérica de las ecuaciones de aguas someras bicapa mediante esquemas de volúmenes finitos*, M.J. Castro (Univ. de Málaga)

- *Esquemas de volúmenes finitos y teorema de Lax*, B. Desprès (Univ. París VI)
- *Simulación de procesos: « digital factory »*, A. Montosa (Airbus España)
- *Modelos matemáticos para el tratamiento de imágenes*, J.M. Mazón (Univ. de Valencia)

Además, tres sesiones de pósteres permitieron a los más jóvenes presentar sus primeros resultados.

El conjunto de las notas de los cursos y las conferencias fue recopilado en un volumen de excelente factura. Este volumen y su contenido fueron objeto de recensión « Mathematical Reviews » bajo la referencia MR2117095 (2005h :00013) 00B15.

5.11.2 Participantes

Esta Escuela reunió a 84 participantes, incluyendo profesores y conferenciantes, procedentes en su mayor parte del sector universitario español, doctorandos, profesores-investigadores jóvenes o experimentados. Estos asistentes procedían de las Universidades de Cádiz (13), Cantabria (1), Córdoba (4), La Coruña (5), Madrid (3, 1 de la Autónoma, 1 de Carlos III y 1 de Complutense), Málaga (10), Navarra (3), Oviedo (2), Santiago de Compostela (10), Sevilla (17), Valencia (1), Valladolid (1), Vigo (3) y Zaragoza (1); de AIRBUS España (1); del INRIA (2) y de la Universidad París VI (3); de la Universidad de Campinas de Brasil (1), de la Universidad Autónoma de México (1) y de la Universidad de Tetuán (2).

5.12 XII Escuela: Castro Urdiales, del 18-9 al 22-9 de 2006

Este año 2006 fue excepcional para la comunidad matemática española, ya que España organizó en agosto el Congreso general de la Unión Matemática Internacional. Se trata del Congreso mayor de los matemáticos, que tiene lugar cada cuatro años, en el transcurso del cuál se entregan las Medallas Fields. Junto al Congreso general, se organizaron un gran número de encuentros, reuniones de trabajo, seminarios, etc. centrados sobre temas concretos que reunían a los especialistas de una misma disciplina. El conjunto de estos eventos, empezando por el Congreso general, tuvieron un gran éxito, tanto por la calidad de las intervenciones como por una organización muy notable.

La organización de la duodécima Escuela en Castro Urdiales menos de un mes después del Congreso de la UMI y de los eventos asociados era un auténtico desafío. Debido a este hecho, la Escuela experimentó una sensible disminución del número de participantes, pero la calidad de la organización y el calor de la acogida hicieron de ella un evento muy logrado y muy apreciado.

Recordemos que esta Escuela fue organizada por la Universidad del País Vasco, bajo la responsabilidad de L. Vega, M. Escobedo y M. Lezaun asistidos por L. A. Fernández de la Universidad de Cantabria. La Escuela tuvo lugar en la ciudad balneario de Castro Urdiales a unos 40 km al oeste de Bilbao,

Figura 15: Cartel de la EHF2006

más precisamente en el recién estrenado Centro Internacional de Estudios Matemáticos, instalado en una soberbia residencia perfectamente restaurada y muy funcional.

Además de las actividades científicas, los organizadores reservaron media jornada para hacernos descubrir el Puente Colgante de Portugalete que, desde hace más de cien año, asegura el traslado de vehículos y peatones de una orilla a otra de la ría. Después nos condujeron al célebre e ineludible Museo Guggenheim de Bilbao, cuya audaz arquitectura ha contribuido fuertemente al renacimiento de una ciudad marcada por delicadas reestructuraciones industriales.

5.12.1 Programa

Esta Escuela contó con cuatro cursos de tres horas cada uno:

- *Integración geométrica*, J. Sanz Serna (Univ. de Valladolid)
- *Optimización de forma y de topología*, G. Allaire (Escuela Politécnica, París)
- *Modelos y algoritmos para problemas de localización y rutas*, E. Fernández Aréizaga (Univ. Politécnica de Cataluña)
- *Una introducción a la modelización matemática de flujos bifásicos dispersos*, P. Villedieu (ONERA, Univ. Paul Sabatier, Toulouse)

Las siguientes ocho conferencias de una hora cada una complementaron los cursos:

- *Estimaciones del error para la aproximación numérica de problemas de control gobernados por EDPs*, M. Mateos (Univ. de Oviedo)
- *Análisis matemático y simulación numérica del ensayo de flexión en tres puntos para materiales cerámicos*, P. Quintela (Univ. Santiago de Compostela)
- *De la Mecánica a la Biología mediante procesos de fragmentación: estado de la cuestión y problemas abiertos*, J. Soler (Univ. de Granada)
- *Blow-up y problemas parabólicos mal puestos*, A. Rodríguez Bernal (Univ. Complutense de Madrid)
- *Ecuaciones con solitones: sinergia entre estudio analítico y simulación*, A. Sánchez (Univ. Carlos III de Madrid)
- *Preservación de invariantes mediante métodos Runge-Kutta*, L. Rández (Univ. de Zaragoza)
- *Métodos de bases reducidas para la aproximación fiable de la solución de problemas no lineales*, Y. Maday (Univ. París VI)

- *Simulación numérica de la interacción sangre-venas en arterias largas*, M. Fernández (INRIA Rocquencourt)

Todos los profesores y conferenciantes pusieron a disposición de los participantes soportes (en papel y electrónico) directamente relacionados con el contenido de su presentación e incluyendo una bibliografía muy completa y actualizada sobre los temas abordados.

5.12.2 Participantes

Los participantes procedían de las Universidades de Cantabria (1), Castilla-La Mancha (1), Córdoba (2), La Coruña (3), Granada (1), Madrid (3, 2 de Carlos III y 1 de la Complutense), Oviedo (2), País Vasco (14), Politécnica de Cataluña (1), Santiago de Compostela (4), Sevilla (4), Valladolid (1), Vigo (1), Zaragoza (2); del Consejo Superior de Investigaciones Científicas (1) ; de la Escuela Politécnica de Palaiseau (1), de la Universidad París VI (1), del INRIA Rocquencourt (1), del ONERA y la Universidad Paul Sabatier (1) y de la Escuela Superior de Ingenieros Leonardo da Vinci (1).

El nivel de los participantes fue muy variando, yendo desde doctorandos a profesores-investigadores con muy distinto nivel de experiencia. Un tercio de entre ellos al menos participa personalmente en colaboraciones regulares con equipos franceses (París VI, INRIA, Escuela Politécnica, Universidad Paul Sabatier...).

5.12.3 Conclusiones

La participación en esta Escuela, por una parte, de los profesores J.I. Montijano y C. Vázquez, presidentes saliente y entrante de SeMA respectivamente, y por otra parte de los profesores Y. Maday y G. Allaire, presidentes de la SMAI (Société de Mathématiques Appliquées et Industrielles) y del Gamni/SMAI (Groupe pour l'Avancement des Méthodes Numériques de l'Ingénieur) respectivamente, permitieron discutir sobre el futuro de estos eventos.

A partir de 2008, las Escuelas serán organizadas por SeMA del lado español y por la SMAI del lado francés, lo que introducirá una mayor simetría y aumentará la visibilidad. La Escuela de 2008 será organizada localmente por la Universidad de Valladolid (`www.uva.es/ehf2008`).

Título:	Influencia del Potencial de Hardy en ecuaciones Elípticas y Parabólicas.
Doctorando:	Ana Primo Ramos.
Director/es:	Ireneo Peral Alonso.
Defensa:	19 de mayo de 2008, Universidad Autónoma de Madrid.
Calificación:	Sobresaliente cum laude por unanimidad.

Resumen:

Los protagonistas de la Tesis son el clásico potencial de Hardy y los términos de orden cero y uno que son potencias de la incógnita o del módulo de su gradiente respectivamente. Todos estos términos actúan en la Ecuación del Calor y la Ecuación de Laplace. Los problemas que se tratan estudian cómo cooperan o compiten el potencial de Hardy con dichas potencias. Es un *perfecto laboratorio* para estudiar algunos casos límite y descubrir fenómenos nuevos. Muchos de los resultados se pueden extender a otros operadores.

Debido a su gran interés físico se considera con preferencia el siguiente problema

$$(P)\quad \begin{cases} u_t - \Delta u = |\nabla u|^p + \lambda \dfrac{u}{|x|^2} + f(x,t) & \text{en } \Omega_T \equiv \Omega \times (0,T), \\ u(x,t) > 0 & \text{en } \Omega_T, \\ u(x,t) = 0 & \text{en } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{si } x \in \Omega, \end{cases}$$

que representa una aproximación viscosa a la ecuación de Hamilton-Jacobi. Se considera $\lambda > 0$, $f(x,t) > 0$, $u_0(x) \geq 0$ funciones en clases convenientes y Ω un dominio acotado de $\mathbb{R}^N$ con $N \geq 3$ y $0 \in \Omega$.

En el caso $l = 0$ y $p = 2$, este modelo aparece en la teoría física del crecimiento de rugosidades en superficies por deposición de partículas por gravedad, donde es conocido como modelo de Kardar-Parisi-Zhang. Una modificación de este problema ha sido considerado por H. Berestycki, S. Kamin, G. Sivashinsky, como un modelo para la propagación de llamas. El caso $1 < p < 2$ es conocido como modelo generalizado de Kardar-Parisi-Zhang. Tanto en el caso estacionario como en el de evolución, si $l = 0$ y $p = 2$, B. Abdellaoui, A. Dall'Aglio e I. Peral, demuestran un resultado de no unicidad de soluciones positivas, dando la caracterización de todas las soluciones en relación a medidas concentradas en conjuntos de capacidad cero. Este fenómeno de no unicidad *salvaje* es producido por el término $|\nabla u|^2$.

Con estos resultados como motivación, se estudia en primer lugar el caso estacionario del problema (P), analizando la influencia conjunta del potencial de Hardy en problemas que tienen un término $|\nabla u|^2$. Si el cuadrado del gradiente aparece en el lado derecho de la ecuación (compitiendo con el operador de Laplace), se demuestra un resultado de no existencia de solución positiva en el sentido más débil posible, es decir, exigiendo las condiciones mínimas necesarias para que la ecuación tenga significado en el sentido de las distribuciones. Como consecuencia de este resultado tan fuerte de no existencia, se produce un fenómeno de explosión de las soluciones de los problemas aproximados (truncando el potencial de Hardy) en todo punto del dominio (el caso de no existencia para $1 < q < 2$ es analizado por B.Abdellaoui e I. Peral en Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. VI (2007), 159-183).

Si el cuadrado del gradiente aparece en el lado izquierdo de la ecuación (cooperando con el operador de Laplace), el resultado es completamente diferente. Se prueba la existencia de solución positiva para todo parámetro λ, sin necesidad de restringir su tamaño y para toda función positiva f en $L^1(\Omega)$, sin imponer condiciones extras de integrabilidad.

El fenómeno de rotura de resonancia y efecto regularizante se analiza en pesos más generales y con una potencia $|\nabla u|^q$, $1 < q < 2$, llamados en la memoria pesos admisibles, entre los que, en particular, se encuentra el potencial de Hardy. Un dato importante: se prueba la optimalidad de la condición impuesta a estos pesos admisibles, utilizando de nuevo el potencial de Hardy como laboratorio de pruebas. Extendemos estos resultados a operadores en forma divergencia, con funciones más generales que dependen de la incógnita y de su gradiente y con datos en los que no imponemos condiciones de signo.

Para analizar la influencia del potencial de Hardy en el caso de evolución, se parte del estudio de la sumabilidad de las soluciones de la Ecuación del Calor con respecto a la sumabilidad del término fuente, completando algunos resultados clásicos.

Al perturbar con el potencial de Hardy, se prueba que las supersoluciones en general no están acotadas y por tanto, el estudio se sale de marcos clásicos. Se estudia en primer lugar el problema semilineal tomando la potencia crítica optimal para el problema elíptico asociado obtenida por Brezis-Dupaigne-Tesei. Se comprueba que esta misma potencia es la crítica para determinar la existencia o la no existencia de solución positiva en la ecuación del calor semilinear perturbada con el potencial de Hardy. El análisis del problema de Cauchy asociado, permite obtener el exponente de Fujita que depende del parámetro λ.

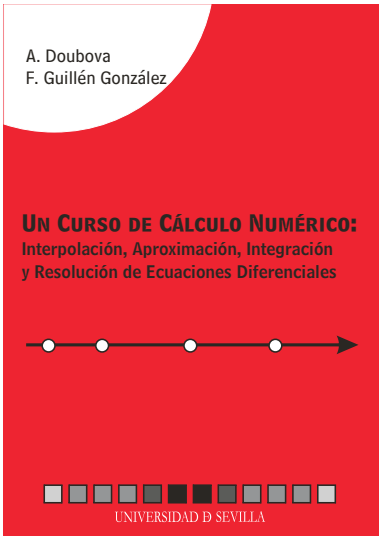
El problema (P) es particularmente interesante. Por un lado, plantea serias dificultades técnicas, apareciendo de forma natural la necesidad de demostrar un principio de comparación que en particular extiende un resultado de Alaa-Pierre para el caso elíptico asociado. Por otro lado, se hallan profundas diferencias con respecto a la ecuación del calor. Si $\lambda > 0$, el exponente crítico obtenido por B. Abdellaoui e I. Peral es también el exponente crítico para el problema parabólico casi-lineal, en el sentido que a partir de él no hay solución positiva para ningún dato inicial no trivial. Sin embargo, para $\lambda = 0$, bajo ciertas hipótesis en el

dato incial, B. Gilding, M. Guedda, R. Kersner demuestran un resultado de existencia global de solución para todo $p > 1$.

Si $\lambda > 0$, en el problema de Cauchy ($\Omega = \mathbb{R}^N$) existe un exponente de tipo Fujita tal que por debajo de él todas las soluciones no negativas explotan en tiempo finito. Sin embargo, para $\lambda = 0$, existe solución para dato no trivial. Los resultados son bastante sorprendentes, además hay una discontinuidad de comportamiento cuando $\lambda \to 0^+$, a diferencia del caso semilineal.

Un Curso de Cálculo Numérico: Interpolación, Aproximación, Integración y Resolución de Ecuaciones Diferenciales.
Anna Doubova y Francisco Guillén González
Secretariado de Publicaciones, Universidad de Sevilla.
ISBN: 978-84-472-0941-5 – 2007

Por Enrique Fernández Cara



Este texto de Cálculo Numérico está orientado a la docencia universitaria en Ciencias e Ingeniería. Los autores dedican una parte importante al Análisis Numérico (planteamiento riguroso de los problemas, convergencia, estimaciones del error, etc.) y presentan también algoritmos eficientes de cálculo, numerosos ejemplos y ejercicios resueltos, así como una relación de ejercicios propuestos al final de cada materia.

La obra consta de cinco capítulos. En los Capítulos 1 y 2 se desarrollan técnicas de interpolación polinómica (global, a trozos y con splines) y de aproximación de funciones (en el sentido de la convergencia uniforme y en el sentido de los mínimos cuadrados, incluyendo el caso discreto de aproximación de un conjunto de puntos en el plano).

En el Capítulo 3 se estudia la integración numérica en una variable (de nuevo con métodos globales y a trozos), más concretamente las fórmulas de cuadratura de tipo interpolatorio, el caso particular de las fórmulas de Newton-Côtes y las fórmulas de cuadratura de Gauss con soporte óptimo.

Finalmente, los Capítulos 4 y 5 están dedicados al estudio de esquemas numéricos para aproximar soluciones de problemas diferenciales ordinarios: problemas de Cauchy en el Capítulo 4 y problemas de contorno en el Capítulo 5. En el caso de los problemas de Cauchy, están recogidos los

resultados analíticos más importantes de los esquemas de un paso (consistencia y orden, estimaciones del esquema, convergencia y estimaciones del error de discretización y estabilidad). En el caso de problemas de contorno, tan sólo se realiza una introducción, desde el punto de vista de la implementación numérica, a algoritmos de dos tipos: los métodos de disparo y los métodos de diferencias finitas.

En conjunto, el libro pretende proporcionar una visión unificada y equilibrada de resultados propios del Análisis Numérico, complementados con el estudio detallado de un buen número de algoritmos.

Tipo de evento:	Congreso
Nombre:	International Workshop on Dynamical Systems and Multidisciplinary Applications (IWDSMA)
Lugar:	Universidad Miguel Hernández de Elche
Fecha:	September 16–19, 2008
Organiza:	
Información:	
E-mail:	dsma@umh.es
WWW:	http://hefesto.umh.es/IWDSMA

Tipo de evento:	Congreso
Nombre:	13th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic, and Verified Numerical Computations (SCAN'2008)
Lugar:	El Paso, TEXAS, USA
Fecha:	September 29–October 3, 2008
Organiza:	
Información:	
E-mail:	mceberio@cs.utep.edu, vladik@utep.edu
WWW:	http://www.cs.utep.edu/interval-comp/scan08.html

Tipo de evento:	Congreso
Nombre:	II Encuentro Ibérico de Matemáticas
Lugar:	Badajoz. Facultad de Ciencias de la Universidad de Extremadura
Fecha:	Del 3 al 5 de octubre de 2008
Organiza:	Sociedad Matemática Portuguesa (SPM), Real Sociedad Matemática Española (RSME), Departamento de Matemáticas de la Universidad de Extremadura
Información:	
E-mail:	imm2@unex.es
WWW:	http://imm2.unex.es/indexes.html

Tipo de evento:	Congreso
Nombre:	10th International Conference on Information and Communications Security (ICICS'08)
Lugar:	Birmingham, United Kingdom
Fecha:	October 20–22, 2008
Organiza:	School of Computer Science, University of Birmingham, in co-operation with HP Laboratories (Bristol, UK), the UK Engineering and Physical Sciences Research Council (EPSRC), and the International Communications and Information Security Association (ICISA).
Información:	
E-mail:	a.j.brown@cs.bham.ac.uk
WWW:	http://events.cs.bham.ac.uk/icics08

Tipo de evento:	Congreso
Nombre:	International Conference in Modeling Health Advances 2008
Lugar:	San Francisco, California, USA
Fecha:	October 22–24, 2008
Organiza:	
Información:	
E-mail:	wcecs@iaeng.org
WWW:	http://www.iaeng.org/WCECS2008/ICMHA2008.html

Tipo de evento:	Congreso
Nombre:	International Conference on Modeling, Simulation and Control (ICMSC'08)
Lugar:	San Francisco, USA
Fecha:	October 22–24, 2008
Organiza:	International Association of Engineers
Información:	
E-mail:	wcecs@iaeng.org
WWW:	http://www.iaeng.org/WCECS2008/ICMSC2008.html

Tipo de evento:	Congreso
Nombre:	Second Workshop on Mathematical Cryptology (WMC 2008)
Lugar:	Universidad de Cantabria, Santander
Fecha:	October 23–25, 2008
Organiza:	
Información:	
E-mail:	jaime.gutierrez@unican.es
WWW:	http://grupos.unican.es/amac/wmc-2008

Tipo de evento:	Congreso
Nombre:	Elliptic and Hyperbolic Equations on Singular Spaces
Lugar:	Mathematical Sciences Research Institute, Berkeley, California, USA
Fecha:	October 27–31, 2008
Organiza:	
Información:	
E-mail:	jz@msri.org
WWW:	http://www.msri.org/calendar/workshops/WorkshopInfo/444/show_workshop

Tipo de evento:	Congreso
Nombre:	Second SIAM Conference on Financial Mathematics and Engineering
Lugar:	Campus of Rutgers, The State University of New Jersey, at the Heldrich Hotel, New Brunswick, NEW JERSEY, USA
Fecha:	November 21–22, 2008
Organiza:	
Información:	
E-mail:	finmath@rci.rutgers.edu; rcarmona@princeton.edu
WWW:	http://www.siam.org/meetings/fm08/index.php; http://www.finmath.rutgers.edu/conferences/

Tipo de evento:	Congreso
Nombre:	International Conference on Partial Differential Equations and Applications in honour of Professor Philippe G. Ciarlet's 70th birthday
Lugar:	City University of Hong Kong, Kowloon, Hong Kong
Fecha:	December 5–8, 2008
Organiza:	
Información:	
E-mail:	mclbj@cityu.edu.hk
WWW:	http://www6.cityu.edu.hk/rcms/ICPDEA2008

Tipo de evento:	Congreso
Nombre:	FEMTEC 2008 (Finite Element Methods in Engineering and Science)
Lugar:	University of Texas at El Paso, USA
Fecha:	December 8–12, 2008
Organiza:	
Información:	
E-mail:	femtec2008@math.utep.edu; solin@utep.edu
WWW:	http://servac.math.utep.edu/femtec_2008/home

Tipo de evento:	Congreso
Nombre:	2008 International Conference on Computer Science and Software Engineering (CSSE 2008)
Lugar:	Wuhan, China
Fecha:	December, 12-14th, 2008
Organiza:	Y. Wang, University of Calgary, Canada; P. Shen, University of California at Irvine, USA; H. B. Zhou, Wuhan University, China
Información:	
E-mail:	csse@highsci.org
WWW:	http://www.highsci.org/csse2008submission/website/csse/index.aspx

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