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sumario

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Estimados socios,

Si bien en el Boletín anterior abríamos este editorial con una referencia ilusionada ante la celebración del XX CEDYA y el X CMA, ahora no podemos dejar de congratularnos todos, una vez concluido el encuentro, por el resultado obtenido a todos los niveles. Dirigimos una calurosa felicitación a nuestros colegas de Sevilla por el trabajo, el esfuerzo y la ilusión que han puesto en la preparación de nuestro congreso. Hemos empezado ya a trabajar en ese volumen monográfico que os anunciábamos con los contenidos científicos del encuentro.

Pero es que además tenemos un motivo adicional para estar contentos y agradecidos, al saber que la organización del próximo CEDYA en 2009 ha recaído sobre nuestro departamento en Castilla-La Mancha. Estamos muy ilusionados por lo que esta designación por parte de S$\vec{e}$MA supone para nuestro departamento y nuestros grupos de trabajo y esperamos no defraudar la confianza depositada.

En relación con los contenidos de este número, la reciente adjudicación del premio Julio Rey Pastor de Matemáticas y Tecnologías de la Información y las Comunicaciones en su edición 2007 a nuestro compañero Enrique Zuazua nos motivó para realizar la entrevista con la que abrimos este boletín. Luego publicamos dos interesantes trabajos de F.J. Sayas y C. Cascante, X. Massaneda, J. Ortega-Cerdà y D. Pascuas, que son acompañados por la primera parte de las conferencias presentadas en el primer "Workshop Iberoamericano de Matemáticas Aplicadas" celebrado en la ciudad de Chillán (Chile) en Agosto de este año. Finalmente presentamos también el trabajo de nuestro compañero José Ramón Fernández, ganador del Premio S$\vec{e}$MA al Joven Investigador 2007.

Esperamos que disfrutéis de los contenidos de este número. Recibid un cordial saludo.

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Queridos socios y socias de SëMA: Quiero aprovechar el último número de 2007 del Boletín para dirigirme a todos vosotros y referirme a algunas novedades especiales relacionadas con la Sociedad.

En primer lugar os informo de que, tras seis años de pertenencia al Consejo Ejecutivo, abandonan el mismo Mikel Lezaun, también hasta hace poco Vicepresidente de la Sociedad, y Enrique Zuazua. Desde SëMA les enviamos un sincero agradecimiento por la calidad de su trabajo y las aportaciones realizadas para la mejora de nuestra Sociedad. Por otra parte, quiero felicitar por su elección a los nuevos miembros del Consejo Ejecutivo, Luis Vega (Universidad del Pais Vasco) y Sergio Amat (Universidad de Cartagena), así como a los que han renovado, Carlos Castro (Universidad Politécnica de Madrid)) y José Antonio Carrillo (ICREA). Recientemente, Carlos Castro ha renovado como Secretario y Rosa Donat ha pasado a ser la nueva Vicepresidenta.

También se han producido renovaciones en el Comité Científico del Boletín, que abandonan José Luis Cruz (Universidad de Córdoba) y Eduardo Casas (Universidad de Cantabria), a los que agradecemos las tareas realizadas en este cometido. Se incorporan como nuevos miembros, Carlos Conca (Universidad de Chile), Amadeus Delshams (Universidad Poltécnica de Cataluña), Martin Gander (Universidad de Ginebra), Vivette Girault (Universidad de París VI), Arieh Iserles (Universidad de Cambridge), Stanley Osher (UCLA), Benoît Perthame (Universidad de París VI), Olivier Pironneau (Universidad de París VI) y Alfio Quarteroni (EPF Lausanne), cuyas aportaciones seguramente redundarán en beneficio del Boletín.

En 2007 la SëMA ha apoyado un buen número de actividades, a cuyos organizadores agradecemos el buen trabajo realizado y el propiciar la difusión de la Sociedad dentro de los distintos eventos. En particular, por la relevancia del número de participantes, destacamos el Congreso Hispano-Francés de Matemáticas, celebrado en julio en Zaragoza, organizado en conjunto con la RSME y SMF, y del que ha sido responsable principal Alberto Elduque.

También en 2007 se ha celebrado el XX CEDYA—X CMA, organizado por el Departamento de Ecuaciones Diferenciales y Análisis Numérico de la Universidad de Sevilla, que ha alcanzado un éxito total en cuanto a la organización, la calidad de los conferenciantes invitados, de las sesiones especiales y de las contribuciones. Desde SëMA enviamos nuestro sincero agradecimiento y felicitación a las personas involucradas en las tareas relativas a la organización del evento, coordinadas por Enrique Fernández Cara. Como muchos sabéis, durante la celebración se informó de que la Universidad de Castilla La Mancha acogerá en 2009 el XXI CEDYA-XI CMA.

En 2008 SëMA seguirá apoyando un buen número de eventos relacionados con la Matemática Aplicada. En particular, por su tradición, os informo de que

la Escuela Hispano-Francesa de 2008 se celebrará en Valladolid del 15 al 19 de Septiembre, encargándose de ella el Departamento de Matemática Aplicada, con la novedad de involucrar también en la organización a la SMAI francesa.

En el apartado de Premios de S$\vec{e}$MA, ya publicamos en este número del Boletín las bases de la convocatoria del IX Premio a la Divulgación de la Matemática Aplicada y del XI Premio al Joven Investigador.

Finalmente, deseo invitaros a todos a participar activamente en S$\vec{e}$MA, aportando ideas y posibles mejoras, pues ello será de gran ayuda para el progreso y desarrollo de nuestra Sociedad.

Un abrazo,

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Presidente de S$\vec{e}$MA
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Presentación

Enrique Zuazua Iriondo nació en Eibar (Guipúzcoa) en 1961. Se licenció en Ciencias Matemáticas por la UPV-EHU en 1984, donde obtuvo posteriormente el doctorado en 1987. En 1988 se doctoró por la Universidad Pierre et Marie Curie, siendo financiados sus estudios por una beca del Gobierno Vasco y una "Allocation de Recherche" de la Cátedra de Jacques Louis Lions en el "Collège de France". En 1990 obtuvo una Cátedra de Matemática en la Universidad Complutense de Madrid y en 2001 se trasladó a la Universidad Autónoma de Madrid. En la actualidad dirige el Instituto IMDEA-Matemáticas, fundación recientemente creada por el Gobierno de la Comunidad Autónoma de Madrid con el objeto de promover la investigación en los aspectos más computacionales, aplicados y multidisciplinares de las Matemáticas.

Es miembro del comité científico de diversos institutos como el CIM (Centro Internacional de Matemáticas) de Coimbra, Centre International de Mathématiques Pures et Appliquées de Niza, el Centre International de Recontres Mathématiques de Luminy, el CERFACS de Toulouse y el Centro de Ciencias Pedro Pascual de Benasque. Ha sido gestor del Programa de Matemáticas del Plan Nacional (2001-2004), y ha dirigido y participado en diversos paneles internacionales del CNRS francés, la DFG alemana, y la European Research Council entre otras agencias, además de investigador principal de proyectos del Plan Nacional y coordinador de nodos de proyectos europeos y de la OTAN desde 1990. En la actualidad es investigador y coordinador del proyecto SIMUMAT de la Comunidad de Madrid e INGENIO MATHEMATICA – i-Math del Programa Consolider-Ingenio 2010. Por último, cabe resaltar que ha sido profesor visitante, entre otros, del Courant Institute en Nueva York, de las Universidades de Minnesota y Rice en los EE.UU., de la Universidad Federal de Río de Janeiro y de diversas Universidades francesas, además de profesor a tiempo parcial de la Ecole Polytechnique de Paris entre 1999 y 2002.

Es editor en jefe de la revista *ESAIM:COCV*, y editor correspondiente del *SIAM Journal on Control and Optimization*, entre otros comités editoriales.

Sus campos de especialización abarcan las Ecuaciones en Derivadas Parciales, el Control de Sistemas y el Análisis Numérico. Su obra ha tenido

una importante repercusión habiendo sido reconocido como "Highly Cited Researcher" por el Instituto ISI (Thomson) en 2004. Recientemente ha sido galardonado con el Premio Euskadi de Ciencia y Tecnología en su edición 2006, y el Premio Julio Rey Pastor de Matemáticas y Tecnologías de la Información y las Comunicaciones en su edición 2007. Como miembro de la S$\vec{e}$MA participa en el comité científico de su boletín y hasta hace poco, ha formado parte de su comité ejecutivo. Además ha sido distinguido en dos ocasiones (2000 y 2003) con el premio S$\vec{e}$MA a la divulgación en Matemática Aplicada.

El premio

B.S.: *¿Qué ha supuesto este reconocimiento tan importante de tu trabajo y de tus méritos?*

E.Z.: Ha sido un gran motivo de satisfacción para la familia, mi círculo de colaboradores más cercanos y también para mi, por supuesto. Lo hemos entendido más como un estímulo para seguir trabajando que como el cierre de un ciclo.

B.S.: *¿Lo entiendes como un mérito más, o lo ves como un reconocimiento especial?*

Sin duda se trata de un reconocimiento especial en muchos aspectos. En primer lugar es un premio concedido en el ámbito nacional y abarcando disciplinas diversas como son las Matemáticas, la Informática y las Telecomunicaciones. Resulta pues muy grato y halagador ser el elegido en esta edición. Se trata además de un premio que reconoce la labor de investigación, tarea a la que me he dedicado de manera intensa durante más de veinte años y constituye una generosa recompensa. Como decía, es además un estímulo muy fuerte y en gran medida inesperado para seguir trabajando.

El trabajo

B.S.: *No es necesario aclarar los campos concretos de trabajo de Enrique Zuazua pues casi todos los socios lo conocen. ¿Podrías indicar un campo más concreto del que te sientes especialmente satisfecho?*

E.Z.: A mi me gusta particularmente el trabajo que hemos realizado en los últimos años en la interacción entre el control de ecuaciones de ondas y otros modelos de EDP's y su aproximación numérica que se recoge en un trabajo publicado en el 2005 en el *SIAM Review* y en las actas del ICM2006. Más recientemente, en colaboración con Carlos Castro y Francisco Palacios, hemos propuesto un algoritmo que resulta muy eficaz para el control de leyes de conservación escalares en presencia de choques, tema importante y que creo que será objeto de relevantes desarrollos en los próximos años. En este último trabajo se combinan ideas

y técnicas de diversos ámbitos y se dan respuestas de utilidad en varias aplicaciones, en particular en el ámbito del diseño inverso en aeronáutica. Estamos pues muy satisfechos por estos resultados y por la acogida que está teniendo en nuestra comunidad.

B.S.: *De las nuevas tendencias o campos dentro de la Matemática Aplicada ¿cuáles te parecen los más importantes por su relevancia o su aplicabilidad de cara al futuro más inmediato? Desde un punto de vista más personal ¿por cuáles te sientes especialmente atraído en este momento para trabajar en los próximos años?*

E.Z.: Desde mi punto de vista, a pesar de los importantísimos avances que se han producido tanto en las Ecuaciones en Derivadas Parciales como del Análisis Numérico en los últimos 50 años queda aún mucho por hacer en la intersección de estos dos campos. Por ejemplo un análisis cuidadoso sobre la capacidad de los métodos numéricos para, más allá de su convergencia entendida en el sentido clásico, reproducir las propiedades cualitativas finas de las EDP's (estabilidad, singularidades, problemas inversos,...) me parece que es un campo que ha de ser estudiado de manera sistemática. Por supuesto los problemas adquieren especial relevancia y se perfilan con más nitidez cuando se abordan desde un punto de vista más multidisciplinar. Nanociencias y Ciencias biomédicas son dos áreas que ya forman parte de las aplicaciones de las Matemáticas que se suman a otros ya más clásicos como la Física, las Ingenierías y la Química Computacional y en cada una de ellas se plantean problemas como los que acabo de mencionar. Por ejemplo, el trabajo sobre el control de choques en leyes de conservación que antes mencionaba está motivado por nuestro interés en el diseño de formas óptimas en aeronáutica y tiene mucho que ver con esa sutil interacción entre los métodos numéricos y las propiedades más dinámicas de las soluciones, en este caso en relación a su sensibilidad con respecto al cambio de valores de parámetros de control.

B.S.: *¿Crees que el trabajo del investigador en Matemáticas es suficientemente comprendido, reconocido, remunerado?¿Qué se puede hacer para mejorar en este sentido?*

E.Z.: Creo que, en general, el trabajo de los investigadores no se entiende bien ni se reconoce suficientemente por parte de nuestra sociedad. En lo que respecta a las remuneraciones ocurre algo semejante y sin duda los gestores de la Ciencia y los políticos que se ocupan de estas cuestiones tendrán que abordar el tema, pues resulta difícil de creer que se pueda formar parte de los países de élite en Ciencia si no se dispone de mecanismos para cuidar y atraer a los mejores investigadores. En lo que respecta al reconocimiento, las Matemáticas no están bien situadas en España y en muchos foros se las considera un área un tanto deficitaria aún sin consolidarse.

Poco a poco los matemáticos vamos teniendo más presencia pero hay elementos objetivos que nos hacen perder mucho peso específico. Entre ellos está el que las Matemáticas españolas tengan un factor de impacto por debajo de la media mundial. Esto contribuye a que desde otras áreas se nos considere como una Ciencia que no ha alcanzado los parámetros de normalidad y calidad necesarios para formar parte del club de los que han de influir de manera decisiva en el devenir de nuestra Ciencia. Creo que es un hecho a tener en cuenta y que la autocomplacencia que a todos nos resulta más cómoda no es una actitud que ayude a avanzar en esta dirección.

La Matemática en España

B.S.: *Mucho se ha hablado a raíz del ICM2006 sobre el empujón de la Matemática en España en los últimos años. ¿Qué falta todavía para seguir avanzando entre los primeros puestos?*

E.Z.: Como decía antes, los números son tozudos y colocan a las Matemáticas españolas por detrás de otras áreas en lo que respecta a nuestro impacto con respecto a los países líderes mundiales. Parece indispensable avanzar de manera decidida en esa dirección. Algunos países como Chile, con políticas decididas de apoyo a la investigación de mayor calidad han conseguido avances más notables que los nuestros en la última década. Todos somos conscientes del efecto benéfico que pueden tener políticas adecuadas. Ocurrió hace veinte años con los sexenios, por ejemplo, que fueron haciendo que la cultura de la publicación internacional empapara nuestra comunidad. Se echa en falta una revisión y renovación de esas políticas. Como decía antes, es mucho más popular considerar y enviar el mensaje de que ya hemos llegado a puerto y que es hora de repartir beneficios. Pero, como decía, los números son tozudos y tenemos aún mucho camino por delante.

Hay sin embargo otros ámbitos de acción en los que se están consiguiendo importantes éxitos. Cabe mencionar la reciente creación de algunos institutos de referencia como el Instituto Mixto del CSIC con las Universidades madrileñas (ICMat) o el Instituto IMDEA-Matemáticas en Madrid, así como el éxito del proyecto i-Math en la convocatoria CONSOLIDER. Se trata de herramientas que contribuirán a cubrir las lagunas que antes mencionábamos. Pero aún nos queda mucho que aprender de otras áreas que cuentan con realizaciones de este calibre desde hace ya mucho tiempo. En concreto en el campus de Cantoblanco, donde trabajo, contamos con centros de Biotecnologías, de Biología Molecular y de Ciencias de Materiales, entre otros, que deberían servirnos de inspiración. Los matemáticos, por naturaleza, tenemos tendencia al trabajo individual o en grupos pequeños pero eso no debería ser

obstáculo para abordar retos más ambiciosos en cooperación. Las iniciativas que antes he mencionado son buenos ejemplos del factor multiplicador que tiene la cooperación entre diferentes agentes y equipos en empresas de envergadura. Son dinámicas que se están aplicando con éxito en los países de nuestro entorno entre los que Gran Bretaña, Francia, Alemania, son sólo algunos ejemplos. Para colocar hoy en día una ciudad, un campus, en el mapa internacional de las Matemáticas, hace falta concentrar talento y fomentar la cooperación. Eso a veces cuesta ser entendido pero es el único camino. Pero ni siquiera en el ámbito político las cosas están claras, con cambios demasiado frecuentes de equipos. La legislatura que ahora termina empezó bien, con un discurso muy rotundo de apoyo a la Ciencia, y hubo algunas iniciativas, como los CONSOLIDER, genuinamente novedosas. Pero ahora que llega a su fin está dejando una cierta sensación de frustración pues se percibe que faltó decisión para consolidar estas políticas. Tal vez sea la cercanía de las elecciones, pero los guiños actuales en la dirección opuesta a lo que son unos niveles altos de exigencia y unos usos de evaluación internacionales, aún cuando se conciben con el noble propósito de contentar al mayor número de gente posible, son incompatibles con abordar los retos que antes mencionábamos.

B.S.: *Más en concreto, parece que la Matemática Aplicada goza de muy buena salud en nuestro país en este momento. ¿Qué se puede hacer para avanzar más en este campo tan nuestro?*

E.Z.: Yo tengo tendencia a ser crítico, empezando por mi propio trabajo. Creo que es la mejor manera de asegurarnos de que el futuro será mejor que lo que dejamos atrás. No puedo más que tener esa misma mirada sobre la Matemática Aplicada. Estamos en condiciones de liderar ese salto cualitativo a la calidad pues contamos con una comunidad amplia pero a la vez lo bastante cohesionada para fomentar estrategias de comunicación y cooperación que nos permitan avanzar más rápidamente. El proyecto CONSOLIDER i-Math es en mi opinión un buen ejemplo de la aportación que la Matemática Aplicada puede hacer al conjunto de las matemáticas orientándolas a temas más multidisciplinares, más computacionales, en los que el impacto de lo realizado siempre es doble por su contenido matemático y su dimensión aplicada.

Con frecuencia nos cuesta reconocer este efecto multiplicador de las estrategias cooperativas y somos víctimas del personalismo excesivo, pero los éxitos recientes marcan con claridad el camino. El Congreso ICM06 fue otro buen ejemplo del que deberíamos aprender. Un grupo reducido de personas fue capaz de movilizar a toda una comunidad matemática para organizar el mejor Congreso Internacional de Matemáticas de la historia. Sin duda el mayor mérito es del Comité Organizador que Manuel de León encabezó con

maestría y creo que deberíamos sacar conclusiones que vayan mucho más allá de lo que fue la actividad del ICM06 en sí.

Nos hemos referido al esfuerzo individual de los científicos y a la necesidad de cooperación para amplificar el rendimiento. Las políticas ministeriales a las que hemos hecho también mención pueden contribuir diseñando herramientas que supongan lanzaderas hacia el éxito de nuestra ciencia. En esto ámbito, como decía, hemos vivido una legislatura un tanto agridulce en la que, a pesar del aumento de la inversión y la firme convicción inicial de nuestros gobernantes, los efectos concretos han sido escasos. De hecho los avances más notables nos han llegado muchas veces de las CC.AA. Parece haberse invertido la importancia de los agentes en la gestión de la Ciencia. Eso nada tiene de malo y es tal vez consecuencia natural del desarrollo del estado autonómico. Sí que resulta sin embargo sorprendente que el gobierno central parezca estar sumido en una permanente crisis, con cambios frecuentes de responsables y de políticas, y más inclinado al populismo cortoplacista que al diseño de políticas tan necesarias a medio y largo plazo. Confío que en la próxima legislatura se produzca una clarificación diáfana en este ámbito. La comunidad científica, salvo algunos elementos concretos que mantienen una dinámica de trabajo poco perturbable, es muy sensible a los golpes de timón. Conviene pues asegurarse que se dan en la buena dirección.

El futuro y los jóvenes

B.S.: *A juzgar por lo que sucede en los países de nuestro entorno, parece que los estudios universitarios de Matemáticas están cayendo en picado. Aunque éste es un fenómeno complejo, ¿qué reflexiones te merece este tema?*

E.Z.: Sin duda alguna nuestra licenciatura resulta costosa para los jóvenes. Se percibe como excesivamente difícil y tal vez no se conoce suficientemente el éxito que tienen los jóvenes licenciados en el mercado laboral. Son unas de las conclusiones de un valioso informe que acaba de publicarse por la ANECA y la RSME. Sin duda hay que redoblar los esfuerzos de difusión entre los jóvenes y revisar permanentemente nuestros currícula para hacerlos más adaptados a las necesidades actuales. Pero yo no soy partidario de diluir el espíritu y los contenidos centrales de nuestra licenciatura. Siempre suelo decir, tras haber visto ya unos cuantos planes de estudios, que el que yo tuve la suerte de cursar en Leioa entre los años 79 y 84 en el que le dedicábamos un curso entero a cuatro materias centrales de las Matemáticas como eran el Análisis, la Geometría, el Álgebra, las Probabilidades o las Ecuaciones Diferenciales, no me parece envidiar en nada a los actuales. Las Matemáticas han de ser

capaces de abrirse al mundo y adecuarse a las exigencias de nuestra sociedad actual sin perder su esencia. Si seguimos trabajando en esa doble dirección los jóvenes volverán a nuestras aulas aunque yo me preocuparía más por la capacidad y el entusiasmo de los que vienen que por su número. Desde luego no nos podemos conformar con la situación que se da con excesiva frecuencia de que los jóvenes con más interés en las Matemáticas y más capacidad acaben cursando otras carreras simplemente porque exigen una mayor nota de corte o porque les orientan a profesiones con más reconocimiento social. En ese sentido creo que la visibilidad que podamos dar a nuestra Ciencia y profesión puede ser muy útil. Es importante trasladar con convicción el mensaje de que la licenciatura de Matemáticas es la más adecuada para los jóvenes que disfrutan con las Matemáticas y que en ella podrán encontrar un futuro rico lleno de oportunidades intelectuales y laborales.

B.S.: *También es inquietante la deserción de jóvenes de valía hacia otros terrenos del saber o de la actividad profesional. ¿Qué principios o medidas generales podrían ayudar a resolver este problema?*

E.Z.: Se está haciendo un gran esfuerzo de difusión entre los más jóvenes. Hay iniciativas como las de ESTALMAT, o las que muchas Universidades adoptan de dar conferencias en los centros de secundaria de la región que están teniendo efectos positivos. Pero yo creo que los propios matemáticos tenemos que hacer un esfuerzo adicional para definir mejor los contornos de nuestra profesión, proyectarla hacia el futuro y comunicarla en nuestro entorno. Si tenemos una licenciatura y un futuro profesional delineado con nitidez será mucho más fácil transmitirlo y a atraer a los jóvenes con más talento.

B.S.: *¿Te aventuras a predecir el futuro a corto plazo de la Matemática española y de la Matemática Aplicada, en particular?*

E.Z.: Yo ya dije hace unos cuantos años que el impacto medio de la Matemáticas española bajaría en breve. Creo que este hecho se ha confirmado en este último año. Hemos bajado ligeramente el impacto medio en el último quinquenio del -7 al -8, lo cual quiere decir que nuestros artículos, en media, tienen un impacto menor en un 8 % que la media mundial, en términos de citas.

Debemos ser capaces de invertir esa tendencia. Para eso es importante que todos redoblemos esfuerzos. Los más senior, que gozamos de mayor estabilidad, apostando por temas de más calado y asumiendo los riesgos de abordar temas más aplicados y multidisciplinares en los que hay menos garantías de éxito y se vislumbran como menos rentables desde el punto de vista de la productividad en términos de publicaciones. Por otra parte es muy importante invertir en los más jóvenes, darles protagonismo pero

es también indispensable que ellos muestren ambición. Creo que en SëMA hemos sido cuidadosos en estos temas, sabiendo asumir responsabilidades y promoviendo la renovación generacional y el relevo. Debemos seguir haciéndolo. Confío en que esta política vaya llegando a otros foros que influyen también en la imagen e impacto global de nuestras Matemáticas. Mala sería la comunidad que, 25 años más tarde, presenta en su portada principal a los mismos agentes sin haber sabido variar sus roles. Muchas veces se suele decir que en Matemáticas no tenemos tradición y puede que sea eso lo que nos está ocurriendo. Veo con demasiada frecuencia a investigadores senior más preocupados en competir con los más jóvenes más que en desempeñar la labor de liderazgo que les corresponde. También en eso tenemos mucho que aprender. Estas cosas ocurren menos en países con más tradición como Francia donde la carrera está mejor diseñada y hay momentos en la misma para todo, desde sacar una plaza de investigador en el CNRS, o INRIA, hasta más tarde regresar a la Universidad como Catedrático y luego asumir labores que exigen aún mayor entrega como es el desarrollo de una escuela, la incursión en temas de investigación novedosos, la dirección de algunos de los laboratorios mixtos CNRS-Universidad, etc.

SëMA

B.S.: *¿Cómo puede mejorar SEMA? ¿Qué aspectos habría que reforzar?*

E.Z.: SëMA tiene una presidencia y un Comité Ejecutivo que son quienes tienen el protagonismo en ésto. En esta entrevista hemos mencionado algunas carencias y temas por abordar y sin duda alguna SëMA tiene mucho que decir en cada uno de ellos. Ahora que realizamos esta entrevista para el Boletín de SëMA sí que me gustaría subrayar el papel que el mismo ha desempeñado en estos ya más de quince años de andadura y animaría a sus editores a seguir en esta línea en la que el Boletín es un cauce de comunicación fluida entre sus socios y en el que se pueden abordar temas de los que habitualmente no se ocupan las revistas de investigación.

B.S.: *¿Querrías añadir alguna otra reflexión?*

E.Z.: Ya que esta entrevista ha sido motivada por el premio Julio Rey Pastor sí que me gustaría mencionar mi agradecimiento a nuestra sociedad y a su dirección actual por impulsar mi candidatura al mismo y por el apoyo que me han dado. Soy muy consciente de la importancia que esto tiene a la hora de hacer prosperar una candidatura. En este caso he sido yo el beneficiado por esta política de promoción y apoyo a nuestros matemáticos, que ya había dado buenos resultados en casos anteriores. Espero que sepamos mantener

este modo de proceder y contribuir así a que el trabajo de nuestros socios sea debidamente reconocido y a dar a conocer mejor la tarea que realizamos desde nuestra disciplina. Eskerrikasko.[1]

[1] Gracias.

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ARTÍCULOS

INFIMUM–SUPREMUM

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Abstract

In this note we introduce and discuss some basic notions about the continuous and discrete infimum–supremum (or LBB) conditions and their relation with right–inverses, discrete liftings and stability of numerical methods. The context will be that of Hilbert space methods in numerical analysis, with a strong operator theory flavor. In passing, we will also mention some historical milestones on the discovery and precise setting of these inequalities.

Key words: *Inf–sup, Babuška–Brezzi condition, stability, pseudoinverse*

In the relatively wide world of the Finite Element Method, including theorists and practitioners, the concept of infimum–supremum condition (there are many other names for it, which we will comment later on) is one of the most important ones, at least in what respects to stability and convergence issues. The purpose of this note is to give a simple explanation of several facts related to this condition at the continuous and discrete levels, including several useful (and sometimes not very well–known) equivalent forms of writing the conditions. The requirements for fully understanding what follows are very few. Here is a list of what the reader should know: the basic notions of Hilbert space theory, including the Riesz–Fréchet theorem and the concept of adjoint of a linear bounded operator; the Banach isomorphism theorem and the version of the Banach–Steinhaus theorem that is sometimes referred to as the Uniform Boundedness Principle.

Foreword. An aspect where the expository character of the paper will be noticeable will be the presence of footnotes. I will be also using the solemn *we* when including the reader and the personal *I* and *you* when giving an opinion or asking the reader to collaborate respectively.

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1 Continuous inf–sup conditions

Let X and M be two Hilbert spaces and consider a bounded linear operator $B : X \to M$. The brackets $(\,\cdot\,,\,\cdot\,)$ will denote indistinctly the inner products of X and M: context will clarify which is which. We have also the induced bounded bilinear form $b : X \times M \to \mathbb{R}$

$$b(u,\mu) := (Bu,\mu).$$

Notice that in the continuity bound

$$|b(u,\mu)| \leq \|B\|\,\|u\|\,\|\mu\|, \qquad \forall (u,\mu) \in X \times M$$

the constant

$$\|B\| := \sup_{0\neq u\in X} \frac{\|Bu\|}{\|\mu\|}$$

is the lowest possible. The well–known Riesz–Fréchet representation theorem says that given the bilinear form, we can reconstruct the operator, since the mapping

$$u \longmapsto (Bu,\,\cdot\,) : M \to \mathbb{R}$$

is linear and bounded from X to M' (the dual space of M) and M' is isometrically isomorphic to M. The adjoint operator $B^* : M \to X$ is constructed by imposing the identity

$$(Bu,\mu) = b(u,\mu) = (u,B^*\mu), \qquad \forall (u,\mu) \in X \times M.$$

In its simplest version (there are many generalizations), an inf–sup condition for the bilinear form b is the following lower bound: there exists $\beta > 0$ such that

$$\sup_{0\neq u\in X} \frac{|b(u,\mu)|}{\|u\|} \geq \beta\|\mu\|, \qquad \forall \mu \in M. \tag{1}$$

Instead of stating and proving a theorem, let us work out some surrounding concepts and inequalities and derive consequences from hypotheses. To organize chaos, I am going to divide what follows into five entries. For those in the hard need of a statement, go simply to the end of the section.

1. The Cauchy–Schwarz inequality states that[1]

$$\|v\| = \sup_{0\neq u\in X} \frac{|(u,v)|}{\|u\|}, \qquad \forall v \in X.$$

Then (1) is equivalent to a lower bound for B^*

$$\|B^*\mu\| \geq \beta\|\mu\|, \qquad \forall \mu \in M. \tag{2}$$

[1] In the classroom, this assertion makes some students raise their eyebrows. Some other show total indifference. Anyway, this is one of the happy occasions when the supremum is a maximum: take $v = u$.

Hence, B^* is injective (this is trivial) and its range $\mathcal{R}(B^*) := \{B^*\mu \,|\, \mu \in M\}$ is a closed subset of X (this is very simple). I have seen inequality (2) as the definition of **bounding operator** (B^* would be the bounding operator). I am not that sure this is a standard notation, so let us ignore it.

2. Given the trivial fact that

$$B^*\mu = 0 \qquad \Longleftrightarrow \qquad (Bu, \mu) = 0, \quad \forall u \in X,$$

i.e., $\mathcal{R}(B)^\perp = \mathcal{N}(B^*)$ ($\mathcal{N}(B^*)$ is the null–space or kernel of B^*), injectivity of B^* is the same thing as density of $\mathcal{R}(B)$ as a subspace of X. Take now $u \in \mathcal{N}(B)^\perp$. We already know that $\mathcal{R}(B^*)$ is closed, that is, $\mathcal{N}(B)^\perp = \mathcal{R}(B^*)$. Therefore

$$\begin{aligned} \|Bu\| &= \sup_{0\neq\mu\in M} \frac{|b(u,\mu)|}{\|\mu\|} = \sup_{0\neq\mu\in M} \frac{|(u,B^*\mu)|}{\|\mu\|} \\ &\geq \beta \sup_{0\neq\mu\in M} \frac{|(u,B^*\mu)|}{\|B^*\mu\|} = \beta \sup_{0\neq v\in\mathcal{R}(B^*)} \frac{|(u,v)|}{\|v\|} = \beta\|u\|, \quad \forall u \in \mathcal{R}(B^*). \end{aligned}$$

We have just proved that (2) implies

$$\|Bu\| \geq \beta\|u\|, \qquad \forall u \in \mathcal{N}(B)^\perp. \tag{3}$$

Since the null–space of B is closed, we have the orthogonal sum $X = \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp$, that ensures that the image of $B_1 := B|_{\mathcal{N}(B)^\perp} : \mathcal{N}(B)^\perp \to M$ is the same as that of B. The operator B_1 is injective and by (3), its image is closed and dense in M, so $\mathcal{R}(B) = M$. If you go quietly to the beginning of this paragraph, you will notice that we have proved that (2) implies that B is surjective.

3. Looking more carefully, you will notice that we have proved that if $\mathcal{R}(B^*)$ is closed, so is $\mathcal{R}(B)$. The argument works in the reverse direction, since $(B^*)^* = B$. Summing up, we have proved the equivalence

$$\mathcal{R}(B) \text{ is closed} \qquad \Longleftrightarrow \qquad \mathcal{R}(B^*) \text{ is closed}$$

and then

$$B \text{ is surjective} \qquad \Longleftrightarrow \qquad \left[\begin{array}{l} B^* \text{ is injective,} \\ \mathcal{R}(B^*) \text{ is closed.} \end{array}\right.$$

4. Let us now begin with B being surjective. As we have already noticed in the previous point, we can restrict B to the orthogonal complement of its null–space and obtain an isomorphism

$$B_1 : \mathcal{N}(B)^\perp \to M.$$

Since every bounded isomorphism between Hilbert spaces has a bounded inverse, we know that the inverse operator

$$B^\dagger := (B_1)^{-1} : M \to \mathcal{N}(B)^\perp \subset X$$

is bounded. If we call $\beta := 1/\|B^\dagger\|$, we obtain the bound

$$\beta\|B^\dagger\mu\| \leq \|\mu\|, \qquad \forall \mu \in M, \tag{4}$$

which, in essence, is just (3) backwards. The operator $B^\dagger$ is the **Moore–Penrose pseudoinverse** of B. What it does is associating to $\mu \in M$, the solution of $Bu = \mu$ that has minimum norm (this is very easy to verify). It is straightforwardly a right–inverse of B, whereas on the left what we obtain is that $B^\dagger B$ is the orthogonal projection onto $\mathcal{N}(B)^\perp$. For more on this pseudoinverse, I strongly recommend the monograph [6].

5. Beginning again with B being surjective and transposing the expression $BB^\dagger = I$ to $(B^\dagger)^* B^* = I$, we easily prove

$$\|\mu\| = \|(B^\dagger)^* B^* \mu\| \leq (1/\beta)\|B^*\mu\|, \qquad \forall \mu \in M,$$

(the norms of $B^\dagger$ and $(B^\dagger)^*$ are the same), that is, (2).

All the preceding arguments give a disorganized proof of the following result.

Theorem 1 *Given a bounded bilinear form $b : X \times M \to \mathbb{R}$ and the related operators B, B^* the following conditions are equivalent:*

(a) *There exists $\beta > 0$ satisfying* (1).

(b) *There exists $\beta > 0$ satisfying* (2).

(c) *B^* is injective and its image is closed.*

(d) *B is surjective.*

(e) *B admits a right–inverse that is bounded by a constant $1/\beta$.*

The constants in (a), (b) *and* (e) *can be taken to be the same*

$$\beta := \inf_{0 \neq \mu \in M} \left[\sup_{0 \neq u \in X} \frac{|b(u,\mu)|}{\|u\|\,\|\mu\|} \right] > 0. \tag{5}$$

The *problem solving* approach that you usually find when you read someone trying to work through these conditions is usually the following: there exists β such that for all $0 \neq \mu \in M$ you can find $0 \neq u \in X$, such that

$$|b(u,\mu)| \geq \beta\|u\|\|\mu\|.$$

This argument tries to get directly (5). The important feature is the equivalence of all points of view.

2 A bit of terminology

All of this is very well–known for many people in the fields of Functional, PDE or Numerical Analysis. Let us go now for names. Condition (1) or its equivalent (5) are often called infimum–supremum conditions (by an obvious reason in the form (5)), or, in short, inf–sup conditions. Sometimes they are referred to as Babuška–Brezzi (BB) conditions or even Ladyzhenskaya–Babuška–Brezzi (LBB) conditions.

I am going to try to clarify some of the names. I must confess, though, that we have reached a point in calling these conditions when it is immaterial who is right: everybody is mixing the names up and too many people have a strong opinion to get things in a single path.

Condition (5), which can be written in the more compact form

$$\inf_{\|\mu\|=1} \Big[\sup_{\|u\|=1} |b(u,\mu)| \Big] > 0 \tag{6}$$

(the constant is not specified) together with the supplementary condition

$$\sup_{\|\mu\|=1} |b(u,\mu)| > 0, \qquad \forall u \in X, \tag{7}$$

form the so–called pair of Banach conditions, that ensure that the operator B is invertible: the first one proves surjectivity and the second injectivity. They are in fact equivalent to invertibility of the operator B. These are the conditions (with different names for everything and with the other order of unknowns) that appear in the much quoted lecture notes of Ivo Babuška [2]. Since B is invertible if and only if B^* is invertible, we can write (1) together with

$$\sup_{0\neq\mu\in M} \frac{|b(u,\mu)|}{\|\mu\|} \geq \beta' \|u\|, \qquad \forall u \in X \tag{8}$$

as an equivalent set of conditions to prove invertibility. At this stage, I hope the reader is convinced that once you have proved (1) and (8), you can assert that both are satisfied with the best of β and β' (best meaning largest). These conditions already appear in a somewhat older paper of Babuška [1], back in 1971. In a recent text of Finite Element theory [7] (there is an English edition of the book for those not acquainted with the language of Racine) the mane of Jindřich Nečas is also associated to the conditions (the original paper is [11] and is already quoted in Babuška's one). I have already mentioned Banach in the naming, which can give you an idea on how early back they go.

The paper by Franco Brezzi that everybody quotes when referring to these conditions is [3]. There the problem is related only to surjectivity of B, i.e., involves only (5), since it is related to problems with mixed structure, that we will examine later on.

The mixed structure appears in the treatment of linear restrictions in quadratic minimization problems, when writing the Lagrange saddle–point form of critical points. It is however natural to flow problems (Stokes and Navier–Stokes) and therefore it is not surprising that the inf–sup condition already

appears in Olga Ladyzhenskaya's works on fluids. Most people quote the English translation [10] of her treatise on fluid dynamics, whose first Russian edition is [9]. Go, however to the ending section for more on this purely historical question.

If you are totally puzzled by now, do not worry: so am I. If you want the whole story from an expert with first–hand knowledge, look at the slides of the talk *The inf–sup condition, the bubble and the subgrid* by Franco Brezzi in his website:

`http://www.imati.cnr.it/~brezzi/papers/history.pdf`

3 Inf–sup becomes discrete

The real difficulty for numericians is to retain the quality of the inf–sup condition

$$\sup_{0\neq u\in X}\frac{|b(u,\mu)|}{\|u\|}\geq\beta\|\mu\|,\qquad\forall\mu\in M,$$

when restricting to finite–dimensional subspaces of X and M. Let thus

$$X_h\subset X,\qquad M_h\subset M$$

be two sequences (or directed sets) of finite–dimensional subspaces, directed in the parameter $h\to 0$. We define the constant

$$\beta_h:=\inf_{0\neq\mu_h\in M_h}\left[\sup_{0\neq u_h\in X_h}\frac{|b(u_h,\mu_h)|}{\|u_h\|\,\|\mu_h\|}\right]\geq 0. \tag{9}$$

Therefore, β_h is the best (largest) possible constant making the bound

$$\sup_{0\neq u_h\in X_h}\frac{|b(u_h,\mu_h)|}{\|u_h\|}\geq\beta_h\|\mu_h\|,\qquad\forall\mu_h\in M_h, \tag{10}$$

true. We define the discrete operators $B_h:X_h\to M_h$ and $B_h^*:M_h\to X_h$ by the relations

$$(B_hu_h,\mu_h)=b(u_h,\mu_h)=(u_h,B_h^*\mu_h),\qquad\forall(u_h,\mu_h)\in X_h\times M_h.$$

(Incidentally, we do not need the Riesz–Fréchet theorem here. It is enough with taking bases of the spaces). We can go back to the abstract theory, that applies to the finite dimensional case (although for this one, we could try a direct and more algebraic proof) and notice that the following statements are equivalent:

- $\beta_h>0$
- B_h is surjective
- B_h^* is injective (the closed range property is always satisfied in finite dimension)

- B_h admits a right–inverse.

The Moore–Penrose pseudoinverse of B_h admits then the bound

$$\|B_h^\dagger\| \leq 1/\beta_h. \tag{11}$$

If β_h is the best possible constant (i.e., it is defined by (9)), we have in fact an equality in (11). Obviously, the conditions imply that

$$\dim X_h \geq \dim M_h.$$

The key to numericians is that we want β_h to stay away from zero as $h \to 0$, i.e., we want

$$\sup_{0 \neq u_h \in X_h} \frac{|b(u_h, \mu_h)|}{\|u_h\|} \geq \beta^* \|\mu_h\|, \qquad \forall \mu_h \in M_h, \tag{12}$$

with $\beta^* > 0$ independent of h. Equivalently, we want to have a uniform bound for the pseudoinverses of the discrete operators B_h or, what is the same, uniformly bounded right–inverses[2]. Condition (12), assuming that β^* does not depend on h is called a **uniform discrete inf–sup (or BB or LBB) condition**. However, most people do not care to write or say something that long and just say that the couple of spaces $X_h \times M_h$ satisfy the inf–sup condition.

The main problem derives from the fact that the discrete version of the inf–sup condition does not follow from the continuous one. Those accustomed to elliptic problems will possibly frown: in the elliptic world, stability is always for free!

A very interesting relation between the discrete and the continuous inf–sup condition was very early on discovered by Michel Fortin [8]. It is often called Fortin's Lemma, Trick or Criterion. It states the following.

Theorem 2 (Fortin's Lemma) *Assume that* (1) *holds and that we have a sequence of operators* $\Pi_h : X \to X_h$ *such that*

$$b(\Pi_h u - u, \mu_h) = 0, \qquad \forall u \in X, \quad \forall \mu_h \in M_h \tag{13}$$

and

$$\|\Pi_h\| \leq C. \tag{14}$$

Then (12) *holds with* $\beta^* := \beta/C$.

The proof of this result is so simple, that the result does not seem to deserve the name of Theorem (hence Lemma?). Just follow the following chain of inequalities

$$\begin{aligned} \beta\|\mu_h\| &\leq \sup_{0 \neq u \in X} \frac{|b(u, \mu_h)|}{\|u\|} = \sup_{0 \neq u \in X} \frac{|b(\Pi_h u, \mu_h)|}{\|u\|} \\ &\leq C \sup_{0 \neq u \in X} \frac{|b(\Pi_h u, \mu_h)|}{\|\Pi_h u\|} = C \sup_{0 \neq u_h \in X_h} \frac{|b(u_h, \mu_h)|}{\|u_h\|}. \end{aligned}$$

[2]If there are uniformly bounded right–inverses, the Moore–Penrose psuedoinverses are uniformly bounded, since they are optimal from the point of view of the norm.

In fact, it is clear at first sight that we do not need Π_h to be a linear operator: it suffices then to change (14) by

$$\|\Pi_h u\| \leq C\|u\|, \qquad \forall u \in X.$$

In general, most people work hard with Fortin's trick to prove that the discrete inf–sup conditions hold for their particular examples. What many people do not know (or they do not care to ask themselves)[3] is that they are working just hard enough, because the existence of Π_h satisfying (13) and (14) is necessary for the uniform discrete inf–sup condition to hold, on condition that the continuous inf–sup condition holds. In fact if $Q_h : M \to M_h$ is the orthogonal projector onto M_h, the operator

$$\Pi_h := B_h^\dagger Q_h B : X \to X_h$$

satisfies the required properties. This is very easily verified, since (13) is equivalent to

$$Q_h B \Pi_h = Q_h B$$

and $B\Pi_h = B_h\Pi_h$. And, obviously, there is a detail missing: does (12) imply (1)? The answer is easy: yes, assuming that $\lim_{h\to 0} Q_h\mu = \mu$ for all μ (it is an easy exercise). And we can obtain $\beta = \beta^*$, which proves that the discrete constant will be usually worse than the continuous one.

A highly conflicting issue when trying to learn (or to teach) inf–sup conditions is to remember which space goes with the inf and which one goes with the sup, that is, of the two following conditions

$$\sup_{0\neq u_h\in X_h} \frac{|b(u_h,\mu_h)|}{\|u_h\|} \geq \beta_1\|\mu_h\|, \qquad \forall \mu_h \in M_h,$$

$$\sup_{0\neq \mu_h\in M_h} \frac{|b(u_h,\mu_h)|}{\|\mu_h\|} \geq \beta_2\|u_h\|, \qquad \forall u_h \in X_h,$$

which is the one we want? I certainly hope that the reader will be by now convinced that using operators clears very much the way through the question. The conditions above are simply equivalent to

$$\|B_h^*\mu_h\| \geq \beta_1\|\mu_h\|, \qquad \forall \mu_h \in M_h$$

and

$$\|B_h u_h\| \geq \beta_2\|u_h\|, \qquad \forall u_h \in X_h,$$

respectively. Therefore, if we are trying to prove something related to the surjectivity of B_h, which is related to the injectivity of B_h^*, the first one is the right option.

The square case simplifies the situation and at the same time, makes things more confusing for beginners. If both spaces have the same dimension, the injectivity of B_h^* is equivalent to its surjectivity and the corresponding Moore–Penrose pseudoinverse is just the inverse. Then, both conditions hold, with the same constant! (the norm of the inverse of the adjoint is the same as that of the inverse of the operator).

[3]This aspect is already proven in the original paper by Fortin.

4 Mixed problems

An **abstract mixed problem** requires an additional operator $A : X \to X$ or its corresponding bilinear form

$$a(u, v) := (Au, v).$$

The problem can be written with operators

$$\left[\begin{array}{ll} (u, \lambda) \in X \times M, & \\ Au + B^*\lambda & = f, \\ Bu & = g, \end{array}\right. \tag{15}$$

for a given right–hand side $(f, g) \in X \times M$. It can equivalently be written with bilinear forms

$$\left[\begin{array}{lll} (u, \lambda) \in X \times M, & & \\ a(u, v) + b(v, \lambda) & = (f, v), & \forall v \in X, \\ b(u, \mu) & = (g, \mu), & \forall \mu \in M. \end{array}\right. \tag{16}$$

Notice first that if (15) is to have a unique solution for arbitrary right–hand sides, i.e., if

$$\left[\begin{array}{ll} A & B^* \\ B & 0 \end{array}\right]$$

is due to be invertible, B has to be surjective (g is arbitrary in (15)), so (1) is a necessary condition beforehand.

A key idea to understand this problem seems to have been given by Douglas Arnold. Before telling the story let us notice that there are two reasons to write the system with the equations given in the current order. The first one is its relation with minimization problems with linear restrictions. If the bilinear form $a(\,\cdot\,,\,\cdot\,)$ is symmetric and non–negative, we are just writing the Lagrange conditions for the minimization problem

$$\text{Minimize} \quad \tfrac{1}{2}a(u, u) - (f, u), \qquad \text{Subject to} \quad Bu = g.$$

The additional unknown λ is just the Lagrange multiplier[4]. In Lagrangian equations, we always make first the derivative with respect to the original variable and then with respect to the multiplier (i.e., the second equation is just the constraint). The second one is more, say, moral. If I write

$$\left[\begin{array}{ll} (u, \lambda) \in X \times M, & \\ Bu & = g \\ Au + B^*\lambda & = f, \end{array}\right.$$

[4]You can read a lot of this type of structure in Gilbert Strang's *Introduction to Applied Mathematics* [13], which from my personal point of view is one of the best reads you can do in your whole career as an applied mathematician. Some people will disagree though.

in a way I am forcing you to see the system as a triangular system and to expect B to be invertible. That is the case when just one element satisfies the restriction and therefore, there is nothing to minimize. It is usually not the case and B is simply surjective.

We go back now to Arnold's way of looking at the system. Let

$$X_0 := \mathcal{N}(B), \qquad X_1 := \mathcal{N}(B)^\perp,$$

and consider the four operators

$$A_{ij} : X_j \to X_i$$

given by the restrictions of components of the bilinear form $a(\,\cdot\,,\,\cdot\,)$ to different pairs os subspaces

$$(A_{ij}u_j, v_i) = a(u_j, v_i), \qquad \forall (u_j, v_i) \in X_j \times X_i.$$

We also consider the operator

$$B_1 : X_1 \to M,$$

which we have already met in the first section. Surjectivity of B is equivalent to invertibility of B_1. Breaking $X \times M$ as $X_0 \times X_1 \times M$ we arrive at an operator equation related to the following matrix of operators

$$\begin{bmatrix} A_{00} & A_{01} & 0 \\ A_{10} & A_{11} & B_1^* \\ 0 & B_1 & 0 \end{bmatrix}.$$

Playing a bit with the blocks (with Gaussian elimination) we arrive at the following decomposition of the previous operator

$$\begin{bmatrix} I & A_{01}B_1^{-1} & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} A_{00} & 0 & 0 \\ 0 & B_1 & 0 \\ A_{10} & A_{11} & B_1^* \end{bmatrix}.$$

The left–most operator is invertible, because it is nothing but a rearrangement of a triangular matrix with identities on the diagonal positions. Though not an endomorphism (it acts from $X_0 \times X_1 \times M$ to $X_0 \times M \times X_1$), the second one is truly a lower triangular operator, since two of its diagonal positions, those occupied by B_1 and B_1^*, correspond to isomorphisms. Its invertibility is equivalent to that of A_{00}. Thus we arrive very easily to the extremely popular theorem of mixed problems.

Theorem 3 *Problem* (16) *is uniquely solvable for arbitrary right–hand sides if and only if: B is surjective and the operator A_{00} is invertible.*

The first hypothesis is just (1). The second one requires the two Banach conditions applied to the bilinear form $a : X_0 \times X_0 \to \mathbb{R}$, or equivalently, well–posedness of problems of the form

$$\left[\begin{array}{l} u \in X_0, \\ a(u,v) = \ell(v), \qquad \forall v \in X_0. \end{array}\right.$$

We can repeat these same ideas at the discrete level and notice that what we need is: a uniform discrete inf–sup condition for $b(\,\cdot\,,\,\cdot\,)$, i.e. (12), and a uniform discrete inf–sup condition for $a(\,\cdot\,,\,\cdot\,)$ restricted to

$$X_h^0 := \{u_h \in X_h \,|\, b(u_h, \mu_h) = 0, \quad \forall \mu_h \in M_h\},$$

that is

$$\sup_{0 \neq u_h \in X_h^0} \frac{|a(u_h, v_h)|}{\|u_h\|} \geq \alpha^* \|v_h\|, \qquad \forall v_h \in X_h^0, \tag{17}$$

with $\alpha^* > 0$ independent of h. Since this restricted bilinear form happens with both variables in the same space, we are in the square case mentioned above, and the order of the variables in (17) is immaterial. Notice that when repeating the same arguments, the new subdivision of spaces $X_h^0 \times X_h^1 \times M_h$ is in a way independent of the one in the continuous setting, since X_h^0 could not be a subspace of X_0. The important fact is however, that in the corresponding factorization, the diagonal blocks need uniformly bounded inverses, but this is exactly what is ensured by the inf–sup conditions (12) and (17).

5 Discrete traces

A somewhat surprising place to find all this is in the treatment of non–homogeneous Dirichlet conditions in finite element computations[5]. A boundary value problem with non–homogeneous Dirichlet conditions falls into the following setting. Let X be a Hilbert space and $\gamma : X \to M$ be a surjective linear bounded operator. Let

$$X_0 := \{u \in X \,|\, \gamma u = 0\} = \mathcal{N}(\gamma).$$

Given now $a : X \times X \to \mathbb{R}$ bilinear and bounded and $f \in X$, we can try to solve the problem

$$\left[\begin{array}{l} u \in X, \qquad \gamma u = g, \\ a(u,v) = (f,v), \qquad \forall v \in X_0. \end{array}\right. \tag{18}$$

For simplicity let us assume that $a(\,\cdot\,,\,\cdot\,)$ is elliptic in X_0.

[5]Let us say that non–homogeneous Dirichlet conditions are usually hidden under the carpet very early on in Finite Element textbooks as something trivial. Well, it is not that trivial after all.

A brief interruption with an example. Think of $X = H^1(\Omega)$, $M = H^{1/2}(\partial\Omega)$, γ the trace operator, so that $X_0 = H^1_0(\Omega)$ and take $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$. The typical definition of $H^{1/2}(\partial\Omega)$ as the set of elements of $L^2(\partial\Omega)$ that are traces of elements of $H^1(\Omega)$ makes the trace operator surjective. The image norm

$$\|g\|_{1/2,\partial\Omega} := \inf\{\|v\|_{1,\Omega} \,|\, \gamma v = g\}$$

makes $H^{1/2}(\partial\Omega)$ closed and the operator γ continuous. In fact, its Moore–Penrose pseudoinverse $\gamma^\dagger : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ is equivalent to solving the standard Dirichlet problem

$$\left[\begin{array}{ll} u \in H^1(\Omega), & \gamma u = g, \\ -\Delta u + u = 0, & \text{in } \Omega. \end{array}\right.$$

In fact, by definition, $\|g\|_{1/2,\partial\Omega} = \|\gamma^+ g\|_{1,\Omega}$. □

Let now X_h be a sequence of finite dimensional subspaces of X (think of finite element spaces) and

$$X_h^0 := X_h \cap \mathcal{N}(\gamma) = X_h \cap X_0.$$

Consider the space $M_h := \gamma X_h \subset M$, so that $\gamma_h := \gamma|_{X_h} : X_h \to M_h$ is surjective. The discrete problem

$$\left[\begin{array}{ll} u_h \in X_h, & \gamma u_h = g_h, \\ a(u_h, v_h) = (f, v_h), & \forall v_h \in X_h^0, \end{array}\right. \tag{19}$$

where $M_h \ni g_h \approx g$ is arbitrary, is always uniquely solvable. Going step by step through the proof of the celebrated Lemma of Jean Céa, we arrive at an estimate

$$\|u - u_h\| \leq C \inf\left\{ \|u - v_h\| \,\middle|\, v_h \in X_h, \quad \gamma v_h = g_h \right\}. \tag{20}$$

(We are not going to mind about C). What is the problem here? Let me say that I find this inequality very unsatisfactory. In our problem there are two sources of approximation error: one related to the ability of the whole of X_h to approximate the exact solution u; another one related to how well we took g_h to approximate g. In (20), they are mixed. I would clearly rather have

$$\|u - u_h\| \leq C^* \left[\inf_{v_h \in X_h} \|u - v_h\| + \|g - g_h\| \right]. \tag{21}$$

Apparently out of nowhere, Ridgway Scott and Shangyou Zhang [12] constructed (for the particular case of Lagrange finite elements on triangles and tetrahedra and applied to the problem we have shown as an example) an operator $\Pi_h : X \to X_h$ satisfying three properties:

(SZ1) It is uniformly bounded: $\|\Pi_h u\| \leq C_1 \|u\|$, for all $u \in X$.

(SZ2) It is a projection onto X_h, that is, $\Pi_h u_h = u_h$ for $u_h \in X_h$.

(SZ3) It preserves the homogeneous condition for γ: if $\gamma u = 0$, then $\gamma \Pi_h u = 0$.

Because of (SZ1) and (SZ2), the operator Π_h is quasi–optimal[6]:

$$\|u - \Pi_h u\| \leq (1 + C_1) \inf_{v_h \in X_h} \|u - v_h\|. \tag{22}$$

Now, we play the dirty trick: surprisingly, the trace of the function

$$\Pi_h(u + \gamma^\dagger(g_h - g))$$

is g_h. Why? Notice that

$$\gamma(u - \gamma^\dagger g + \gamma^\dagger g_h - \gamma_h^\dagger g_h) = \gamma u - g + g_h - g_h = 0.$$

This fact plus (SZ2) and (SZ3) proves that

$$\gamma \Pi_h(u - \gamma^\dagger g + \gamma^\dagger g_h) = \gamma \Pi_h \gamma_h^\dagger g_h = \gamma \gamma_h^\dagger g_h = g_h.$$

This means that we can input $v_h := \Pi_h(u + \gamma^\dagger(g_h - g))$ as a function to compare with in (20) and obtain

$$\|u - u_h\| \leq C \Big[\|u - \Pi_h u\| + C_1 \|\gamma^\dagger\| \, \|g - g_h\| \Big],$$

which, together with (22) gives the desired (21).

As many others before, I thought the argument was amazing (you can find several reformulations of it; this one is my own, with lots operators and pseudoinverses) but difficult to come out with. The question is not how you devise Π_h but: is it too much to expect to have Π_h? Well, again, basically it is not. Look at these three points.

1. Notice that the pseudoinverse of the discrete trace[7] is given by the definition

$$\gamma_h^\dagger g_h := w_h, \qquad \text{where} \qquad \left[\begin{array}{ll} w_h \in X_h, & \gamma w_h = g_h, \\ (w_h, v_h) = 0, & \forall v_h \in X_h^0. \end{array} \right. \tag{23}$$

(This is just the fact that $\gamma_h^\dagger g_h$ is orthogonal to the discrete kernel of the trace X_h^0). Since $\gamma^\dagger g_h - \gamma_h^\dagger g_h = \gamma^\dagger g_h - w_h \in X_0$, we can take $v_h := \Pi_h(\gamma^\dagger g_h - \gamma_h^\dagger g_h) = \Pi_h \gamma^\dagger g_h - w_h$ in problem (23) and get

$$\|w_h\|^2 \leq C_1 \|\gamma^\dagger\| \, \|g_h\| \, \|w_h\|,$$

that is,

$$\|\gamma_h^\dagger g_h\| \leq C_1 \|\gamma^\dagger\| \|g_h\|, \qquad \forall g_h \in M_h. \tag{24}$$

This means that existence of Π_h guarantees the existence of a uniformly bounded right–inverse of the discrete trace, a stable discrete lifting.

[6]The constant $1 + C_1$ is simple to get. You can come down to C_1 by using a Lemma by Kato: *two complementary projections that are non–trivial, have the same norm* (see the extremely simple note [14] to convince yourself).

[7]I am calling here γ a trace, although you have to notice that we are in an abstract setting.

2. The mathematician within will immediately ask for the reciprocal. Of course, it holds. The following argument appears in a short note by myself and Víctor Domínguez [5]. We need the orthogonal projections

$$P_h^0 : X \to X_h^0, \qquad Q_h : M \to M_h.$$

We can then define

$$\Pi_h := P_h^0 + \gamma_h^\dagger Q_h \gamma : X \to X_h.$$

Since $\|\Pi_h\| \leq 1 + \|\gamma_h^\dagger\| \, \|\gamma\|$, uniform boundedness of $\gamma_h^\dagger$ implies property (SZ1) for Π_h. If $\gamma u = 0$, then $\Pi_h u = P_h^0 u \in X_h^0$, and (SZ3) also holds. Finally, let $u_h \in X_h$ and

$$w_h := \Pi_h u_h = P_h^0 u_h + \gamma_h^\dagger Q_h \gamma u_h = P_h^0 u_h + \gamma^\dagger \gamma u_h.$$

It is simple to see that

$$\left[\begin{array}{l} w_h \in X_h, \quad \gamma w_h = \gamma u_h, \\ (w_h, v_h) = (u_h, v_h), \quad \forall v_h \in X_h^0, \end{array} \right.$$

son $w_h = u_h$ and we have (SZ2).

3. Can we close the loop with the improved Céa estimate? Yes, we can. Assume that we have an estimate (21) for the discretization of (18) by (19). Take $f = 0$ and $g = 0$ (so $u = 0$). Then, the solution to (19) satisfies

$$\|u_h\| \leq C \left[\inf_{v_h \in X_h} \|v_h\| + \|g_h\| \right] = C \|g_h\|,$$

and we have a uniformly bounded right–inverse of the trace.

We can wrap up these ideas in form of a theorem.

Theorem 4 *The following three statements are equivalent:*

(a) *There exist operators* $\Pi_h : X \to X_h$ *satisfying properties* (SZ1–SZ3).

(b) *There exist uniformly bounded right–inverses of* $\gamma : X_h \to M_h$.

(c) *There exists a bound like* (21) *for the approximation of* (18) *by* (19).

The reader will wonder: where is the inf–sup condition here? It is hidden in the uniformly bounded right–inverse, only here the natural concept is the discrete operator [8]. The corresponding bilinear form is simply

$$(\gamma u, \lambda), \qquad (u, \lambda) \in X \times M.$$

[8] Let me mention here a celebrated paper related to inf–sup conditions without any inf–sup condition in it. The article by Michel Crouzeix and Pierre–Arnaud Raviart [4] not only established the rules to numerically work with the Stokes problem but also originated the world–famous Crouzeix–Raviart elements. It was published when the discrete BB conditions were just appearing. In fact, it uses right inverses and not inf–sup conditions to deal with the discretization of the divergence constraint in the Stokes problem.

For those who are used to applying Lagrange multiplier techniques to impose essential boundary conditions, let us remark that the bracket in the preceding expression is the inner product of M and not any other representation of duality for M. The point here is seeing how very lateral thinking (like the definition of a Scott–Zhang type operator) works sometimes when dealing directly with bilinear forms can be at least complicated.

6 What to do when the theory fails

In some practical cases, engineers use pairs of spaces where the inf–sup condition (12) cannot be verified. However, they might have a good feeling about that pair of spaces. What many practitioners do is taking the beginning of the sequence of spaces, computing the constant β_h in (9) and seeing if it stagnates sufficiently far away from zero. This is taken as a good symptom for the asymptotic behavior[9], and the method is applied with 'finer grids'. Let us show here at least a simple way of computing β_h. First we need bases: $\varphi_1, \ldots, \varphi_n$ of X_h and $\mu_1, \ldots, \mu_m$ of M_h. Then we consider the matrix $\mathbf{B}$

$$\mathbf{B}_{ij} := b(\varphi_j, \mu_i),$$

and the Gram matrices $\mathbf{R}$ and $\mathbf{M}$

$$\mathbf{R}_{ij} := (\varphi_i, \varphi_j), \qquad \mathbf{M}_{ij} := (\mu_i, \mu_j).$$

Inequality (10) is equivalent to

$$\sup_{\mathbf{0}\neq\mathbf{u}\in\mathbb{R}^n} \frac{|\mathbf{m}\cdot(\mathbf{B}\mathbf{u})|}{|\mathbf{R}^{1/2}\mathbf{u}|} \geq \beta_h |\mathbf{M}^{1/2}\mathbf{m}|, \qquad \forall \mathbf{m}\in\mathbb{R}^m. \tag{25}$$

There are several elements to clarify here: the symbol $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^m$ as well as the absolute value; the dot is used for the Euclidean inner product in both spaces; finally $\mathbf{R}^{1/2}$ and $\mathbf{M}^{1/2}$ denote the positive definite square roots of $\mathbf{R}$ and $\mathbf{M}$ respectively. Do not worry, we do not have to compute them: this is just the beginning of the argument. It is simple to check that (25) is equivalent to

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbb{R}^n} \frac{|(\mathbf{R}^{-1/2}\mathbf{B}^\top\mathbf{M}^{-1/2}\mathbf{n})\cdot\mathbf{v}|}{|\mathbf{v}|} \geq \beta_h |\mathbf{n}|, \qquad \forall \mathbf{n}\in\mathbb{R}^m, \tag{26}$$

or to the simpler

$$|\mathbf{R}^{-1/2}\mathbf{B}^\top\mathbf{M}^{-1/2}\mathbf{n}| \geq \beta_h |\mathbf{n}|, \qquad \forall \mathbf{n}\in\mathbb{R}^n. \tag{27}$$

[9]With good sense, the mathematician claims that the beginning of the sequence says nothing about its asymptotic behavior. True enough. Against this argument, the engineer will oppose experience and a good–natured optimism. And he will forget an argument that would serve him well: since he is not going to compute up to the limit, why should he worry so much about the limiting behavior of the stability constant?

All of this is equivalent to

$$\beta_h := \inf_{\mathbf{0}\neq\mathbf{n}\in\mathbb{R}^n} \frac{|\mathbf{R}^{-1/2}\mathbf{B}^\top\mathbf{M}^{-1/2}\mathbf{n}|}{|\mathbf{n}|}$$

which is the expression of the smallest singular value of the matrix $\mathbf{R}^{-1/2}\mathbf{B}^\top\mathbf{M}^{-1/2}$. Elementary algebra shows that we are computing the smallest value β_h such that

$$\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{m} = \beta_h\,\mathbf{M}\mathbf{m}$$

has non–trivial solutions. A simple specialization (wait for incoming comments after the algorithm) of the Inverse Power Method can be used to compute β_h. We begin with $\mathbf{x}_0 \in \mathbb{R}^m$ normalized so that $\mathbf{x}_0 \cdot (\mathbf{M}\mathbf{x}_0) = 1$. Then, for each iteration (denoted with the index $k \geq 1$) we do:

$$\begin{aligned}
\mathbf{y}_k &:= \mathbf{M}\mathbf{x}_k \\
\mathbf{z}_{k+1} &:= (\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top)^{-1}\mathbf{y}_k \\
\xi_k &:= \mathbf{z}_{k+1}\cdot(\mathbf{M}\mathbf{z}_{k+1}) \\
\beta_{k+1} &:= (\mathbf{z}_{k+1}\cdot\mathbf{y}_k)/\xi_k \\
\mathbf{x}_{k+1} &:= (1/\sqrt{\xi_k})\,\mathbf{z}_{k+1}.
\end{aligned}$$

A stopping criterion can be based on the observation of the sequence β_k. The second line of the algorithm requires solving the system

$$(\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top)\mathbf{z}_{k+1} = \mathbf{y}_k.$$

If $\mathbf{B}^\top$ has maximum range by columns[10], then $\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top$ is symmetric positive definite. A conjugate gradient iteration seems the best choice to solve this system: each iteration of the CG method requires two matrix products plus the solution of a system with $\mathbf{R}$ as matrix, which is costly but avoids having to invert $\mathbf{R}$ to create the matrix $\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top$. The best choice to begin the CG iteration is the value of the preceding iteration of the power method, i.e., $\mathbf{z}_k$.

What can we do if $\beta_h = 0$, that is, when the spaces make the discrete operator B_h non–surjective? The strategy above, using CG–iterations, has some advantages. The CG method converges for positive semidefinite matrices to the least–squares solution of the system which is the closest to the initial iteration[11]. The method will go on till ξ_k explodes or β_k gets very near zero.

7 Should we be obsessed with inf–sup conditions?

Discrete uniform inf–sup conditions are of the utmost importance in dealing with Petrov–Galerkin methods with hilbertian techniques. What does this mean?

[10] that is $\beta_h > 0$ or equivalently, the discrete operator B_h is surjective

[11] I can see heads been scratched... This is true and little known outside some specific circles, like the one of people used to ill–posed problems, where it is well–known that the CG method is a regulariser.

Take a general problem, related to an invertible operator $A : Y \to Z$ (for instance, in the sense of section 4, $Y = Z = X \times M$). The exact problem, with the bilinear form $a(u, v) := (Au, v)$ is

$$\left[\begin{array}{l} u \in Y, \\ a(u, v) = \ell(v), \qquad \forall v \in Z, \end{array}\right.$$

and its discretization, related to finite–dimensional couples $Y_h \times Z_h$ (with $\dim Y_h = \dim Z_h$) is

$$\left[\begin{array}{l} u_h \in Y_h, \\ a(u_h, v_h) = \ell(v_h), \qquad \forall v_h \in Z_h, \end{array}\right.$$

Let us assume that this last sequence of problems is always uniquely solvable. **Stability** of the method is any of the following three equivalent statements:

(S1) There exists $\alpha > 0$ (independent of h) such that

$$\sup_{0 \neq u_h \in Y_h} \frac{|a(u_h, v_h)|}{\|u_h\|} \geq \alpha \|v_h\|, \qquad \forall v_h \in Z_h.$$

(S2) There exists $C_1 > 0$ (independent of h) such that for all u and h

$$\|u_h\| \leq C_1 \|u\|.$$

(S3) There exists $C_2 > 0$ (independent of h) such that for all u and h

$$\|u - u_h\| \leq C_2 \inf_{y_h \in Y_h} \|u - y_h\|.$$

The constants can be taken $C_1 = C_2 = 1/\alpha$[12]. Because of the estimate (S3), stability plus the **approximation property**

$$\inf_{y_h \in Y_h} \|u - y_h\| \overset{h \to 0}{\longrightarrow} 0, \qquad \forall u \in Y, \tag{28}$$

imply **convergence** of the method for arbitrary right–hand sides

$$u_h \overset{h \to 0}{\longrightarrow} u, \qquad \forall u \in Y.$$

Taking the solution operator $u \mapsto u_h$, it is easy to see that convergence for arbitrary right–hand sides implies the approximation property (28) but also, as a simple consequence of Banach–Steinhaus's theorem, stability, expressed as the estimate (S2).

So: is it important to have an inf–sup condition? It is, if we are going to deal with arbitrary right–hand sides. When is this the case? Well, sometimes

[12] That $C_1 = 1/\alpha$ is an exercise. To see that $C_1 = C_2$, apply Kato's lemma as explained above

it is not. For instance, scattering problems typically have smooth right–hand sides, or in finite element computations data are piecewise smooth or square integrable. Some other times, the problem in hand is the linearization of a non–linear problem and we find it in iterations. Right–hand sides correspond then to errors and they tend to be non–smooth. Inverse problems produce also sequences of problems where all types of right–hand sides have to be produced.

I have heard, however, heated discussions of cases where we obtain convergence without the inf–sup condition (again, I mean the discrete uniform inf–sup condition) being satisfied. Where can you find this and why? In adaptive situations! In that case, the sequence of spaces is not given a priori, but produced as a part of the computational process. That sequence guarantees convergence for that particular right–hand side, but not for general right–hand sides. It is therefore not obliged to give a well–balanced (stable) scheme. So is it not important after all? Well, the inf–sup condition guarantees that you are capturing correctly the sprectrum of the continuous problem and your computations are as badly conditioned as the original problem, but never more. Without it, the discrete problem can be more severely ill–conditioned than the exact one. But there is another but: ill–conditioning provokes a bad performance of numerical methods for the algebraic systems, but when you are working in an adaptive scheme, you are already beginning with a good approximation of the solution... Is there and end to this discussion? I am afraid not. I am only trying to emphasize that there are several points of view and many people can be right saying almost opposite things: they are simply looking at different problems.[13]

8 The origins of the inf–sup condition: literature

The two inf–sup conditions guaranteeing invertibility, as shown in [1], look like this[14]:

> **Theorem 2.1** Let H_1 and H_2 be two Hilbert (complex and complete) spaces with scalar product $(\,\cdot\,,\,\cdot\,)_{H_1}$ (resp. $(\,\cdot\,,\,\cdot\,)_{H_2}$). Further let $B(u,v)$ be a bilinear form on $H_1 \times H_2$, $u \in H_1$, $v \in H_2$ such that
>
> $$|B(u,v)| \leq C_1 \|u\|_{H_1} \|v\|_{H_2}, \tag{2.1}$$
>
> $$\sup_{\substack{u \in H_1 \\ \|u\|_{H_1} \leq 1}} |B(u,v)| \geq C_2 \|v\|_{H_2}, \tag{2.2}$$
>
> $$\sup_{\substack{v \in H_2 \\ \|v\|_{H_2} \leq 1}} |B(u,v)| \geq C_3 \|u\|_{H_1}, \tag{2.3}$$

[13] If you move away from Hilbertian techniques, for instance when you deal with Finite Difference discretizations, you see that the paradigm *stability+approximation=convergence* is easily abandoned. Some other properties, like monotonicity, are then relevant.

[14] Apart from some alignments, I have respected the original writing as much as possible in the following partial quotations. I have not corrected misprints in the originals.

with $C_2 > 0$, $C_3 > 0$, $C_1 < \infty$. [etc]

After that, you can imagine, there is existence and uniqueness. The paper, received in 1969, mentions Nečas's paper, shown below. The same result in Nečas [11], in French and giving credit to the famous theorem (or lemma) by Lax and Milgram, which this result generalizes has the following appearance

> Considérant les espaces complexes et les opérateurs différentiels elliptiques, le théorème de P.D. Lax et A. Milgram (cf. p. ex. L. Nirenberg [1]) paraît être très utile pour la méthode variationelle. D'abord nous en signalons une généralisation facile:
>
> THÉORÈME 3.1 *Soiet H_1 et H_2 deux espaces de Hilbert, $B(u, v)$ forme bilinéaire, $v \in H_1$, $u \in H_2$ jouissant des propriétés:*
>
> $$|B(v,u)| \le c_1 \|v\|_{H_1} \|u\|_{H_2}, \tag{3.1}$$
>
> $$\sup_{\substack{v \in H_1 \\ \|v\|_{H_1} \le 1}} |B(v,u)| \ge c_2 \|v\|_{H_2}, \tag{3.2}$$
>
> $$\sup_{\substack{u \in H_2 \\ \|u\|_{H_2} \le 1}} |B(v,u)| \ge c_3 \|v\|_{H_1}. \tag{3.3}$$
>
> *Alors étant donné une fonctionnelle $F(v)$ sur H_1, il existe précisément un élément u de H_2 tel que $B(v, u) = F(v)$ et on a*
>
> $$|u|_{H_2} \le c\,|F|. \tag{3.4}$$

Note two things: (a) Nečas prefers using the second argument of the bilinear form for the unknown[15] and thus is obliged to produce the strange feeling of having $u \in H_2$ and $v \in H_1$ in very non–alphabetical order; (b) in the invertibility constant, the result does not recognise the influence of c_2 or c_3.

With the distinctive flavor of the typing machine, the Lecture Notes of Babuška and Aziz [2], give the two conditions in the asymmetrical way to which we are now used and with inf–sup together.

> Theorem 5.2.1. If
> (I) H_1 and H_2 are two real Hilbert spaces with scalar product $(\,\cdot\,,\,\cdot\,)_{H_1}$ and $(\,\cdot\,,\,\cdot\,)_{H_2}$ respectively.
> (II) $B(u, v)$ is a bilinear form on $H_1 \times H_2$, $u \in H_1$, $v \in H_2$ such that

[15] This is usually better if one is going to construct the FEM equations afterwards, since the number of equation is the first of the two indices in a matrix and corresponds to testing, which would come more naturally in the first position.

(5.2.1) $$|B(u,v)| \leq C_1 \|u\|_{H_1} \|v\|_{H_2},$$

(5.2.2) $$\inf_{\substack{u \in H_1 \\ \|u\|_{H_1}=1}} \sup_{\substack{v \in H_2 \\ \|v\|_{H_2} \leq 1}} |B(u,v)| \geq C_2 > 0\ ,$$

(5.2.3) $$\sup_{u \in H_1} |B(u,v)| > 0, \quad v \neq 0,$$

[etc]

Note that we still are speaking of invertibility. These conditions are essentially (6, 7), but there is a slight difference. Can you see it? Many years later, in reviewing the question, the English edition of the book of Alexandre Ern and Jean–Luc Guermond [7] looks like this.

Theorem 2.6 (Banach–Nečas–Babuška). *Let W be a Banach space and let V be a reflexive Banach space. Let $a \in \mathcal{L}(W \times V; \mathbb{R})$ and $f \in V'$. Then, problem* (2.1) *is well–posed if and only if:*

(BNB1) $$\exists \alpha > 0, \quad \inf_{w \in W} \sup_{v \in V} \frac{a(u,v)}{\|w\|_W \|v\|_V} \geq \alpha,$$

(BNB2) $$\forall v \in V, \quad (\forall w \in W, a(w,v) = 0) \quad \Longrightarrow \quad (v = 0).$$

...

The name of Nečas serves as an interpolator, gluing two B's together: those of Banach and Babuška. However right to the origins, the name is not catching. Notice also that we have moved to the case of a reflexive Banach space for testing and that, given the fact that we are in real spaces, the absolute value is not needed in the inf–sup fraction.

Let us go to the other B in LBB. Although dealing exclusively with problems with mixed form (they are called saddle–point problems), the celebrated paper of Brezzi [3] clarifies the question of the inf–sup conditions right in the section of preliminaries:

Theorem 0.1.– *Let X, Y be real Hilbert spaces; let $\mathcal{T}(x,y)$ be a continuous bilinear form on $X \times Y$ and let T be the continuous linear operator from X into Y' associated to $\tau(x,y)$ defined by:*

$$\langle Tx, y \rangle = \mathcal{T}(x,y) \qquad \forall x \in X, y \in Y.$$

For all $k > 0$ the three following statements are equivalent:

i) $$\sup_{x \in X - \{0\}} \frac{\mathcal{T}(x,y)}{\|x\|} \geq k\|y\| \qquad \forall y \in Y,$$

ii) $$\|T'y\| \geq k\|y\| \qquad \forall y \in Y,$$

iii) *$\exists S \in \mathcal{L}(Y', X')$ such that $TS = I$ (identity) on Y' and $\|S\| \leq k^{-1}$.*

Notice here that the right–inverse is clearly identified as possibility iii) in writing what an inf–sup condition is. In fact, if you look at the proof, you will see that the right–inverse that is defined is (surprise!) the Moore–Penrose pseudoinverse, although not given with this name. For those interested in stylistic question let me remark that Brezzi does not bother to give different names to the norms (neither do I) but takes care of getting the annoying zero out of the testing quotient: the singularity of the zero testing is, said in the language of classical analysis, avoidable, but it is still there.

One page later, a theorem like the ones before is derived as a corollary, by applying this Theorem 0.1 to the bilinear form and its transposed. Recognition of the feeling that *all of this is in the air* comes afterwards in the form of a remark:

> REMARK 0.1.– The results contained in theorem 0.1 and in corollary 0.1 are of classical type and might not be new. For instance part I) $\Rightarrow$ III) of corollary 0.1 was used by Babuska [3].[16]

The surprise for me, and I guess it will be the same for many more knowledgeable readers, is the fact that the book of Ladyzhenskaya most people are quoting does no deal with inf–sup conditions. The monograph in question deals with stationary and non–stationary Stokes and Navier–Stokes problems, where the divergence restriction imposes in a natural way the mixed form of the problem and makes it reasonable to assume that there is something done with respect to surjectivity of the divergence operator (which, in the transposed block becomes recovery of the pressure). However, the book works extensively with layer potentials, obtaining the zero–divergence condition in a natural way, and the reader will be surprised in finding nothing that resembles even an inf–sup condition in this book. Since this is not a paper on history, I will leave to the really interested reader the task of finding where in the wide production of the great Russian mathematician this condition appears.

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COMPUTERIZED TOMOGRAPHY AND THE RADON TRANSFORM

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Abstract

We give an overview of some of the mathematical foundations of Computerized Tomography as well as of some of the tools and problems that arise from the mathematical model discussed and its numerical implementation.

Key words: *Computerized tomography, Radon transform, Inversion formulas, Sampling theory.*
AMS subject classifications: *44A12 92C55 65R32 94A20*

1 Introduction

This expository paper is devoted to computerized tomography (CT), a discipline that provides techniques of reconstruction of a function from its integrals over lines. The paper is based on the lecture notes of a seminar that was run at the University of Barcelona in 2005. The main bibliographical sources were the books by Frank Natterer [13] and [14], and the references therein.

The word "tomography" is derived from the Greek words *tomos* (slice) and *graphein* (to write), and it refers to an imaging technique that allows to obtain cross-sectional images of an object from its projections.

A simple physical model for CT is as follows. If I_0 is the initial intensity of an X-ray beam, $f(x)$ denotes the X-ray attenuation coefficient of the object at the point x, L is the ray along which the radiation propagates, and I the X-ray intensity detected after having passed through the body, we have

$$I = I_0 e^{-\int_L f(x)\,dx}.$$

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In this integral and in all subsequent ones the measure dx corresponds to the k-Lebesgue measure where k is the dimension of underlying manifold L.

Consequently,

$$g(L) := \int_L f(x)\, dx = \log \frac{I_0}{I}.$$

The function g, defined over the set of lines going through the object, is the two dimensional *Radon transform of* f, and the main problem in CT is to reconstruct f from the values $g(L)$. In practice only a finite number of lines are considered, depending on the scanning geometry and the resolution of the scanner. In 1917, the German mathematician Johann Radon obtained an inversion formula for the reconstruction of f from the data g. However, this inversion formula is not suitable for numerical computations.

CT began in 1963 with the work of the South African (and later American) physicist Allan M. Cormack. He obtained a new inversion formula based on a development of the function f in spherical polynomials, see Section 2.5. The first commercially viable CT scanner was invented by the British engineer Godfrey N. Hounsfield at Thorn EMI Central Research Laboratories. Hounsfield conceived his scanner in 1967, but the first prototype (which used X-rays) was not built until 1971. In 1972 its existence was publicly announced, thus starting the commercial era for computerized tomography devices. The original prototype took 160 parallel readings through 180 directions, each 1 degree apart. Each scan would take a little over five minutes and once all of them were acquired, a large computer (using an algebraic reconstruction technique) needed 2.5 hours to process the final images. In 1979 the Nobel prize in Medicine was awarded to Cormack and Hounsfield.

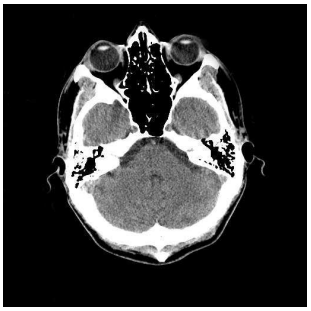

Figure 1: CT scan of a human brain after being hit by a car, by Andrew Ciscel. Original at http://www.flickr.com/photos/ciscel/124548696

In the forthcoming pages we will provide an overview of the mathematical foundations of this model.

Nevertheless there are other similar geometric transformations, with their corresponding inversion formulas, that have also been of great use in other models. Among these we mention the single photon emission computed

tomography (SPECT), the synthetic aperture radar (SAR) and the 3D dimensional models (see [14, §5.5] and the references therein).

SPECT is a medical imaging technique used in nuclear medicine which was developed in the early 1960's by David Edwards and Roy Kuhl. It is used for clinical diagnosis of brain diseases and for tumor imaging in clinical oncology. In order to conduct the scan, a short-lived radioactive isotope (usually fluorine-18) is injected in the blood or inhaled. As the radioisotope decays, it emits a positron which after travelling up to a few millimeters, encounters and annihilates with an electron, thus producing a pair of gamma photons. These photons are detected by a gamma camera collimated to detect only the photons coming in a given direction. The radiation intensity measured by the detector along the line L is given by

$$I = \int_L f(x) e^{-\int_{L(x)} \mu(y)\, dy} dx.$$

where $L(x)$ is the halfline in the direction of L starting at x, f is the distribution of the radiopharmaceutical and μ is the attenuation coefficient of the studied tissue. The value of μ is assumed to be known, and very often it is computed by a simultaneous standard tomography performed by the same machine. The goal is to compute $f(x)$, i.e. the distribution of the radioisotopes. The mathematical model of this process is a weighted Radon transform (for more information see [14, §3.2]). It is only very recently that explicit inversion formulas for this transform have been found, see [16] and [3]. The description of the range of the attenuated transform has been obtained in [15] while the range of the usual Radon transform is classical, see subsection 2.3.

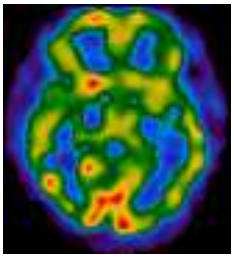

Figure 2: SPECT scan of a cocaine abused brain, Brigham and Women's Hospital, Harvard Medical School. Original at
http://brighamrad.harvard.edu/education/online/BrainSPECT

SAR is a form of radar in which sophisticated post-processing of data is used to produce a very narrow effective beam. In spite of the fact that it can only be used by moving instruments over relatively immobile targets, it has seen wide applications in remote sensing and mapping. In SAR, a plain terrain is surveyed by a single radar antenna attached to the side of an aircraft flying with constant speed along a straight trajectory. The quantity to be imaged

is the ground reflectivity, which is modelled by a function f. The strength of the radar signal reflected by a surface element dS at point x in the plane with distance r from the antenna is $1/r^2\ f(x)dS$. The reflected signal received at time t from those points x in the plane whose distance to the antenna is $tc/2$, where c denotes the speed of light. Thus, letting $y \in \mathbb{R}^2$ denote the ground coordinates of the plane at time t and $h \in \mathbb{R}$ its height, the total reflected signal at time t is

$$\frac{1}{r^2} \int_{|y-x|=\sqrt{r^2-h^2}} f(x)\, dx,\ r = \frac{tc}{2}.$$

This gives rise to an integral transform of f, the so-called average reflectivity function

$$g(y,r) = \int_{S^1} f(y + r\theta) d\theta.$$

For more details see [14, §3.6.1] and the references therein.

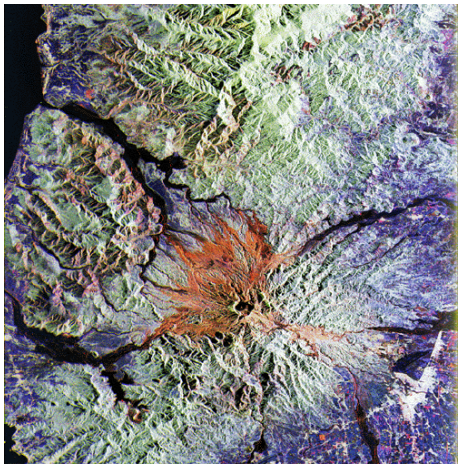

Figure 3: SAR image of Pinatubo volcano (NASA and JPL lab), original at http://veimages.gsfc.nasa.gov/526/PIA01709_md.jpg

The outline of the paper is the following. In Section 2 we give the definition and basic properties of the Radon transform. In particular, we describe its range and we prove its injectivity. We also provide three different inversion formulas. The first one is based on the Fourier transform. The second one uses the backprojection transform and fractional powers of the Laplacian. Finally, the last one (but the first ever used in CT) is obtained by expansion in spherical harmonics.

In Section 3 we deal with the problems associated to the fact that, in practice, the function must be recovered from only a finite number of values of its Radon transform. We first study how many samples are theoretically needed to recover a function with a given resolution. Then we study, in a semidiscrete

situation, how many directions are needed to determine the function with a prescribed resolution, assuming that we know the Radon transform in all the slices in the given directions.

Finally, in Section 4 we provide two numerical methods to invert the Radon transform. One is the filtered backprojection, which is the standard nowadays. The other is the so-called gridding method, which is based on the Fourier inversion formula.

2 The Radon transform

2.1 Definition and preliminaries

The Radon transform of a given function on $\mathbb{R}^n$, $n \geq 2$, is a function defined on the set of all hyperplanes of $\mathbb{R}^n$. Every hyperplane is determined by a normal vector to the hyperplane, $\theta \in S^{n-1}$, and its (signed) distance from the origin $s \in \mathbb{R}$, so that it can be written as

$$\theta_s := \{\, x \in \mathbb{R}^n \,:\, x \cdot \theta = s \,\}.$$

Note that $\theta_s = x + \theta^\perp = s\theta + \theta^\perp$, for any $x \in \theta_s$, where $\theta^\perp := \theta_0$ is the hyperplane perpendicular to θ passing through the origin.

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function which is integrable on every hyperplane of $\mathbb{R}^n$; for example, $f \in C(\mathbb{R}^n)$ satisfying the growth condition $|f(x)| = O(|x|^{-a})$, for some $a > n-1$. Then the *Radon transform of* f is the complex function $\mathcal{R}f$, defined on the set of hyperplanes of $\mathbb{R}^n$, whose value at a hyperplane equals the integral of f on that hyperplane, i.e.

$$\mathcal{R}f(\theta, s) = \mathcal{R}_\theta f(s) := \int_{\theta_s} f(x)\, dx = \int_{\theta^\perp} f(s\theta + y)\, dy \qquad (\theta \in S^{n-1}, s \in \mathbb{R}).$$

It is clear that $\mathcal{R}f$ is an even function on the cylinder $Z := S^{n-1} \times \mathbb{R}$, i.e. $\mathcal{R}f(-\theta, -s) = \mathcal{R}f(\theta, s)$, for every $(\theta, s) \in Z$.

The following result on continuity and growth of the Radon transform is easily obtained by integrating on polar coordinates over $\theta^\perp$ and using the standard continuity theorem of parametric integrals. We use the following terminology: a function $f \in C(\mathbb{R}^n)$ is *rapidly decreasing on* $\mathbb{R}^n$ if $|f(x)| = O(|x|^{-k})$ for every $k \in \mathbb{N}$. Similarly, $g \in C(Z)$ is *rapidly decreasing on* Z if $|g(\theta, s)| = O(|s|^{-k})$ for every $k \in \mathbb{N}$.

Proposition 1 *If $f \in C(\mathbb{R}^n)$ satisfies the growth condition $|f(x)| = O(|x|^{-a})$, for some $a > n-1$, then $\mathcal{R}f \in C(Z)$ and $|\mathcal{R}f(\theta, s)| = O(|s|^{n-1-a})$. In particular, if f is rapidly decreasing on $\mathbb{R}^n$, then $\mathcal{R}f$ is rapidly decreasing on Z.*

Next we consider the smoothness of the Radon transform. Let $D_j f = \frac{\partial f}{\partial x_j}$. By differentiation under the integral sign one can prove the following result.

Proposition 2 *Assume $f \in C^1(\mathbb{R}^n)$ satisfies the growth conditions*

$$|f(x)| = O(|x|^{-a}) \quad \text{and} \quad |D_j f(x)| = O(|x|^{-b}) \quad (j = 1, \dots, n),$$

for some exponents $a > n-1$ and $b > n$. Let $\mathscr{R}f$ be the extension of $\mathcal{R}f$ to $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ with homogeneity of degree -1, i.e. $\mathscr{R}f(\theta, s) := \frac{1}{|\theta|}\mathcal{R}f(\frac{\theta}{|\theta|}, \frac{s}{|\theta|})$. Then $\mathscr{R}f$ is C^1 on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$,

$$\frac{\partial \mathscr{R}f}{\partial s}(\theta, s) = \frac{1}{|\theta|^2} \sum_{j=1}^{n} \frac{\theta_j}{|\theta|} \mathcal{R}(D_j f)\left(\frac{\theta}{|\theta|}, \frac{s}{|\theta|}\right) \qquad (\theta \in \mathbb{R}^n \setminus \{0\},\ s \in \mathbb{R}), \quad (1)$$

and

$$\frac{\partial \mathscr{R}f}{\partial \theta_j}(\theta, s) = -\frac{\partial \mathscr{R}(x_j f)}{\partial s}(\theta, s) \qquad (\theta \in \mathbb{R}^n \setminus \{0\},\ s \in \mathbb{R},\ j = 1, \dots, n). \quad (2)$$

Most of the results on the Radon transform that we are going to discuss hold for continuous or smooth functions with some decay at infinity, but, in order to simplify the exposition, from now on we will only consider functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Recall that $\mathcal{S}(\mathbb{R}^n)$ is composed of all the functions $f \in C^\infty(\mathbb{R}^n)$ such that f and any of its partial derivatives are rapidly decreasing on $\mathbb{R}^n$. It is clear that Propositions 1 and 2 imply the following:

Corollary 3 *The Radon transform $\mathcal{R}$ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(Z) := \mathcal{S}(\mathbb{R}^{n+1})_{/Z}$.*

We consider the normalization of the Fourier transform given by:

Definition 1 *The Fourier transform of f is*

$$\widehat{f}(\zeta) = \mathcal{F}f(\zeta) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\zeta} f(x)\, dx,$$

and its inverse

$$\widetilde{f}(x) = \mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\zeta} f(\zeta)\, d\zeta.$$

Convolutions and Fourier transforms of functions in $\mathcal{S}(Z)$ are taken with respect to the second variable, that is, if $h, g \in \mathcal{S}(Z)$ then

$$(h * g)(\theta, s) := \int_{\mathbb{R}} h(\theta, s-t) g(\theta, t)\, dt \quad \text{and} \quad \widehat{h}(\theta, \sigma) := \int_{\mathbb{R}} e^{-2\pi i s\sigma} h(\theta, s)\, ds,$$

for any $\theta \in S^{n-1}$ and $s, \sigma \in \mathbb{R}$.

Next theorem states the main relationship between Fourier and Radon transforms.

Theorem 4 (Fourier projection-slice theorem) *If $f \in \mathcal{S}(\mathbb{R}^n)$ then*

$$(\mathcal{R}f)\widehat{\ }(\theta, \sigma) = \widehat{f}(\sigma\theta) \qquad (\theta \in S^{n-1}, \sigma \in \mathbb{R}). \quad (3)$$

Proof. By definition

$$(\mathcal{R}f)\widehat{\ }(\theta,\sigma) = \int_{\mathbb{R}} \left(\int_{\theta^\perp} e^{-2\pi i \sigma\theta\cdot(s\theta+y)} f(s\theta+y)\, dy \right) ds = \widehat{f}(\sigma\theta),$$

where the last identity follows by integrating on slices. □

Observe that (3) says that the Fourier transform of the "θ-projection" $\mathcal{R}_\theta f$ of f coincides with the "θ-slice" $\widehat{f}(\cdot\,\theta)$ of the Fourier transform of f.

Corollary 5 *The Radon transform $\mathcal{R} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(Z)$ is injective.*

Corollary 6 *If $f, g \in \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{R}(f * g) = (\mathcal{R}f) * (\mathcal{R}g)$.*

2.2 Uniqueness

We have just shown a "global" uniqueness result for the Radon transform: any function in $\mathcal{S}(\mathbb{R}^n)$ is determined by its Radon transform (Corollary 5). In many practical applications the Radon transform of a function is only known on a proper subset of its domain Z, and it is natural to ask whether these data determine the function, or at least its restriction to a certain subset of $\mathbb{R}^n$. In this section we present two results that deal with this kind of "local" uniqueness.

Theorem 7 (Support theorem or Hole theorem) *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let K be a convex compact set (a "hole") in $\mathbb{R}^n$. If $\mathcal{R}f(\theta, s) = 0$ for each hyperplane θ_s not meeting K, then $f \equiv 0$ on $\mathbb{R}^n \setminus K$.*

Proof. We only consider the compactly supported case, that is, $f \in C_c^\infty(\mathbb{R}^n)$. In this case, we are going to follow Strichartz's approach [22] which considerably simplifies the general proof due to Helgason [4, p.10].

Since K is a compact convex set, K is the intersection of all the closed balls which contain it, so we may assume that K is a closed ball. By making a translation we may also suppose that it is centered at the origin, i.e. $K = \overline{B}(0, R)$. Now we only have to show that $\mathcal{R}(x_j f)(\theta, s) = 0$, for $j = 1, \ldots, n$, $\theta \in S^{n-1}$ and $s > R$. This is so because then, by induction, $\mathcal{R}(pf)(\theta, s) = 0$, for every polynomial p and every hyperplane θ_s which does not meet $\overline{B}(0, R)$, and by Weierstrass approximation theorem, $f \equiv 0$ on those hyperplanes. This means that $f \equiv 0$ on $\mathbb{R}^n \setminus \overline{B}(0, R)$.

Pick $s > R$. By making a rotation, if necessary, we may assume that θ is the North Pole $\theta^* = (0, \ldots, 0, 1)$. Then

$$\mathcal{R}(x_n f)(\theta^*, s) = \int_{\theta_s^*} x_n f(x)\, dx = s \int_{\theta_s^*} f(x)\, dx = \mathcal{R}f(\theta^*, s) = 0,$$

so we only have to check that $\mathcal{R}(x_j f)(\theta^*, s) = 0$, for $1 \le j < n$, i.e.

$$\int_{\mathbb{R}^{n-1}} y_j f(y, s)\, dy = 0 \qquad (1 \le j < n). \tag{4}$$

Fix $1 \leq j < n$. Since $s > R$, $\mathcal{R}f(\theta, t) = 0$, for every $\theta \in S^{n-1}$ and $t > s$, so we may differentiate and, by (1) and (2), we obtain that

$$0 = \frac{\partial \mathscr{R} f}{\partial \theta_j}(\theta^*, t) = -\mathcal{R}(x_j D_n f)(\theta^*, t) = -\int_{\mathbb{R}^{n-1}} y_j D_n f(y, t)\, dy.$$

(Observe that in the second identity we have used the fact that $f \in C_c^\infty(\mathbb{R}^n)$ and the fundamental theorem of the calculus.) Now an integration on t along the interval $(s, +\infty)$ produces (4). □

Notice that in the above theorem we assume the vanishing of the Radon transform for $s \in \mathbb{R}$ outside a bounded interval and for every direction $\theta \in S^{n-1}$. Our second uniqueness result assumes that the Radon transform vanishes only on a certain set of directions, but for every $s \in \mathbb{R}$.

Theorem 8 *Let $S \subset S^{n-1}$ be a set of unicity for the homogeneous polynomials in $\mathbb{R}^n$ (i.e. the only homogeneous polynomial p in $\mathbb{R}^n$ which vanishes identically on S is $p \equiv 0$). If $f \in C_c(\mathbb{R}^n)$ satisfies $\mathcal{R}_\theta f \equiv 0$, for every $\theta \in S$, then $f \equiv 0$.*

Proof. Since $f \in C_c(\mathbb{R}^n)$, $\widehat{f}$ extends to an entire function on $\mathbb{C}^n$, so $\widehat{f}$ is a real-analytic function on $\mathbb{R}^n$. We write its power series expansion as

$$\widehat{f}(\zeta) = \sum_{k=0}^{\infty} p_k(\zeta) \qquad (\zeta \in \mathbb{R}^n),$$

p_k being a homogeneous polynomial of degree k. Then (3) shows that

$$\sum_{k=0}^{\infty} \sigma^k p_k(\theta) = \widehat{f}(\sigma\theta) = (\mathcal{R}f)\widehat{\ }(\theta, \sigma) = 0 \qquad (\theta \in S,\, \sigma \in \mathbb{R}).$$

Therefore $p_k \equiv 0$ on S so $p_k \equiv 0$ on $\mathbb{R}^n$, since S is a set of unicity for the homogeneous polynomials. Hence $\widehat{f} \equiv 0$ and we conclude that $f \equiv 0$. □

2.3 The range of the Radon transform

We say that a rapidly decreasing continuous function g on Z satisfies the *Helgason-Ludwig consistency conditions* if for every $m \in \mathbb{N}$ there is a homogeneous polynomial p_m on $\mathbb{R}^n$ of degree m (unless $p_m \equiv 0$), such that

$$\int_{\mathbb{R}} s^m g(\theta, s)\, ds = p_m(\theta) \qquad (\theta \in S^{n-1}).$$

It turns out that these conditions essentially characterize the range of the Radon transform on $\mathcal{S}(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$.

Theorem 9 *$\mathcal{R}(\mathcal{S}(\mathbb{R}^n))$ is the space of all even functions $g \in \mathcal{S}(Z)$ that satisfy the Helgason-Ludwig consistency conditions.*

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$. We already know that g is even and belongs to $\mathcal{S}(Z)$. Let us check that it also satisfies the Helgason-Ludwig consistency conditions: For every $m \in \mathbb{N}$ and $\theta \in S^{n-1}$, we have

$$\begin{aligned}\int_{\mathbb{R}} s^m g(\theta, s)\, ds &= \int_{\mathbb{R}} \left(\int_{\theta^\perp} ((s\theta + y)\cdot\theta)^m f(s\theta + y)\, dy \right) ds \\ &= \int_{\mathbb{R}^n} (x\cdot\theta)^m f(x)\, dx = p_m(\theta),\end{aligned}$$

where

$$p_m(y) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\int_{\mathbb{R}^n} x^\alpha f(x)\, dx \right) y^\alpha \qquad (y \in \mathbb{R}^n)$$

is a homogeneous polynomial on $\mathbb{R}^n$ which is either of degree m or $p_m \equiv 0$.

The remaining part of the proof is much more involved. Let $g \in \mathcal{S}(Z)$ be an even function which satisfies the Helgason-Ludwig consistency conditions. Observe that if $f \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\mathcal{R}f = g$ then (3) implies that

$$\widehat{f}(\sigma\theta) = \widehat{g}(\theta, \sigma) \qquad (\sigma > 0,\, \theta \in S^{n-1}). \tag{5}$$

It will be enough to prove that

$$F(\sigma\theta) := \widehat{g}(\theta, \sigma) \qquad (\sigma > 0,\, \theta \in S^{n-1})$$

extends to a function $F \in \mathcal{S}(\mathbb{R}^n)$. This is so because $F = \widehat{f}$ for some $f \in \mathcal{S}(\mathbb{R}^n)$ which satisfies $\mathcal{R}f = g$, due to (5) and the fact that $\widehat{g}$ is even (because so is g).

First note that $\widehat{g} \in \mathcal{S}(Z) = \mathcal{S}(\mathbb{R}^{n+1})_{/Z}$ and so $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$. In order to show that F has a C^∞ extension to $\mathbb{R}^n$ it is enough to prove that all the partial derivatives of F are locally bounded at the origin. Let $\xi = \sigma\theta$, where $\sigma > 0$ and $\theta \in S^{n-1}$. A lengthy calculation shows that the following expression for a partial derivative with respect to ξ in terms of the coordinates σ and θ:

$$\frac{\partial^q}{\partial \xi^\alpha} = \sum_{1 \le |\beta| + k \le q} \frac{c_{\beta,k}(\theta)}{\sigma^{q-k}} \frac{\partial^{|\beta|+k}}{\partial\theta^\beta \partial\sigma^k} \qquad (\alpha \in \mathbb{N}^n,\, |\alpha| = q \ge 1), \tag{6}$$

where the $c_{\beta,k}$'s are polynomials. In order to prove the local boundedness of $\frac{\partial^q F}{\partial \xi^\alpha}$ at the origin we multiply by $g(\theta, s)$ the decomposition

$$e^{-2\pi i s} = \sum_{m=0}^{q-1} \frac{(-2\pi i s)^m}{m!} + e_q(s) \qquad (s \in \mathbb{R}),$$

and integrate with respect to s. We obtain

$$F(\sigma\theta) = P_q(\sigma\theta) + \int_{\mathbb{R}} e_q(\sigma s)\, g(\theta, s)\, ds,$$

where P_q is a polynomial of degree smaller than q, because g satisfies the Helgason-Ludwig consistency conditions. Now observe that $\frac{\partial^q P_q}{\partial \xi^\alpha} = 0$, and so, differentiating under the integral sign and using (6) we get

$$\frac{\partial^q F}{\partial \xi^\alpha}(\sigma\theta) = \sum_{1\le|\beta|+k\le q} \int_{\mathbb{R}} \frac{c_{\beta,k}(\theta)}{\sigma^{q-k}}\, s^k e_q^{(k)}(\sigma s)\, \frac{\partial^{|\beta|} g}{\partial \theta^\beta}(\theta, s)\, ds.$$

Since $e_q^{(k)}(s) = (-2\pi i)^k e_{q-k}(s)$ and $|e_q(s)| \lesssim |s|^q$ (by Taylor's formula with integral remainder), it follows that

$$\left|\frac{\partial^q F}{\partial \xi^\alpha}(\sigma\theta)\right| \lesssim 1.$$

Hence F has a C^∞ extension to $\mathbb{R}^n$. Moreover,

$$\sup_{\substack{\sigma>1\\ \theta\in S^{n-1}}} \sigma^N \left|\frac{\partial^q F}{\partial \xi^\alpha}(\sigma\theta)\right| \le \sum_{1\le|\beta|+k\le q} \sup_{\substack{\sigma>1\\ \theta\in S^{n-1}}} \sigma^N \left|\frac{\partial^{|\beta|+k}\widehat{g}}{\partial\theta^\beta \partial\sigma^k}(\theta,\sigma)\right| < \infty,$$

for every $N \in \mathbb{N}$, which shows that the extension of F belongs to $\mathcal{S}(\mathbb{R}^n)$, and the proof is complete. □

A direct consequence of the previous result and the support theorem (Theorem 7) is the following

Corollary 10 *$\mathcal{R}(C_c^\infty(\mathbb{R}^n))$ is the space of all even functions $g \in C_c^\infty(Z) := C_c^\infty(\mathbb{R}^{n+1})_{/Z}$ that satisfy the Helgason-Ludwig consistency conditions.*

2.4 Inversion formulas for the Radon transform

The inversion algorithms rely on explicit inversion formulas for the Radon transform. Among the different existing inversion formulas we mention three. The first one is the Fourier inversion formula, which is the base of the gridding algorithm that will be analyzed in subsection 4.2. The second one involves the backprojection. It is precisely the relation of the backprojection and the Fourier transform that is in the basis of the filtered backprojection algorithm developed in 4.1. Finally, in subsection 2.5 we obtain an inversion formula from the expansion of a function f and its Radon transform $\mathcal{R}f$ in spherical harmonics. These last formulas for $n = 2$ were obtained by Cormack and Kershaw (see [2] and [6]).

Theorem 11 (Fourier inversion formula) *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$. Then*

$$f(x) = \frac{1}{2}\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\sigma|^{n-1}\, e^{2\pi i\sigma(x\cdot\theta)}\, \widehat{g}(\theta,\sigma)\, d\sigma\right) d\theta \qquad (x \in \mathbb{R}^n),$$

where $d\theta$ denotes the surface measure on S^{n-1}.

Proof. The Fourier inversion formula,

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\zeta}\, \widehat{f}(\zeta)\, d\zeta \qquad (x \in \mathbb{R}^n),$$

the Fourier projection-slice theorem (Theorem 4) and an integration in polar coordinates show that

$$f(x) = \int_0^\infty \sigma^{n-1} \left(\int_{S^{n-1}} e^{2\pi i x\cdot(\sigma\theta)}\, \widehat{f}(\sigma\theta) \right) d\sigma = \int_{S^{n-1}} F(\theta, x)\, d\theta,$$

where

$$F(\theta, x) := \int_0^\infty \sigma^{n-1}\, e^{2\pi i \sigma(x\cdot\theta)}\, \widehat{g}(\theta, \sigma)\, d\sigma.$$

Since $\int_{S^{n-1}} F(-\theta, x)\, d\theta = \int_{S^{n-1}} F(\theta, x)\, d\theta$, we have

$$f(x) = \frac{1}{2} \int_{S^{n-1}} (F(\theta, x) + F(-\theta, x))\, d\theta.$$

Moreover, it is clear that

$$F(\theta, x) + F(-\theta, x) = \int_{-\infty}^\infty |\sigma|^{n-1}\, e^{2\pi i \sigma(x\cdot\theta)}\, \widehat{g}(\theta, \sigma)\, d\sigma,$$

hence we are done. □

The previous inversion formula expresses f in terms of $(\mathcal{R}f)\widehat{\ }$, from which we are going to deduce an inversion formula expressing f in terms of $\mathcal{R}f$. In that formula the backprojection operator will play a fundamental role.

The *backprojection* of a suitable function g on $Z = S^{n-1} \times \mathbb{R}$ (for example, $g \in C(Z)$) is the function

$$\mathcal{R}^\# g(x) := \int_{S^{n-1}} g(\theta, x\cdot\theta)\, d\theta \qquad (x \in \mathbb{R}^n). \tag{7}$$

Observe that if $g = \mathcal{R}f$ then $g(\theta, x\cdot\theta)$ is the integral of f on the hyperplane passing through the point $x \in \mathbb{R}^n$ which is orthogonal to $\theta \in S^{n-1}$, so $\mathcal{R}^\# g(x)$ is the "mean" of the integrals of f on the hyperplanes passing through x. This observation suggests somehow the important role that the backprojection will play in the inversion formulas for the Radon transform.

The following statement summarizes some elementary properties of the backprojection.

Proposition 12

1. *If* $g \in C(Z)$ *then* $\mathcal{R}^\# g \in C(\mathbb{R}^n)$.
2. *If* $g \in C^1(Z)$ *then* $\mathcal{R}^\# g \in C^1(\mathbb{R}^n)$ *and* $D_j \mathcal{R}^\# g = \mathcal{R}^\#(\theta_j \frac{\partial g}{\partial s})$, *for* $1 \le j \le n$.

3. *If* $g \in C^\infty(Z)$ *then* $\mathcal{R}^\# g \in C^\infty(\mathbb{R}^n)$ *and* $D^\alpha \mathcal{R}^\# g = \mathcal{R}^\#(\theta^\alpha \frac{\partial^{|\alpha|} g}{\partial s^{|\alpha|}})$, *for* $\alpha \in \mathbb{N}^n$.

4. *The backprojection and the Laplacian conmute: If* $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ *and* $\square = \frac{\partial^2}{\partial s^2}$ *are the usual Laplacians on* $\mathbb{R}^n$, $n \geq 2$, *and* $\mathbb{R}$, *respectively, then*

$$\mathcal{R}^\#(\square g) = \Delta(\mathcal{R}^\# g) \qquad (g \in C^\infty(Z)).$$

5. $\mathcal{R}^\#$ *is the real formal adjoint operator of* $\mathcal{R}$*: if* $f \in \mathcal{S}(\mathbb{R}^n)$ *and* $g \in \mathcal{S}(Z)$ *then*

$$\int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}f(\theta, s)\, g(\theta, s)\, ds\, d\theta = \int_{\mathbb{R}^n} f(x)\, \mathcal{R}^\# g(x)\, dx.$$

In particular,

$$f * (\mathcal{R}^\# g) = \mathcal{R}^\#(\mathcal{R}f * g) \qquad (f \in \mathcal{S}(\mathbb{R}^n), g \in \mathcal{S}(Z)).$$

Recall that the *Hilbert transform* of $\psi \in \mathcal{S}(\mathbb{R})$ is the function

$$H\psi(t) := \frac{i}{\pi}\ \mathrm{PV} \int_{-\infty}^{\infty} \frac{f(t-s)}{s}\, ds \qquad (t \in \mathbb{R})$$

It is well-known that $H\psi$ is a slowly increasing C^∞ function on $\mathbb{R}$ whose Fourier transform as a tempered distribution on $\mathbb{R}$ is $(H\psi)\widehat{\ } = \mathrm{sgn} \cdot \widehat{f}$, where $\mathrm{sgn}(t) = 1$, for $t \geq 0$, and $\mathrm{sgn}(t) = -1$, for $t < 0$. Now we have the following inversion formula.

Theorem 13 *Let* $f \in \mathcal{S}(\mathbb{R}^n)$ *and* $g = \mathcal{R}f$. *Then*

$$f(x) = \frac{1}{2\,(2\pi i)^{n-1}} \mathcal{R}^\#(\Lambda g)(x) \qquad (x \in \mathbb{R}^n), \tag{8}$$

where

$$\Lambda g = \begin{cases} \dfrac{\partial^{n-1} g}{\partial s^{n-1}}, & \text{if } n \text{ is odd,} \\ H\left(\dfrac{\partial^{n-1} g}{\partial s^{n-1}}\right), & \text{if } n \text{ is even,} \end{cases}$$

and H *denotes the Hilbert transform acting on the real variable* s.

Proof. According to Theorem 11 we only have to prove that

$$\int_{-\infty}^{\infty} |\sigma|^{n-1}\, e^{2\pi i \sigma(x\cdot\theta)}\, \widehat{g}(\theta, \sigma)\, d\sigma = \frac{1}{(2\pi i)^{n-1}} \Lambda g(\theta, x \cdot \theta).$$

This identity follows from the Fourier inversion formula after checking that $|\sigma|^{n-1}\, \widehat{g}(\theta, \sigma) = (2\pi i)^{1-n} (\Lambda g)\widehat{\ }(\theta, \sigma)$. □

Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$ as in Corollary 8. Note that, given $\theta \in S^{n-1}$ and $x \in \mathbb{R}^n$, the derivative $\frac{\partial^{n-1} g}{\partial s^{n-1}}(\theta, x \cdot \theta)$ is determined by the values $g(\theta, t)$,

for $t \in \mathbb{R}$ in a neighborhood of $x \cdot \theta$. But recall that the hyperplanes containing $x \in \mathbb{R}^n$ are those of the form $\theta_{x\cdot\theta}$, for $\theta \in S^{n-1}$. Therefore, to compute

$$\int_{S^{n-1}} \frac{\partial^{n-1} g}{\partial s^{n-1}}(\theta, x \cdot \theta)\, d\theta \qquad (x \in \mathbb{R}^n)$$

we only need the values of the integrals of f along the hyperplanes which meet a neighborhood of x. Hence the inversion formula (8) shows that, if n is odd, $f(x)$ can be computed just from the integrals of f along all the hyperplanes through a neighborhood of x. In this sense, the inversion of the Radon transform for odd dimensions is a local problem.

Observe that for even dimensions formula (8) does not allow the same conclusion, because the Hilbert transform is not local, that is, the value of the Hilbert transform $H\psi$ of a function ψ at t cannot be computed by using only the values of ψ on a neighborhood of t.

For historical reasons let us mention that in the even dimensional case formula (8) can be written in the following way:

Corollary 14 *Let $f \in \mathcal{S}(\mathbb{R}^n)$, where n is even, and $g = \mathcal{R}f$. Then*

$$f(x) = c_n \int_0^\infty F_x^{(n-1)}(r)\, \frac{dr}{r} \qquad (x \in \mathbb{R}^n),$$

where

$$c_n = 2\, \frac{(-1)^{\frac{n}{2}}}{(2\pi)^n} \qquad \text{and} \qquad F_x(r) = \int_{S^{n-1}} g(\theta, x \cdot \theta + r)\, d\theta.$$

For $n = 2$ this is Radon's original inversion formula from 1917:

$$f(x) = -\frac{1}{\pi} \int_0^\infty \frac{dF_x(r)}{r} \qquad (x \in \mathbb{R}^2).$$

It is easy to check that (8) can be written in a compact way using fractional powers of $-\Box$ as

$$f = \frac{1}{2\,(2\pi)^{n-1}}\, \mathcal{R}^{\#}((-\Box)^{(n-1)/2} g).$$

On the other hand, we close this section by stating a deeper and more subtle inversion formula for the Radon transform in terms of the backprojection and fractional powers of $-\Delta$:

Theorem 15 *If $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$ then*

$$f = \frac{1}{2\,(2\pi)^{n-1}}\, (-\Delta)^{(n-1)/2} (\mathcal{R}^{\#} g).$$

For this kind of inversion formulas and all the beautiful deep theory related to them we refer the reader to [4] and [5].

2.5 Inversion formulas with spherical harmonics

We will obtain now an inversion formula for the Radon transform via the development in spherical harmonics.

A spherical harmonic Y_j of degree l is the restriction to S^{n-1} of a harmonic homogeneous polynomial of degree l on $\mathbb{R}^n$. The space $L^2(S^{n-1})$ has an orthogonal decomposition given by $L^2(S^{n-1}) = \bigoplus_{l=0}^{\infty} H_l$, where H_l denotes the vector space of spherical harmonics of degree l. If $N(n,l) = dim\, H_l$, and $\{Y_{l,k}\}_{k=1}^{N(n,l)}$ is an orthonormal basis of H_l, for $f \in L^2(S^{n-1})$ we have

$$f = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} f_{l,k} Y_{l,k},$$

where $f_{l,k} = \int_{S^{n-1}} f(w) Y_{l,k}(w)\, dw$.

In particular, if $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$,

$$f(x) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} f_{l,k}(|x|)\, Y_{l,k}\left(\frac{x}{|x|}\right),$$

$$g(\theta, x) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} g_{l,k}(s) Y_{l,k}(\theta)\,.$$

The inversion formula gives an expression of $f_{l,k}$ in terms of an integral transform of $g_{l,k}$ and a Gegenbauer polynomial. For $\lambda > -1/2$, the Gegenbauer polynomial C_l^{λ} of degree l is the orthogonal polynomial in $[-1,1]$ with respect to the weight $(1-x^2)^{\lambda-1/2}$ normalized so that $C_l^{\lambda}(1) = 1$. C_l^{λ} is even if l is even and odd if l is odd.

Theorem 16 *Let $f \in \mathcal{S}(\mathbb{R}^n)$. For $r > 0$ we have*

$$f_{l,k}(r) = C(n) r^{2-n} \int_r^{\infty} (s^2 - r^2)^{(n-3)/2} C_l^{(n-2)/2}\Big(\frac{s}{r}\Big) g_{l,k}^{n-1}(s)\, ds$$

where $c(n) = \frac{(-1)^{n-1}}{2\pi^{n/2}} \frac{\Gamma((n-2)/2)}{\Gamma(n-2)}$ if $n > 2$ and $c(2) = -1/\pi$.

Proof. We briefly sketch the different steps of the proof.

1. First it can be shown that the functions $g_{l,k}$ and $f_{l,k}$ satisfy the following Abel-type integral equation, namely,

$$g_{l,k}(s) = |S^{n-2}| \int_s^{+\infty} f_{l,k}(r) r^{n-2} C_l^{(n-2)/2}\Big(\frac{s}{r}\Big)\Big(1 - \frac{s^2}{r^2}\Big)^{(n-3)/2} dr,$$

where $|S^{n-2}|$ denotes the surface area of S^{n-2} and $|S^0| = 2$. The formula is derived from the Funck-Hecke theorem, stating that for any

$h \in L^1((1-r^2)^{(n-3)/2}dr)$,

$$\int_{S^{n-1}} h(\theta w)Y_l(w)dw = C(n,l)Y_l(\theta),$$

where $C(n,l) = |S^{n-2}| \int_{-1}^{1} h(t)C_l^{(n-2)/2}(t)(1-t^2)^{(n-3)/2}\,dt$. This is a reproducing formula for the space of spherical harmonics. The above integral formula can be expressed as a multiplicative convolution. Given $\varphi, \psi \in L^1(dr/r)$ let $\varphi * \psi(s) = \int_0^\infty \varphi(r)\psi(s/r)\ dr/r$. Then $g_{l,k}(s) = (r^{n-1}f_{l,k}) * b$, where

$$b(x) = |S^{n-2}| \begin{cases} C_l^{(n-2)/2}(x)(1-x^2)^{(n-3)/2}, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

2. The Mellin transform $M\varphi : (0,\infty) \to \mathbb{R}$, given by

$$M\varphi(s) = \int_0^\infty \varphi(x)x^{s-1}dx,$$

satisfies the identity $M(\varphi * \psi) = M\varphi \cdot M\psi$. Moreover $M(r^p\varphi)(s) = M\varphi(s+p)$ for $s > 0$, $p \geq 0$. Consequently,

$$M(r^{n-2}f_{l,k})(s) = M(r^{n-1}f_{l,k})(s-1) = \frac{M(g_{l,k})(s-1)}{M(b)(s-1)}.$$

But

$$\frac{1}{M(b)(s-1)} = \frac{1}{C_1}\frac{\Gamma(s+n-2)}{\Gamma(s-1)}Ma(s),$$

where $C_1 = |S^{n-2}|\Gamma(1/2)\Gamma((n-1)/2)$, and the function a is given by

$$a(x) = c_2 \begin{cases} (1-x^2)^{(n-2)/2}Ce^{(n-2)/2}(1/x), & \text{if } x < 1, \\ 0, & \text{if } x \geq 1, \end{cases}$$

$$c_2 = \begin{cases} \frac{\Gamma((n-3)/2)}{\Gamma(n-2)}, & \text{if } n > 2, \\ 2, & \text{if } n = 2. \end{cases}$$

We finally deduce that, since $M\varphi'(s) = (1-s)M\varphi(s-1)$,

$$\begin{aligned} M(r^{n-2}f_{l,k})(s) &= \frac{1}{C_1}\frac{\Gamma(s+n-2)}{\Gamma(s-1)}Ma(s)\frac{Mg'_{l,k}(s)}{1-s} \\ &= \frac{(-1)^{n-1}}{C_1}Ma(s)M(r^{n-2}g_{l,k}^{(n-1)}(s)) \\ &= \frac{(-1)^{n-1}}{C_1}M(a * r^{n-2}g_{l,k}^{(n-1)})(s)\,. \end{aligned}$$

The injectivity of the Mellin transform gives finally the desired formula.

□

3 The Radon transform discretization

In most of the applications of the Radon transform we only have a finite number of samples and we must reconstruct an arbitrary function from them. In order to perform the reconstruction we will reduce the function space we consider and we will need to introduce some tolerance error. The general scheme is as follows:

- We identify the space of functions we will deal with. The restrictions must be both, "natural" and technically suitable.
- Once we have identified the space to study, we need to consider several schemes of straight lines (which correspond to our samples) and we need some error estimates showing that we have gathered enough information to recover the function, at least theoretically.
- We need to provide some algorithms that reconstruct the function from the samples selected previously. It is not always the case that the best theoretical estimates can be achieved.
- Finally, one should make a study of the numerical stability of the algorithms used. In practice, most of the times, this is reduced to some numerical experiments.

We start by the first point. The function that we will analyze will have compact support in the unit ball $\bar{B}(0,1)$. This is a reasonable assumption in view of the applications to Computerized Tomography. We assume moreover that we are only interested in the details up to certain scale. This is usually modelled by considering only bandlimited functions, that is, functions f with Fourier transform $\widehat{f}$ supported in a ball $B(0,\Omega)$ (the smallest such Ω is called the bandwidth of f). The finest details that such functions can deal with are of scale $1/\Omega$. It is not possible for a function to be simultaneously of compact support and bandlimited, unless it vanishes identically. It is possible nevertheless to have functions essentially bandlimited in the following sense

Definition 2 *A function $f \in L^2$ is essentially Ω-bandlimited at level ε if*

$$\int_{B(0,\Omega)} |\widehat{f}|^2 \geq (1-\varepsilon)\|f\|_2^2.$$

Similarly, a function $f \in L^2$ is essentially R-supported at level ε if

$$\int_{B(0,R)} |f|^2 \geq (1-\varepsilon)\|f\|_2^2.$$

Definition 3 *The functions in L^2 that are Ω-bandlimited form a Hilbert space of functions that can be extended as entire functions, the Paley-Wiener space PW_Ω.*

There is a "folklore theorem" in signal theory that says that the dimension of the space of functions Ω-bandlimited with support in $B(0,R)$ is $|B(0,R)||B(0,\Omega)| = v_n^2(R\Omega)^n$, where $v_n = \pi^{n/2}\Gamma(1+n/2)$ is the volume of the unit ball in $\mathbb{R}^n$. This gives a lower bound on the number of samples that determine a function in this subspace.

Landau, Slepian and Pollak in a long series of papers (see [7], [8], [17], [18], [19], [20]) have formalized this result using essentially bandlimited (or essentially supported) functions. The result that we introduce derives from their work and it is the following. Let $PW^{\varepsilon}_{\Omega,R}$ denote the functions that are simultaneously essentially R-supported and essentially Ω-bandlimited both at level ε.

Theorem 17 *There are subspaces $H_{\Omega,R} \subset L^2(\mathbb{R}^n)$ with dimension satisfying*

$$\lim_{(R\Omega)\to\infty} \frac{\dim(H_{\Omega,R})}{|B(0,R)||B(0,\Omega)|} = 1$$

such that, for any $\varepsilon > 0$,

$$\limsup_{(R\Omega)\to\infty} \sup_{f\in PW^{\varepsilon}_{\Omega,R}} \frac{\operatorname{dist}(f, H_{\Omega,R})}{\|f\|_2} = 0.$$

The result is optimal, in the sense that there are no other subspaces $h_{\Omega,R}$ with the same property and such that $\liminf_{(R\Omega)\to\infty} \dim(h_{\Omega,R})/\dim(H_{\Omega,R}) < 1$.

To build the required subspaces we introduce the *concentration operator* $C_{\Omega,R}:\ PW_\Omega \to PW_\Omega$ defined as

$$C_{\Omega,R}(f) = P_\Omega[\chi_R \cdot f],$$

where χ_R stands for the characteristic function of the ball $B(0,R)$ and

$$P_\Omega(f) = \mathcal{F}^{-1}(\chi_\Omega \cdot \mathcal{F}(f)).$$

Let us see some of the properties of this operator needed in the proof of the theorem.

Proposition 18 *The operator $C_{\Omega,R}$ is bounded, compact, selfadjoint, positive and injective. Let $\lambda_k = \lambda_k(\Omega,R) > 0$, $k \in \mathbb{N}$, denote its eigenvalues arranged in decreasing order, we also have:*

(a) The trace of $C_{\Omega,R}$ is $\sum \lambda_k = |B(0,R)||B(0,\Omega)|$.

(b) The trace of $C^2_{\Omega,R} = C_{\Omega,R} \circ C_{\Omega,R}$ is $\sum \lambda_k^2 = |B(0,R)||B(0,\Omega)| \pm o((R\Omega)^n)$ as $(R\Omega) \to \infty$.

Proof. The selfadjointness and the boundedness of $C := C_{\Omega,R}$ are trivial. To prove the positivity, notice that for any function $f \in PW_\Omega$ we have

$$\langle C(f), f\rangle = \langle \chi_R \cdot f, f\rangle = \int_{B(0,R)} |f|^2. \tag{9}$$

Thus $C(f)$ estimates how much f is concentrated in $B(0,R)$. Equality (9) implies also that C is injective, because no $f \neq 0$ in PW_Ω can vanish in $B(0,R)$, due to the holomorphicity of f. Moreover, C has the following integral representation:

$$C(f)(y) = \int K(x,y) f(x)\, dx,$$

where $K(x,y) = \mathcal{J}_\Omega(y-x) \cdot \chi_R(x)$,

$$\mathcal{J}_\Omega(x) = \mathcal{F}^{-1}[\chi_\Omega](x) = \frac{\Omega^n J_{\frac{n}{2}}(2\pi|\Omega x|)}{|\Omega x|^{n/2}},$$

and J_α is the Bessel function of the first kind of order α. Then the compactness of C follows from the estimate

$$\iint K(x,y)^2\, dxdy = \iint \mathcal{J}_\Omega(x-y)^2 \chi_R(x)\, dxdy = |B(0,R)||B(0,\Omega)|. \quad (10)$$

By the spectral representation theorem, C has a complete orthonormal system of eigenvectors $f_k \in PW_\Omega$ with associated eigenvalues λ_k. Equality (9) applied to the eigenvalues gives

$$\lambda_k = \int_{B(0,R)} |f_k|^2.$$

Therefore $\lambda_k \in (0,1)$ is the relative concentration of the vector f_k in the ball $B(0,R)$. All functions $f \in PW_\Omega$ can be written as $f = \sum c_k f_k$, with $\|f\|^2 = \sum |c_k|^2$ and then

$$C(f) = \sum \lambda_k c_k f_k.$$

Since the functions $f_k(x)$ form an orthonormal basis of PW_Ω the reproducing kernel for this space is

$$\mathcal{J}_\Omega(x-y) = \sum f_k(x)\overline{f_k(y)}. \quad (11)$$

The convergence is in L^2 in each variable separately and thus uniform (since the f_k's belong to PW_Ω). Applying the operator C we have

$$C(\mathcal{J}_\Omega(\cdot - y))(x) = \sum \lambda_k f_k(x)\overline{f_k(y)},$$

and putting $y = x$ and integrating in x, we obtain

$$\int C(\mathcal{J}_\Omega(\cdot - x))(x)\, dx = \sum \lambda_k = \operatorname{trace} C.$$

Therefore the trace is

$$\begin{aligned} \operatorname{trace} C &= \iint \mathcal{J}_\Omega(y-x)\mathcal{J}_\Omega(x-y)\chi_R(x)\, dxdy \\ &= \int \mathcal{J}_\Omega(0)\chi_R(x)\, dx = |B(0,\Omega)||B(0,R)|. \end{aligned}$$

Similarly, to compute trace(C^2) we use again (11) (we denote by $C^x(f)$ the operator $C_{\Omega,R}$ acting on the function f as a function of x):

$$\begin{aligned}\sum \lambda_k^2 &= \int C^x(\overline{C^y(\mathcal{J}_\Omega(x-y))(t)})(t)\,dt \\ &= \iiint \mathcal{J}_\Omega(t-x)\mathcal{J}_\Omega(y-t)\mathcal{J}_\Omega(x-y)\chi_R(x)\chi_R(y)\,dxdydt.\end{aligned}$$

Integrating first in t and using the reproducing property of $\mathcal{J}_\Omega$ it follows that

$$\text{trace}(C^2) = \iint |\mathcal{J}_\Omega(x-y)|^2\chi_R(x)\chi_R(y)\,dxdy.$$

Let us see, finally, that

$$\text{trace}(C^2) = |B(0,\Omega)||B(0,R)| \pm o(|B(0,\Omega)||B(0,R)|).$$

Indeed, trace$(C^2) = I - J$, where

$$\begin{aligned} I &= \iint |\mathcal{J}_\Omega(x-y)|^2\chi_R(x)\,dxdy \quad \text{and} \\ J &= \iint |\mathcal{J}_\Omega(x-y)|^2\chi_R(x)(1-\chi_R(y))\,dxdy. \end{aligned}$$

Now $I = |B(0,\Omega)||B(0,R)|$, by(10). To estimate J we use a simple estimate of the Bessel functions. It follows that $|\mathcal{J}_\Omega(x)|^2 \lesssim \Omega^{2n}/(1+|\Omega x|^{n+1})$, so

$$J \lesssim \int_{|x|<R}\int_{|y|>R} \frac{\Omega^{2n}\,dydx}{1+(\Omega|x-y|)^{n+1}} \lesssim \int_0^{R\Omega} \frac{t^{n-1}\,dt}{1+(R\Omega-t)} \simeq (R\Omega)^{n-1}\log(R\Omega),$$

and the proof is complete. □

From these inequalities it can be observed that some eigenvalues are close to one (about $|B(0,R)||B(0,\Omega)|$ of them) and the remaining ones are very small. More precisely:

Proposition 19 *For any $0 < t < 1$ it holds*

$$\begin{aligned}&|B(0,R)||B(0,\Omega)| - \frac{1}{1-t}o(|B(0,R)||B(0,\Omega)|) \le \\ &\quad \#\{k:\ \lambda_k(\Omega,R) > t\} \le |B(0,R)||B(0,\Omega)| + \frac{1}{t}o(|B(0,R)||B(0,\Omega)|), \quad (12)\end{aligned}$$

when $(R\Omega) \to \infty$.

Before proving the proposition, observe that it implies that, for any $0 < \varepsilon < 1$,

$$\#\{k:\ \varepsilon < \lambda_k(\Omega,R) < 1-\varepsilon\} = o(|B(0,R)||B(0,\Omega)|),$$

and this means that most of the eigenvalues are either very close to 1 (bigger than $1-\varepsilon$) or very close to 0 (smaller than ε). The amount of eigenvalues close

to 1 (in between $1-\varepsilon$ and 1) is of the order $|B(0,R)||B(0,\Omega)|$, the intermediate eigenvalues (in between ε and $1-\varepsilon$) are much fewer than the big ones and, on the other hand, the small ones (smaller than ε) are infinite.

Proof. To prove this result it is convenient to introduce the following notation: let $\mu = \mu_{\Omega,R}$ denote the positive measure in the interval $[0,1]$ obtained by adding the Dirac masses of the eigenvalues λ_k, i.e.

$$\mu = \sum_k \delta_{\lambda_k(\Omega,R)}.$$

In these terms the statement we want to prove is that, for any $t \in (0,1)$,

$$\mu((t,1)) = |B(0,R)||B(0,\Omega)| \pm o(|B(0,R)||B(0,\Omega)|).$$

We will use an argument that appears in [9]. By the previous proposition the moments of first and second order of μ are known:

$$\begin{aligned} &\int_0^1 x\, d\mu(x) = |B(0,R)||B(0,\Omega)|, \\ &\int_0^1 x^2\, d\mu(x) = |B(0,R)||B(0,\Omega)| \pm o(|B(0,R)||B(0,\Omega)|) \end{aligned} \tag{13}$$

Consider the parabolas $q_t(x) = \frac{x(x-t)}{1-t}$, which have $q_t(0) = q_t(t) = 0$ and $q_t(1) = 1$, and $p_t(x) = \frac{-x^2+(1+t)x}{t}$, with $p_t(t) = p_t(1) = 1$ and $p_t(0) = 0$; see the picture:

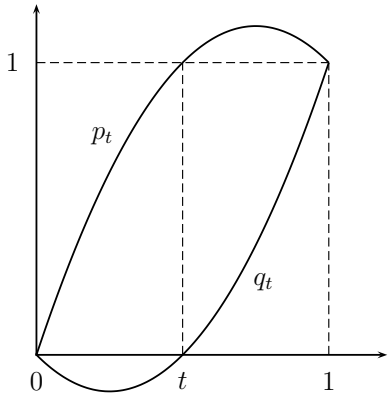

Figure 4: Auxiliary parabolas

It follows that

$$\int_0^1 q_t(x)\, d\mu(x) \le \mu([t,1]) \le \int_0^1 p_t(x)\, d\mu(x).$$

Since

$$\int_0^1 p_t(x)\,d\mu(x) = -\frac{1}{t}\int_0^1 x^2\,d\mu(x) + \frac{1+t}{t}\int_0^1 x\,d\mu(x),$$

and

$$\int_0^1 q_t(x)\,d\mu(x) = \frac{1}{1-t}\int_0^1 x^2\,d\mu(x) - \frac{t}{1-t}\int_0^1 x\,d\mu(x),$$

the estimates of the moments (13) yield (12) as desired. □

Now we prove Theorem 17. It follows from Proposition 19 that there exist $n(R\Omega), N(R\Omega) \in \mathbb{N}$, $n(R\Omega) \leq N(R\Omega)$, so that $N(R\Omega)/(|B(0,R)||B(0,\Omega)|) \to 1$, $n(R\Omega)/N(R\Omega) \to 1$, as $R\Omega \to \infty$, and simultaneously $\lambda_{N(R\Omega)}(\Omega, R) \to 0$ and $\lambda_{n(R\Omega)}(\Omega, R) \to 1$.

We take as $H_{\Omega,R}$ the vector space generated by $f_1, \ldots f_{N(R\Omega)}$. Any function f of unit norm in the Paley-Wiener space can be written as $f = \sum v_k f_k$ with $\sum |v_k|^2 = 1$. On the other hand, if f is essentially R-supported at level ε and $R\Omega$ is big enough so that $\lambda_{N(R\Omega)}(\Omega, R) \leq \varepsilon$, we have

$$(1-\varepsilon) \leq \int_{B(0,R)} |f|^2 = \langle C[f], f\rangle = \sum \lambda_k |v_k|^2 \leq \varepsilon + \sum_{k<N(R\Omega)} |v_k|^2,$$

and therefore

$$\sum_{k\geq N(R\Omega)} |v_k|^2 \leq 2\varepsilon.$$

Thus any function f essentially R-supported at level ε and of norm 1 can be approximated by its Fourier sum in the basis of functions of "prolate spherical" type f_k:

$$\|f - \sum_{k<N(r,\Omega)} \langle f, f_k\rangle f_k\|^2 \leq 2\varepsilon.$$

From this we deduce the main statement, because any function $f \in PW^{\varepsilon}_{\Omega,R}$ can be approximated by a function $\widetilde{f} = P_\Omega(f) \in PW_\Omega$ which is essentially R-supported at level 2ε, and this $\widetilde{f}$ can be approximated by functions in $H_{\Omega,R}$.

To conclude we are going to see that the result is optimal. Let us denote by $G_{\Omega,R}$ the space generated by $f_1, \ldots f_{n(R\Omega)}$. For any ε, take $R\Omega$ big enough so that $\lambda_{n(\Omega,R)} \geq 1-\varepsilon$. Any function $f \in G_{\Omega,R}$ is bandlimited and essentially R-supported at level ε, so the dimension of any space $h_{\Omega,R}$ that is at ε-distance from all vectors in $G_{\Omega,R}$ must be at least $n_{\Omega,R}$.

3.1 The semidiscrete case

In this section we study how many "directions" are sufficient to determine completely a function essentially bandlimited. We will assume that in each direction we know the integrals along all hyperplanes perpendicular to the given direction. In this sense, this is a semidiscrete case, as we only discretize the directions. In the next section we will consider a totally discrete case. The

semidiscrete study is due to Logan, see [11] and [12], although the sketch of the proof that we give is an original simplification different from theirs.

Let us start by a lemma in function theory of independent interest.

Lemma 20 *Let $\alpha \in \mathbb{R}$. For any $\varepsilon > 0$ there is an integer $k_0 \geq 0$ such that for any integer $k \geq k_0$ and any function g with $\operatorname{supp} \widehat{g} \subset [-1,1]$ and $g^{(j)}(0) = 0$, for $j = 0, \dots, k$, one has*

$$\int_{-(1-\varepsilon)k/2}^{(1-\varepsilon)k/2} |g(x)|^2 (1+|x|)^\alpha \, dx \leq \varepsilon \int_{\mathbb{R}} |g(x)|^2 (1+|x|)^\alpha \, dx,$$

provided the right hand side is finite.

Proof. We can assume that

$$\int_{\mathbb{R}} |g(x)|^2 (1+|x|)^\alpha \, dx = 1.$$

By the Paley-Wiener theorem we see that g is an entire function with a growth control of type $|g(z)| \leq K(1+|z|)^m e^{2\pi|\Im z|}$. As g vanishes k times at the origin we have a representation of the form

$$g(x) = \frac{1}{2\pi i} \int_{\Gamma_k} \frac{g(w) \sin^k(2\pi z/k)}{(w-x) \sin^k(2\pi w/k)} \, dw,$$

for all $x \in (-k/2, k/2)$, where Γ_k is a square centered at 0 of sidelength k with sides parallel to the axes.

In the square Γ_k the quotient $\frac{g(w)}{\sin^k(2\pi w/k)}$ is bounded by $C(1+|w|)^m$. Thus, if $x \in [(1-\varepsilon)k/2, (1-\varepsilon)k/2]$,

$$|g(x)| \leq C \left| \sin\Big(\frac{2\pi x}{k}\Big) \right|^k k^m \log\Big(\frac{1}{k/2 - |x|}\Big) \lesssim (1-\varepsilon)^k k^m \log(\epsilon k).$$

Therefore $|g(x)|$ converges uniformly to zero in the interval and the lemma is proved. □

For $m \in \mathbb{Z}$, $m \geq 0$, let H'_m be the space of all harmonic polynomials h of degree not greater than m that are even if the degree of h is even and odd if the degree of h is odd.

Definition 4 *A set $A \subset S^{n-1}$ is m-resolvent if it is a uniqueness set for H'_m.*

Theorem 21 *Let A be an m-resolvent set and let $f \in C_c^\infty(B(0,1))$. If $\mathcal{R}_\theta f \equiv 0$ for all $\theta \in A$, then*

$$\int_{|\zeta| \leq (1-\varepsilon)m/2} |\widehat{f}(\zeta)|^2 d\zeta \leq C_m \|f\|_2^2,$$

where the constant $C_m > 0$ only depends on m and satisfies $C_m \to 0$, as $m \to \infty$.

Proof. If $\mathcal{R}_\theta f \equiv 0$ for all $\theta \in A$, then for any $k \in \mathbb{N}$,

$$\mu_k(\theta) = \int_{\mathbb{R}} s^k \mathcal{R}_\theta f(s)\, ds = 0 \text{ for all } \theta \in A.$$

On the other hand, the range theorem assures that either $\mu_k \equiv 0$ or it is a homogeneous polynomial of degree k. This polynomial, when restricted to S^{n-1}, coincides with a harmonic polynomial of degree k that it is even if k is even and odd if k is odd (see [21, Theorem 2.1, p.139]). Thus, since A is an m-resolvent set, $\mu_k \equiv 0$ for $k \leq m$. This implies that

$$\frac{\partial^k}{\partial \sigma^k}(\widehat{\mathcal{R}_\theta f})(0) = 0, \quad k = 0, \ldots, m.$$

By assumption $\operatorname{supp} f \subset B(0,1)$, hence we may apply Lemma 20 with $g = \widehat{\mathcal{R}_\theta f}$ and deduce that

$$\int_{(1-\varepsilon)m/2}^{(1-\varepsilon)m/2} \sigma^{n-1} |\widehat{\mathcal{R}_\theta f}(\sigma)|^2\, d\sigma \leq C_m \int \sigma^{n-1} |\widehat{\mathcal{R}_\theta f}(\sigma)|^2\, d\sigma.$$

The proof is finished by integrating in the variable θ and using Theorem 4. □

The main application of the theorem is the following corollary that roughly speaking says that an $(m/2)$-bandlimited function is determined by its Radon transform on a set of directions that is m-resolvent.

Corollary 22 *If we know the Radon transform of an essentially bandlimited function with band $B(0,(1-\varepsilon)m/2)$ at level $r < 1 - C_m$ in an m-resolvent set of directions A, then we know it everywhere.*

Proof. Indeed, if there was another function g also essentially bandlimited with the same Radon transform in the directions of A, the difference $h = f - g$ would be essentially bandlimited and its Radon transform would vanish in the directions of A. Applying the theorem we see that $\int_{|w|<(1-\varepsilon)m/2} |\widehat{h}|^2 \leq C_m \|\widehat{h}\|^2 < \|\widehat{h}\|^2$ and since h is essentially $(1-\varepsilon)m/2$-bandlimited, we get $\int_{|w|>(1-\varepsilon)m/2} |\widehat{h}|^2 < \|\widehat{h}\|^2$. This implies that $h \equiv 0$. □

This corollary requires the exact knowledge of the Radon transform of f in all the directions of A. It would be desirable a similar result with only an "approximate" knowledge of the Radon transform in the directions of A, since, in practice, we only know its value on a finite number of slices in each of the directions of A.

Let us see how many samples we need to apply this Theorem. For a function essentially $m/2$ bandlimited we require the knowledge of its values on an m-determinant set A_m of directions. In the case we are dealing with, since the function is unidimensional and bandlimited with band $[-m/2, m/2]$, in each of the directions we need $2m$ samples to determine it in the interval $[-1,1]$ (this can be computed applying Theorem 17 to one dimensional signals

and we get $2m = \mathrm{vol}([-m/2, m/2])\,\mathrm{vol}([-1,1]))$. Therefore, all in all, we need $2m\,\mathrm{card}(A_m)$ samples. On the other hand, the dimension of H'_m is $\dim(H'_m) = 2\binom{m+n-1}{n-1} = 2\frac{m^{n-1}}{(n-1)!}(1+o(1/m))$. Therefore, we need a minimum of $\frac{4m^n}{(n-1)!}$ samples to apply the Theorem.

On the other hand, the number of needed samples according to the theorem of Landau, Pollack and Slepian (Theorem 17) is

$$\mathrm{vol}(B(0,m/2))\,\mathrm{vol}(B(0,1)).$$

In particular when the dimension is $n=2$ we need at least $(\pi^2/4)m^2 \simeq 2.467m^2$ samples. It seems that the sufficient condition that we get is of a factor $4/2.467$ redundant, therefore there is some room for improvement.

4 Reconstruction algorithms

Many algorithms have been proposed for the reconstruction of a function f from its Radon transform $g = \mathcal{R}f$, or more precisely, from a finite family of values $g_{j,l} = \mathcal{R}f(\theta_j, s_l)$. In this section we describe two reconstruction methods which are widely used in applications: the *filtered backprojection* algorithm which was proposed by Shepp and Logan in [10] and the *gridding method* (Fourier method) that was introduced in radio astronomy by [1]. Each of these schemes has several variants, depending on the scanning geometry and the specific numerical implementation.

In order to simplify the presentation we consider only the 2-dimensional case. We assume that the function f to be reconstructed is supported in $\mathbb{D} = B(0,1)$, belongs to $L^2(\mathbb{D})$ and has essential bandwidth Ω. Finally, we assume that the data are obtained scanning with *parallel standard geometry*, i.e. with a certain number p of equidistributed directions and taking in each of them $2q+1$ equispaced samples. The data have thus the following form:

$$g_{j,l} = g(\theta_j, s_l) \qquad \theta_j = e^{i\varphi_j},\ \varphi_j = j\pi/p, \quad j = 0,\dots,p-1 \qquad \Delta\varphi = \pi/p$$

$$s_l = l/q \qquad\qquad l = -q,\dots q \qquad\qquad \Delta s = 1/q. \tag{14}$$

Finally, it is important to remember that the mathematical models used in CT are idealizations of the complex relations between the object studied and the measured data. In numerical computations discrepancies occur for various reasons: the positive diameter of the X-ray source, the lack of accuracy of detectors, errors in the numerical approximation, etc...

4.1 The filtered backprojection

For its accuracy and fast implementation this is the reconstruction algorithm most widely used in applications. It has at least two advantages over frequency

domain interpolation schemes. First, the reconstruction procedure can be started as soon as the first projection has been measured. This clearly speeds up the process and reduces the amount of data stored at any time. The second reason is numerical: in the reconstruction process some kind of interpolation is often necessary. It turns out that interpolation in the space domain is usually simpler (linear interpolation is often enough) than in the Fourier domain, where more sophisticated methods are required.

The filtered backprojection is based in the following relation (see 5 in Proposition 12)

$$(\mathcal{R}^{\#}v) * f = \mathcal{R}^{\#}(v * \mathcal{R}f) \qquad (f \in \mathcal{S}(\mathbb{R}^n),\ v \in \mathcal{S}(Z)),$$

where $\mathcal{R}^{\#}$ is the backprojection operator defined by (7) at Section 2.4.

For $g = \mathcal{R}f$, and letting $V = \mathcal{R}^{\#}v$, the previous identity is

$$(V * f)(x) = \mathcal{R}^{\#}(v * g)(x) = \int_{S^{n-1}} (v * g)(\theta, x \cdot \theta)d\theta. \tag{15}$$

The key feature of the filtered backprojection algorithm is the choice of a so-called *point-spread function* V approximating the Dirac mass δ_0. Then the left-hand side of the identity above approximates $f(x)$ (in terms to be precised in each case). Once v is determined, using that $\mathcal{R}^{\#}v = V$, the integral on the right-hand side of the identity has to be discretized (something that depends on how the data have been obtained).

Identity (15) explains the name of the algorithm: first the data g are filtered with v (this gives $v * g$) and then the backprojection $\mathcal{R}^{\#}$ is applied.

Due to the nature of the backprojection operation, if the data were reconstructed directly, with no filtering, artifacts would appear in the reconstructed images. Moreover, because of the random nature of radioactivity, there is an inherent noise in the data that tends to make the reconstructed images rough.

As said above, the crucial point in this scheme is the choice of V. Convolution is a computationally intensive task so it is better to avoid it when possible. Since the convolution in the spatial domain is equivalent to a multiplication in the frequency domain, the filtering by V in the spatial domain can be performed by a simple multiplication by $\widehat{V}$ in the frequency domain.

Usually V is chosen so that $V * f$ deletes or de-emphasizes high frequencies, which are mostly observation noise. Since f has (essential) bandwidth Ω, one looks for V such that

$$(V * f)\widehat{\ }(\zeta) \simeq \begin{cases} \widehat{f}(\zeta), & \text{if } |\zeta| \le \Omega, \\ 0, & \text{if } |\zeta| > \Omega. \end{cases}$$

The relationship between V and v is explicit through the following distributional identity [13, Theorem 2.4]: if $g \in \mathcal{S}(Z)$ is even then

$$(\mathcal{R}^{\#}g)\widehat{\ }(\zeta) = 2|\zeta|^{1-n}\widehat{g}(\zeta/|\zeta|, |\zeta|).$$

In practice only radial symmetric functions $V(x) = V(|x|)$ are considered. Then v does not depend on θ and it is an even function of s. In this particular situation the identity above gives

$$\widehat{V}(\zeta) = 2|\zeta|^{1-n}\widehat{v}(|\zeta|), \tag{16}$$

where $\widehat{V}$ indicates the 1-dimensional Fourier transform.

In the usual cases the point-spread function V can be computed explicitly from $\widehat{V}$.

In order to reconstruct accurately functions f with essential bandwidth Ω we can take, for instance $\widehat{V}(\zeta) = \mathcal{X}_{B(0,\Omega)}(\zeta)$. More generally, consider a filter $\widehat{\phi}(\sigma)$ close to 1 when $|\sigma| \leq 1$ and with $\widehat{\phi}(\sigma) = 0$ for $|\sigma| > 1$, and define

$$\widehat{V}_\Omega(\zeta) = \widehat{\phi}\left(\frac{|\zeta|}{\Omega}\right).$$

According to (16), the corresponding function v_Ω (such that $\mathcal{R}^{\#}v_\Omega = V_\Omega$) is determined by the identity

$$\widehat{v}_\Omega(\sigma) = \frac{1}{2}|\sigma|^{n-1}\widehat{\phi}\left(\frac{|\sigma|}{\Omega}\right). \tag{17}$$

Multiplication by $|\sigma|^{n-1}$ in the Fourier domain corresponds (except for a constant) to the operation $\frac{\partial^{n-1}}{\partial s^{n-1}}$ when n is odd, and $H\frac{\partial^{n-1}}{\partial s^{n-1}}$ when n is even (see Theorem 13). Thus (17) can be viewed as the Fourier counterpart of the Radon reconstruction formulas (see (8)).

In applications many different $\widehat{\phi}$'s have been proposed. It seems, however, that there is no justification for any specific choice other than the expiremental results. In other words, the choice of $\widehat{\phi}$ is still more an art than a science.

Next, we show three common filters in dimension $n = 2$.

(a) *Ram-Lak filter.* Introduced in this context by Ramachandran and LakshmiNarayanan (1971). It is associated to the standard low-pass filter $\widehat{\phi}(\sigma) = \mathcal{X}_{[0,1]}(\sigma)$. Here (17) yields $\widehat{v}_\Omega(\sigma) = 1/2|\sigma|\mathcal{X}_{[0,1]}(|\sigma|/\Omega)$, hence

$$v_\Omega(s) = \int_{\mathbb{R}} \widehat{v}_\Omega(\sigma)e^{2\pi i\sigma s}d\sigma = \frac{1}{2}\int_{-\Omega}^{\Omega} |\sigma|e^{2\pi i\sigma s}d\sigma.$$

Splitting the integral for $\sigma > 0$ and $\sigma < 0$, and integrating by parts we get

$$\begin{aligned}\int_{-\Omega}^{\Omega} |\sigma|e^{2\pi i\sigma s}d\sigma &= 2\Omega^2\frac{\sin(2\pi\Omega s)}{2\pi\Omega s} + 2\frac{\cos(2\pi\Omega s) - 1}{(2\pi s)^2} \\ &= 2\Omega^2\Big(\operatorname{sinc}(2\pi\Omega s) - \frac{1}{2}\big(\operatorname{sinc}(\pi\Omega s)\big)^2\Big),\end{aligned}$$

where $\operatorname{sinc}(x) = \sin(x)/x$ is the cardinal sinus, and finally,

$$v_\Omega(s) = \Omega^2 u(2\pi\Omega s), \qquad \text{where} \quad u(s) = \operatorname{sinc}(s) - \frac{1}{2}\left(\operatorname{sinc}\left(\frac{s}{2}\right)\right)^2.$$

(b) *Cosine filter.* Here $\widehat{\phi}(\sigma) = \cos(\frac{\sigma\pi}{2})\mathcal{X}_{[0,1]}$ and the corresponding filter is

$$v_\Omega(s) = \frac{\Omega^2}{2}\left(u\left(2\pi\Omega s + \frac{\pi}{2}\right) + u\left(2\pi\Omega s - \frac{\pi}{2}\right)\right), \qquad \text{where } u \text{ is as in (a).}$$

(c) *Shepp-Logan filter.* Now $\widehat{\phi}(\sigma) = \operatorname{sinc}(\frac{\sigma\pi}{2})\mathcal{X}_{[0,1]}$ and

$$v_\Omega(s) = \frac{2\Omega^2}{\pi}u(2\pi\Omega s), \qquad \text{where} \quad u(s) = \begin{cases} \frac{\pi/2 - s\sin s}{(\pi/2)^2 - s^2}, & \text{if } s \neq \pm\pi/2, \\ 1/\pi, & \text{if } s = \pm\pi/2. \end{cases}$$

Discretization of (15). In a first instance the convolution integral of (15) has to be discretized:

$$(v_\Omega * g)(\theta, s) = \int_{\mathbb{R}} v_\Omega(s-t)g(\theta,t)\,dt = \int_{-1}^{1} v_\Omega(s-t)g(\theta,t)\,dt.$$

According to (16), v_Ω has bandwidth Ω, while g as a function of s is essentially bandlimited (by the usual Fourier relation (3)). Thus, except for a negligible error (g is only essentially bandlimited), Shannon's Theorem [13, Theorem 4.2] can be applied to $f_1(t) = v_\Omega(s-t)$, $f_2(t) = g(\theta,t)$ and the grid $(\Delta s)\mathbb{Z}$, with $\Delta s \leq 1/(2\Omega)$. This yields

$$(v_\Omega * g)(\theta, s) = \Delta s \sum_{l=-q}^{q} v_\Omega(s - s_l)\; g(\theta, s_l). \tag{18}$$

Notice that with our normalization of the Fourier transform the critical density in Shannon's theorem is $1/(2\Omega)$. Next step consists of discretizing the backprojection

$$(V * f)(x) = \mathcal{R}^{\#}(v * g)(x) = \int_0^{2\pi} (v * g)(\theta, x\cdot\theta)d\varphi, \qquad \text{where } \theta = e^{i\varphi}.$$

A computation shows that the π-periodic function $h(\varphi) = (v * g)(\theta, x\cdot\theta)$ has essential bandwidth $4\pi\Omega$, in the sense that

$$\widehat{h}(k) = \frac{1}{2\pi}\int_0^{2\pi} (v * g)(\theta, x\cdot\theta)e^{-ik\varphi}d\varphi$$

is negligible for $|k| > 4\pi\Omega$ [13, p.84-85]. Thus we can apply Shannon's theorem [13, Theorem 4.2], at the cost of only a negligible error: if $\Delta\varphi \leq 1/(2\Omega)$ we obtain the approximation

$$\begin{aligned}(V * f)(x) &= \int_0^{2\pi} (v * g)(\theta, x\cdot\theta)d\varphi = \frac{\pi}{p}\sum_{j=0}^{2p-1} (v * g)(\theta_j, x\cdot\theta_j) \\ &= \frac{2\pi}{p}\sum_{j=0}^{p-1}(v * g)(\theta_j, x\cdot\theta_j),\end{aligned}$$

where the last identity follows by π-periodicity.

This together with (18), and always taking $\max\{\Delta\varphi, \Delta s\} \leq 1/(2\Omega)$, yields

$$(V * f)(x) = \frac{2\pi}{p} \sum_{j=0}^{p-1} \Delta s \sum_{l=-q}^{q} v_\Omega(x \cdot \theta_j - s_l) g(\theta_j, s_l) \tag{19}$$
$$= \frac{2\pi}{p} \Delta s \sum_{j=0}^{p-1} \sum_{l=-q}^{q} v_\Omega(x \cdot \theta_j - s_l) g(\theta_j, s_l).$$

The algorithm, as given by (19), is computationally too demanding. It requires $O(pq)$ operations for each $f(x)$, and since f has (essential) bandwidth Ω it is necessary to compute $f(x)$ in a lattice with stepsize $1/(2\Omega)$. This gives a total number of operations of order $O(\Omega^2 pq) \simeq O(\Omega^4)$. This complexity can be reduced with a linear interpolation. Since $v_\Omega * g$ has bandwidth Ω it is determined by $(v_\Omega * g)(\theta_j, s_l)$, which can be computed with $O(pq^2)$ operations. Then the values $(v_\Omega * g)(\theta_j, x \cdot \theta_j)$ required to compute $V * f$ are obtained from the previous ones by linear interpolation. This reduces the number of operations to $O(\Omega^3)$.

Final algorithm

Step 1. For every direction θ_j, $j = 1, \ldots, p$ take the discrete convolution

$$h_{j,k} = \Delta s \sum_{l=-q}^{q} v_\Omega(s_k - s_l)\, g_{j,l} \qquad (k = -q, \ldots, q).$$

Step 2. For each x compute the discrete backprojection using a linear interpolation of the values obtained in Step 1:

$$f_A(x) = \frac{2\pi}{p} \sum_{j=0}^{2p-1} (1 - \eta) h_{j,k} + \eta h_{j,k+1},$$

where $k = k(j,x) = \left[\frac{x \cdot \theta_j}{\Delta s}\right]$, $\eta = \eta(j,x) = \frac{x \cdot \theta_j}{\Delta s} - \left[\frac{x \cdot \theta_j}{\Delta s}\right]$ and $[a]$ denotes the integer part of a.

4.2 The gridding method

This is considered at present the most accurate Fourier reconstruction method. Its main feature is the use of a weight function W, close to 1 in the reconstruction region $\mathbb{D}$, vanishing when $|x| \geq a > 1$, and with Fourier transform concentrated near 0. Then, the product $W \cdot f$ is close to f in the reconstruction region and we can therefore approximate f as soon as we can approximate the Fourier

transform

$$(W \cdot f)\widehat{\ }(\zeta) = (\widehat{W} * \widehat{f})(\zeta) = \int_{\mathbb{R}^2} \widehat{W}(\zeta - \eta)\widehat{f}(\eta)\, d\eta$$
$$= \int_0^\infty \int_{S_1} \widehat{W}(\zeta - \sigma\theta)\widehat{f}(\sigma\theta)\, \sigma d\theta\, d\sigma$$
$$= \int_0^\infty \int_{S^1} \widehat{W}(\zeta - \sigma\theta)\widehat{g}(\theta, \sigma)\, \sigma d\theta\, d\sigma, \tag{20}$$

where the last identity follows from (3).

First $(W \cdot f)\widehat{\ }(\zeta)$ is evaluated in a cartesian grid; this explains the name of this method. With an inverse 2-dimensional FFT one obtains $W \cdot f$, and finally f is recovered by diving by W.

Since $(W \cdot f)\widehat{\ }$ has bandwidth 1 (f is supported in $\mathbb{D}$), in order to apply Shannon's theorem we need to consider a lattice with stepsize 1/2. Thus we want to evaluate $\{(W \cdot f)\widehat{\ }(k/2)\}_{k\in\mathbb{Z}^2}$.

The success of this method resides in an appropriate discretization of the integral (20). The values θ_j in this discretization are determined by the data (14), since the Fourier transform $\widehat{g}$ does not affect the first variable. Then the trapezoidal rule with stepsizes $\Delta\varphi = \pi/p$ in the angle and $\Delta\sigma$ in the remaining variable yields the approximation

$$(W \cdot f)\widehat{\ }(k/2) = \sum_{l=0}^{\infty} \sum_{j=0}^{2p-1} \widehat{W}(k/2 - l\Delta\sigma\theta_j)\widehat{g}(\theta_j, l\Delta\sigma)\, (l\Delta\sigma)\frac{\pi}{p}\, \Delta\sigma. \tag{21}$$

In the derivation of this formula, however, we ignore that the integral is defined only in $[0, \infty)$, and not in the whole $\mathbb{R}$. This results in the annihilation of all the information stored in the cell represented by $l = 0$ (the information given by $\widehat{g}(\theta_j, 0)$). This produces artifacts and must be corrected.

Discretization of bandlimited functions defined in $[0, \infty)$. Denote $\widetilde{f} = \mathcal{F}^{-1} f$ and start with the classical *Poisson's summation formula*

$$\sum_{l\in\mathbb{Z}} f(l\Delta\sigma) = \frac{1}{\Delta\sigma} \sum_{l\in\mathbb{Z}} \widetilde{f}\big(\frac{l}{\Delta\sigma}\big) \qquad (f \in \mathcal{S}(\mathbb{R})).$$

In the second sum separate the term corresponding to $l = 0$, which is $\frac{1}{\Delta\sigma}\widetilde{f}(0) = \frac{1}{\Delta\sigma}\int_{\mathbb{R}} f(\sigma)d\sigma$, so that

$$\int_{\mathbb{R}} f(\sigma)\, d\sigma = \Delta\sigma \sum_{l\in\mathbb{Z}} f(l\Delta\sigma) - \sum_{l\in\mathbb{Z}\setminus\{0\}} \widetilde{f}\big(\frac{l}{\Delta\sigma}\big). \tag{22}$$

This general formula will be used for $f(\sigma) = g(\sigma)\ \mathrm{sgn}(\sigma)$, with g such that $g(0) = 0$. As pointed out before Theorem 13, the Hilbert transform of f satisfies the relationship $(Hf)\widehat{\ }(\zeta) = \mathrm{sgn}(\zeta)\widehat{f}(\zeta)$. Therefore

$$(H\widetilde{g})\widehat{\ }(\zeta) = \mathrm{sgn}(\zeta)\widehat{\widetilde{g}}(\zeta) = f(\zeta) = \widehat{\widetilde{f}},$$

and $\widetilde{f} = H\widetilde{g}$. Then (22) yields

$$\int_{\mathbb{R}} g(\sigma)\ \mathrm{sgn}(\sigma)\ d\sigma = \Delta\sigma \sum_{l\in\mathbb{Z}} g(l\Delta\sigma)\ \mathrm{sgn}(l\Delta\sigma) + r,$$

where

$$r = \sum_{l\in\mathbb{Z}\setminus\{0\}} (H\widetilde{g})\big(\frac{l}{\Delta\sigma}\big) = \frac{i}{\pi} \sum_{l\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} \frac{\widetilde{g}(s)}{\frac{l}{\Delta\sigma} - s} ds = \frac{i}{\pi} \sum_{l=1}^{\infty} \int_{\mathbb{R}} \widetilde{g}(s) \frac{2s\ ds}{(\frac{l}{\Delta\sigma})^2 - s^2}.$$

Using that $s\widetilde{g}(s) = \frac{1}{2\pi i}\widetilde{g}'(s)$ we obtain finally

$$r = \frac{1}{\pi^2} \sum_{l=1}^{\infty} \int_{\mathbb{R}} \widetilde{g}'(s) \frac{ds}{(\frac{l}{\Delta\sigma})^2 - s^2}.$$

If g has bandwidth A and $\Delta\sigma << 1/A$ the factor s^2 in the denominator is negligible with respect to $(l/\Delta\sigma)^2$ and we obtain, approximately

$$r \simeq \frac{1}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{(l/\Delta\sigma)^2} \int_{-A}^{A} \widetilde{g}'(s) ds \simeq \frac{1}{\pi^2} (\Delta\sigma)^2 \sum_{l=1}^{\infty} \frac{1}{l^2} g'(0) = \frac{1}{6} (\Delta\sigma)^2 g'(0).$$

Hence, for g with $g(0) = 0$ we have

$$\int_{\mathbb{R}} g(\sigma)\, \mathrm{sgn}(\sigma)\, d\sigma = \Delta\sigma \sum_{l\in\mathbb{Z}} g(l\Delta\sigma)\ \mathrm{sgn}(l\Delta\sigma) + \frac{1}{6} (\Delta\sigma)^2 g'(0).$$

Use this formula for $g(\sigma) = \sigma G(\sigma)$, where $G(\sigma) = \int_{S^1} \widehat{W}(\zeta - \sigma\theta)\widehat{g}(\theta, \sigma) d\theta$ is the even function appearing in (20). Since G is even,

$$\int_0^{\infty} \sigma G(\sigma)\, d\sigma = \Delta\sigma \sum_{l=1}^{\infty} l\Delta\sigma G(l\Delta\sigma) + \frac{(\Delta\sigma)^2}{12} G(0),$$

which can be written in a unified form as

$$\int_0^{\infty} \sigma G(\sigma)\, d\sigma = \Delta\sigma \sum_{l=0}^{\infty} \sigma_l\ G(l\Delta\sigma), \qquad \text{where } \sigma_l = \begin{cases} l\Delta\sigma, & \text{if } l > 0, \\ \frac{\Delta\sigma}{12}, & \text{if } l = 0. \end{cases}$$

In our case we have thus

$$\int_0^{\infty} \int_{S^1} \widehat{W}(\zeta - \sigma\theta)\widehat{g}(\theta, \sigma)\sigma\ d\theta\ d\sigma = \Delta\sigma \sum_{l=0}^{\infty} \sigma_l \int_{S^1} \widehat{W}(\zeta - l\Delta\sigma\theta)\widehat{g}(\theta, l\Delta\sigma)\ d\theta.$$

This yields a new discretization of $(W \cdot f)\widehat{\ }(k/2)$:

$$\begin{aligned} (W \cdot f)\widehat{\ }(k/2) &= \Delta\sigma \sum_{l=0}^{\infty} \sigma_l \int_{S^1} \widehat{W}(k/2 - l\Delta\sigma\theta)\widehat{g}(\theta, l\Delta\sigma)\ d\theta \\ &= \Delta\sigma \sum_{l=0}^{\infty} \sigma_l \sum_{j=0}^{2p-1} \widehat{W}(k/2 - l\Delta\sigma\theta_j)\widehat{g}(\theta_j, l\Delta\sigma) \frac{\pi}{p}. \end{aligned}$$

Notice that this discretization differs from that in (21) only in the term corresponding to $l = 0$, which corrects the Riemann sum for the error introduced by truncating the integration domain at 0.

In order to find a good stepsize for $\Delta\sigma$ it is necessary to determine the bandwidth of the integrand in (20) as a function of σ. Let us see that the function $\sigma \mapsto \sigma\widehat{W}(\zeta - \sigma\theta)\widehat{g}(\sigma\theta)$ has bandwidth $a + 1$.

The function $\sigma \mapsto \widehat{g}(\theta, \sigma)$ has obviously bandwidth 1, hence it is enough to prove that $\sigma \mapsto \widehat{W}(\zeta - \sigma\theta)$ has bandwidth a. By the distributional identity

$$\delta(x \cdot \theta - s) = \int_{\mathbb{R}} \widehat{\delta}(\sigma) e^{2\pi i(x\cdot\theta - s)\sigma} = \int_{\mathbb{R}} e^{2\pi i(x\cdot\theta - s)\sigma} d\sigma,$$

we see that, for $|s| > a$,

$$\begin{aligned}\int_{\mathbb{R}} \widehat{W}(\zeta - \sigma\theta) e^{-2\pi i s\sigma} d\sigma &= \int_{\mathbb{R}} \Big\{ \int_{|x|\le a} W(x) e^{-2\pi i x\cdot(\zeta - \sigma\theta)} dx \Big\} e^{-2\pi i s\sigma} d\sigma \\ &= \int_{|x|\le a} W(x) e^{-2\pi i x\cdot\zeta} \int_{\mathbb{R}} e^{2\pi i(x\cdot\theta - s)\sigma} d\sigma dx \\ &= \int_{|x|\le a} W(x) e^{-2\pi i x\cdot\zeta} \delta(x\cdot\theta - s) dx = 0.\end{aligned}$$

This and the previous computations suggest the choice of a stepsize $\Delta\sigma << 1/(a+1)$.

Final algorithm

Step 1. Discretization of $\widehat{g}(\theta_j, N\Delta\sigma)$, with $\Delta\sigma << 1/(a+1)$ and $N = -Q, \ldots, Q$, where Q is such that $N\Delta\sigma \subset [-\Omega, \Omega]$ (i.e. $N \le \Omega/\Delta\sigma$):

$$\widehat{g}_{j,N} = \frac{1}{q} \sum_{l=-q}^{q} e^{-2\pi i N\Delta\sigma \frac{l}{q}} g_{j,l} \qquad (j = 0, \ldots, p-1\ ;\ N = -Q, \ldots, Q).$$

Step 2. For every $k \in \mathbb{Z}^2$, $|k| \le q$, compute the discretization of $(W\cdot f)\widehat{\ }(k/2)$:

$$z_k = \frac{\pi}{p} \Delta\sigma \sum_{N=0}^{Q} \sum_{j=0}^{2p-1} \sigma_N \widehat{W}(k/2 - N\Delta\sigma\theta_j)\, \widehat{g}_{j,N}.$$

Since $\widehat{W}$ decays rapidly, only few terms are relevant in this sum (those corresponding to the values with $|k/2 - N\Delta\sigma\theta_j| \le C$).

Step 3. Compute an approximation of $f(m/q)$, $m \in \mathbb{Z}^2$, $|m| \le q$ by taking the inverse Fourier transform of $(W \cdot f)\widehat{\ }$ and dividing by W:

$$f_m = \frac{1}{W(\frac{m}{q})} \sum_{|k|\le q} e^{2\pi i\pi k\cdot\frac{m}{q}} z_k.$$

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WORKSHOP IBEROAMERICANO DE MATEMÁTICAS APLICADAS CHILLÁN (CHILE), 6–8 DE AGOSTO DE 2007

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Estimado lector:

En este volumen se incluyen algunos de los textos de las conferencias presentadas en el primer "Workshop Iberoamericano de Matemáticas Aplicadas", realizado en la ciudad de Chillán (Chile), del 6 al 8 de Agosto de 2007.

En dicho evento participaron investigadores de diversas universidades de Brasil, Chile, España, Francia e India, así como algunos estudiantes de Pre-Grado interesados por las Matemáticas. En un ambiente muy distendido y agradable, tuvimos la oportunidad de intercambiar durante un par de dias información sobre varios problemas de campos tan diversos como la teoría del control, la homogeneización, las ecuaciones en derivadas parciales no lineales, los problemas de interacción fluido-estructura, teoría fuzzy, entre otros.

Creemos que la celebración de este evento fue un gran éxito y que esta impresión es compartida por todos los participantes. También tenemos la firme convicción que este primer Workshop verá su continuación en años sucesivos.

Los trabajos seleccionados en este volumen son los siguientes:

1. J.L. Boldrini, G. Planas, Some thoughts on mathematical modeling of solidification and melting.

2. Y. Chalco-Cano, M.A. Rojas-Medar, H. Román-Flores, Sobre ecuaciones diferenciales difusas con derivada generalizada.

3. B. Climent-Ezquerra, F. Guillén-González, M.A. Rojas-Medar, A review on reproductivity and time periodicity for incompressible fluids.

4. P. Cumsille, T. Takahashi, Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid.

El resto de trabajos presentados aparecerá en un próximo volumen de este Boletín.

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SOME THOUGHTS ON MATHEMATICAL MODELING OF SOLIDIFICATION AND MELTING

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Abstract

We present some ideas that have been used along the years to model the dynamics of phase changes, specially those involving solid and liquid phases. We briefly comment on sharp interface models and then discuss the case of diffuse interface models, in special the one using the phase field methodology. It would be impossible to present here all the relevant models and results that have been studied; we will thus reduce our scope drastically, considering only few situations having some of the main aspects that a reasonable model for a solidification process should have and for which a mathematical analysis have been done.

Key words: *phase changes, solidification, diffuse interfaces, phase field*
AMS subject classifications: *80A22 74Nxx*

1 Introduction

Mathematical problems arising from the modeling of phase transitions processes have been studied since the late 19th century when J. Stefan formulated the solution for finding the temperature distribution and freezing-front history of a solidifying slab of water.

Since then, particularly in the last 30 years, the problem bearing his name and others related to phase change have been extended to include such complex phenomena as the solidification of alloy systems and supercooling.

The study of the dynamics of phase boundaries can be grouped into two broadly categories:

(a) sharp interfaces

(b) diffuse interfaces

A close relationship may be established between these two methodologies for certain cases [9].

2 Sharp Interface Methodology

The simplest sharp interface model is the classical Stefan formulation [45, 1]. A material, which may be in either of two phases, e.g., solid or liquid, occupies a spatial region $\Omega \subset R^n$ separated by an interface $\Gamma(t)$ (an regular hypersurface). Let $T_m \in R$ be the melting temperature at equilibrium, i.e., the temperature at which both phases may coexist in equilibrium separated by the interface. The absolute temperature $T(x,t)$ must then satisfy the heat diffusion equation

$$C_v T_t = K\Delta T \qquad \text{in} \quad \Omega \setminus \Gamma(t),$$

where C_v is the specific heat and K the thermal conductivity divided by density.

On interface, the rate of change in latent heat equals the amount by which the heat flux jumps across the interface. Therefore, we have the following conditions

$$\begin{aligned} T &= T_m \qquad \text{on} \qquad \Gamma(t), \\ lv &= -K\big[\nabla T \cdot \hat{n}\big]_-^+ \qquad \text{on} \qquad \Gamma(t), \end{aligned}$$

where l is the latent heat, v is the (normal) velocity to the interface, $\hat{n}$ is the unit normal at $\Gamma(t)$ and $[\cdot]_-^+$ denotes the jump in the quantity as one crosses the interface from solid to liquid. Thus the sharp-interface problem is stated as finding $T(x,t)$ and $\Gamma(t)$ subject to suitable initial and boundary conditions.

Thus, Stefan formulation leads to **free boundary value problems**, which may be present certain important difficulties, as we describe in the next subsection.

2.1 Sharp Interface Methodology: difficulties

(a) For sharp interface models, it is difficult to incorporate in a natural (i.e, in a physically sound) way several more complex physical phenomena which may be relevant in several solidification processes. Examples of these phenomena, are certain supercooling and superheating effects, finiteness of the interface thickness, surface tension effects and so on.

(b) The requirement that the sharp interfaces are regular may be lost as time goes by.

(c) The interfaces may evolve to complex geometries (like dentrities), that make very difficult to perform numerical simulations for such models.

(d) In many real solidification/melting processes, the interfaces are actually not sharp; there may be even large transitions regions.

3 Diffuse Interfaces Methodology

Thus, the main idea of the diffuse interfaces methodology is to take in consideration from the beginning that real physical interfaces always have some

thickness, maybe small, and also a structure. That is, they actually **transition layers**.

Such interfaces are then assumed to be determined by the values of certain specified variables, that may have physical meaning (like densities, enthalpy, solid fraction, and so on, depending on the problem being consider).

More generally, auxiliary variables **phase fields**, that are related to the physical ones by suitable relations or equations, can be used

Thus, the diffuse interface methodology use a different approach in which the sharp-interface is replaced by continuous variations which are measured by certain variables (in the simplest case, just one), and the regions of different phases are determined by level sets for these variables.

For instance, being $\phi(x,t)$ a phase field variable, where $x \in \Omega \subset R^n$, the domain where the solidification/melting process occurs, and $t \in [0,T]$, the interval of time of interest, in a certain problem the **solid region** at time t may be identified with

$$\Omega_s(t) = \{x \in \Omega : \ \phi(x,t) \leq 0\},$$

for instance, whereas the **non-solid (melted) region** is identified with

$$\Omega_{ml}(t) = \{x \in \Omega : \ \phi(x,t) > 0\}.$$

Actually, the melted region $\Omega_{ml}(t)$ may be split in two other regions determined by another value of ϕ, say value 1 to fix the idea.

Thus, one has the **liquid region**

$$\Omega_l(t) = \{x \in \Omega : \ \phi(x,t) \geq 1\},$$

and the **mushy region** (transition region).

$$\Omega_m(t) = \{x \in \Omega : \ 0 < \phi(x,t) < 1\}.$$

Thus, the diffusive interfaces methodology, at least in what is concerned with phase change itself, avoids the free boundary value problem characteristic of Stefan formulations, by using level sets of certain fields coupled with the other physical fields of the model at hand.

3.1 An example of the diffuse interface methodology: the enthalpy method

With this methodology, one use the enthalpy as the field that determines the phase. For instance, a classical relationship between the enthalpy w and the temperature θ is given by the following inclusion $\beta(\cdot)$, which is a strictly monotone increasing graph.

$\beta(\theta) = c_s\theta$ for $\theta < 0$, with $c_s > 0$,
$\beta(\theta) = L + c_l\theta$ for $\theta > 0$, with $c_l > 0$,
$\beta(0) = [0, L]$.

Here, $L > 0$ is the latent heat and $c_s > 0$ and $c_l > 0$ are characteristic constants of the solid and liquid materials, respectively. In this case, the enthalpy corresponds to the total internal energy, that is, the vibration energy plus the energy used to break the crystal structure (in the melt).

Equations for a simple enthalpy model A typical equation is the following:

$$\begin{aligned} &\frac{\partial w}{\partial t} - \Delta K(\theta) = 0 \quad \text{in } \Omega, \\ &w \subseteq \beta(\theta), \end{aligned}$$

where $K(\cdot)$ is a suitable function.

The regions of different phases are determined as follows.

The solid region:

$$\Omega_s = \{\text{points where the enthalpy is nonpositive}\},$$

The liquid region

$$\Omega_l = \{\text{points where the enthapy is larger than or equal to } L\},$$

The mush region:

$$\Omega_m = \{\text{points with positive enthalpy that are less than } L\}.$$

Thus, the non-solid (melted) region is then given by:

$$\Omega_{ml} = \Omega_m \cup \Omega_l = \{\text{points where the enthalpy is positive}\}.$$

4 Phase Field Methodology

Phase field models have several advantages:

(a) These models may incorporate in a systematic way several phenomena which are difficult to incorporate in Stefan formulations.

(b) The phase-field method has emerged as a powerful tool that enables the modeling of complex growth structures occurring during phase transitions, like dendritic patterns formed in pure metals and in binary alloy solidification processes.

4.1 Some ideas concerning the derivation of the equations of phase field models

However, when one uses phase fields, it is necessary to obtain their equations to complete the system of equations for the model.

The main idea is the following: the usual physical fields are derived using the usual balance laws; phase field equations are derived with the help of the free energy functional associated to the physical process being studied.

A simple example Let us consider the derivation of the standard phase-field model for pure materials.

Assume for simplicity that the solidification/melting process is isothermal, that is constant temperature θ.

Consider the free energy in terms of a phase field ϕ:

$$\mathcal{F}(\phi,\theta)=\int_\Omega\Big[\frac{1}{2}\epsilon^2|\nabla\phi|^2+F(\phi)-\theta\phi\Big]dx$$

The first term is the energy associated to the interfaces (transition layers); the second term is the energy potential associated to the phase state; the third term is the energy associated to the energy of vibration of molecules (thermal energy.)

Here, $\epsilon>0$ is related to the thickness of the interfacial layer,

It is usual to use the classical two-well potential $F(\phi)=\frac{1}{8}(\phi^2-1)^2$ as the potential associated to the phase state. In this case, the minimum points of the potential are $\bar{\phi}=-1$, and $\hat{\phi}=1$.

Then,

$\Omega_s(t)=\{x\in\Omega:\ \phi(x,t)\le\bar{\phi}\}$,

$\Omega_l(t)=\{x\in\Omega:\ \phi(x,t)\ge\hat{\phi}\}$,

$\Omega_m(t)=\{x\in\Omega:\ \bar{\phi}<\phi(x,t)\le\hat{\phi}\}$,

Then, the time evolution of the phase-field is governed by the kinetic equation

$$\tau\phi_t=\epsilon^2\Delta\phi-F'(\phi)+\theta.$$

In the case of the classical two-wells potential $F'(\phi)=\frac{1}{2}(\phi^3-\phi)$.

Observe that the right-hand side of the previous equations corresponds exactly to the Euler-Lagrange equation associated to $\mathcal{F}(\phi,\theta)$. This is not casual. In fact, the previous equation is obtained by the requirement that the total free-energy $\mathcal{F}(\phi,\theta)$ decays along the trajectories of the system, and moreover:

$$\frac{d\mathcal{F}(\phi,\theta)}{dt}=-\frac{1}{\tau}\int_\Omega|\phi_t|^2dx,$$

under suitable boundary conditions for ϕ, like homogeneou Neumann and Dirichlet boundary conditions.

The kinetic equation could also be derived by requiring that ϕ is a critical point of the action

$$\int_0^t\int_\Omega\Big[\tau|\phi_t|^2-(\frac{1}{2}\epsilon^2|\nabla\phi|^2+F(\phi)-\theta\phi)\Big]dxdt$$

There are alternative derivations. Penrose and Fife use an entropy functional which gives a kinetic equation for the phase field ensuring monotonic increase of the entropy in time.

Another remark is that is possible in some cases to perform a rigorous asymptotic analysis as $\epsilon\to 0+$ and related phase field model with Stefan like models.

Thus, phase field models have a sound physical basis and provide simple and elegant descriptions of phase transition processes. Moreover, it is more versatile than the enthalpy method because it allows effects such as supercooling to be included.

4.2 Phase-field models for pure materials

Phase-field models may be used to describe phase transitions with other effects. For instance, a phase field model for the solidification/melting of a pure material due to thermal effects (that is, with variable temperature,) may be obtained by coupling the kinetic equation for the phase with a simple energy balance equation:

$$\tau\phi_t = \epsilon^2\Delta\phi + f(\phi) + \theta \text{ in } \Omega\times(0,\infty),$$
$$(\theta + \tfrac{l}{2}\phi)_t = \theta_t + \tfrac{l}{2}\phi_t = K\Delta\theta \quad \text{in } \Omega\times(0,\infty),$$

4.3 Phase-field models for solidification of binary alloys

Several phase-field models have also been developed for binary alloys. One example is the following:

$$\alpha\epsilon^2\frac{\partial\phi}{\partial t} - \epsilon^2\Delta\phi = \frac{1}{2}(\phi-\phi^3) + \beta\,(\theta - c\theta_A - (1-c)\theta_B) \text{ in } \Omega\times(0,\infty),$$

$$C_V\frac{\partial\theta}{\partial t} + \frac{l}{2}\frac{\partial\phi}{\partial t} = \nabla\cdot[K_1(\phi)\nabla\theta] \quad \text{in } \Omega\times(0,\infty),$$

$$\frac{\partial c}{\partial t} = K_2\,(\Delta c + M\nabla\cdot[c(1-c)\nabla\phi]) \text{ in } \Omega\times(0,\infty),$$

$$\frac{\partial\phi}{\partial n} = 0, \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,\infty),$$

$$\phi(0) = \phi_0, \theta(0) = \theta_0, \qquad c(0) = c_0 \qquad \text{in } \Omega.$$

5 Phase-field models with convection in the melt

Up to now it was implicitly assumed that during the solidification/melting process, there was no flow of the molten material.

In many practical situations, however, this assumption is not satisfactory because such flows may in exist and affect in important ways the outcome of the process of phase change. In fact, melt convection adds new length and time scales to the problem and results in morphologies that are potentially much different from those generated by purely diffusive heat and solute transport. Moreover, not only does convection influence the solidification pattern, but the evolving microstructure can also trigger unexpected and complicated phenomena. This suggests that models that do not consider melt convection may have some limitations.

Phase-field models with convection in the melt: mathematical difficulties

One must realize, however, that the inclusion of this possibility brings another very difficult aspect to an already difficult problem. For this, it is enough to observe that such a flow must occur only in an a priori unknown non-solid region, and thus one may be left with a rather difficult free boundary-value problem.

Moreover, it is not completely simple to include such phenomenon in the previous models.

5.1 First possibility: the equation for the fluid motion modified by a Carman-Koseny type term

Important assumptions:

(a) a phase field methodology is used to model phase change;

(b) the solid phase is rigid and stationary.

(c) there is functional relationship between the solid fraction and the phase field.

(d) the equation for the velocity is a modified Navier-Stokes equations including buoyancy forces modeled by Boussinesq approximation and a Carman-Koseny type term to model the flow in mushy regions.

Since these modified Navier-Stokes equations only hold in a priori unknown non-solid regions, we actually have a free boundary value problem.

One example of such model is the following.

$$\begin{aligned}
&\frac{\partial \varphi}{\partial t} - \alpha \Delta \varphi = a\varphi + b\varphi^2 - \varphi^3 + \theta && \text{in } Q,\\
&\frac{\partial \theta}{\partial t} - \Delta \theta + v.\nabla \theta = \frac{\ell}{2}\frac{\partial f_s}{\partial \varphi}(\varphi)\frac{\partial \varphi}{\partial t} && \text{in } Q,\\
&\frac{\partial v}{\partial t} - \nu \Delta v + (v.\nabla)v + \nabla p + J(f_s(\varphi))v = \vec{\sigma}\ \theta && \text{in } Q_{ml},\\
&\operatorname{div} v = 0 && \text{in } Q_{ml},\\
&v = 0 && \text{in } \overset{0}{Q}_s,
\end{aligned} \tag{1}$$

subject to the following boundary conditions

$$\begin{aligned}
&\frac{\partial \varphi}{\partial n} = 0, \quad \theta = 0 && \text{on } S,\\
&v = 0 && \text{on } S_{ml}.
\end{aligned} \tag{2}$$

and also to the following initial conditions

$$\begin{aligned}
&\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x) && \text{in } \Omega,\\
&v(x,0) = v_0(x) && \text{in } \Omega_{ml}(0),
\end{aligned} \tag{3}$$

where φ_0, θ_0 and v_0 are suitably given functions such that for compatibility v_0 is identically zero outside $\Omega_{ml}(0)$.

The original Carman-Koseny expression for $J(f_s)$, derived by using the hypothesis that the mushy zones behave like moving porous media, is

$$J(f_s) = \frac{C_m f_s^2}{(1-f_s)^3}, \tag{4}$$

where C_m is a positive constant depending on the material, and f_s is the solid fraction

The solid fraction $f_s = f_s(\phi)$ is a known smooth function of the phase field, with values in $[0,1]$.

5.2 Second possibility: the equation for the fluid motion modified by using a viscosity depending on the phase

An example of a such model is the following.

$$\varphi_t - \alpha^2 \Delta\varphi + u.\nabla\varphi = \varphi(\varphi-1)(1-2\varphi) - |\nabla\varphi|\, c \quad \text{in} \quad Q, \tag{5}$$

$$c_t - \Delta c + u.\nabla c - \operatorname{div}(k(\varphi)\nabla\varphi) = 0 \quad \text{in} \quad Q, \tag{6}$$

$$\varphi = 0, \quad c = 0 \quad \text{on} \quad S, \tag{7}$$

$$\varphi(x,0) = \varphi_0(x), \quad c(x,0) = c_0(x) \quad \text{in} \quad \Omega, \tag{8}$$

and such that:

$$u \quad \text{is a rigid motion in each connected component of } Q_s, \tag{9}$$

$$u_t - \operatorname{div}(\nu(\varphi)\nabla u) + (u.\nabla)u + \nabla p = \vec{\sigma}\, c \quad \text{in } Q_{ml}, \tag{10}$$

$$\operatorname{div} u = 0 \quad \text{in } Q_{ml}, \tag{11}$$

$$u = 0 \quad \text{on } \partial Q_{ml} \cap S, \tag{12}$$

$$\operatorname{tr}(u|_{Q_{ml}}) = \operatorname{tr}(u|_{Q_s}) \quad \text{on } \partial Q_{ml} \cap \partial Q_s, \tag{13}$$

$$u(x,0) = u_0(x) \text{ in } \Omega_{ml}(0). \tag{14}$$

Although the previous equations are coupled, here we have grouped them in two sets to emphasize the fact that the two groups have different characters; this grouping also to facilitates our tentative of explaining them. Let us start, by describing variables and the parameters appearing in the problem. Besides the physical variables defined earlier, the parameter $\alpha > 0$ is a constant related to the width of the transitions layers; for simplicity of exposition, we assumed the diffusion coefficient for c to be one; $k(\cdot) > 0$ is a given function related to the segregation coefficient of the solute due to difference of phases; p is the hydrostatic pressure, $\nu(\cdot) > 0$ is a given function related to the viscosity; $\vec{\sigma}$ is a constant vector associated to the Boussinesq approximation for the buoyancy forces appearing due to difference in solute concentration; again for simplicity of exposition we took the reference concentration to be zero. The initial data φ_0, c_0 and u_0 are suitably given functions.

Observe that the equations in the first group hold in Q, while this is not so for (10) and (11), since the flow only is possible for molten material. Thus, to attain a better understanding of the second group of equations and conditions, we must define the subsets of Q that appear there.

The sets Q_l, Q_m and Q_s correspond respectively to the liquid, mushy and solid regions and are defined as follows:

$$\begin{aligned} &Q_l = \{(x,t) \in Q\,;\, \varphi(x,t) = 0\},\\ &Q_s = \{(x,t) \in Q\,;\, \varphi(x,t) = 1\},\\ &Q_m = \{(x,t) \in Q\,;\, 0 < \varphi(x,t) < 1\},\\ &Q_{ml} = Q \setminus Q_s = Q_m \cup Q_l. \end{aligned} \tag{15}$$

Moreover, for each time $t \in [0,T]$, we define $\Omega_s(t) = \{x \in \Omega\,;\, \varphi(x,t) = 1\}$, $\Omega_{ml}(t) = \Omega \setminus \Omega_s(t)$, $S_{ml} = \{(x,t) \in \overline{Q}\ ;\ x \in \partial\Omega_{ml}(t)\}$ and $\mathcal{C}_{(x,t)}$ is the connected component of a point (x,t) in $\overset{o}{Q}_s$.

With this notation, (10) and (11) only hold in Q_{ml}; (12) means that, as an usual viscous fluid, flow of the molten material occurring next to the vessel wall must stick to it; besides the vessel is supposed be still.

Condition (9) translates the implicity assumption that the parts of solidified material behave like rigid material. It may happen that parts of solidified material be advected by the flow of the surrounding molten material, but if this is so, it must move in rigid motion.

5.3 A theorem of existence

Technical Hypotheses:

($\mathbf{H}_0$) $\Omega \subset R^N$, $N = 2$ or 3 is a bounded domain with C^2-boundary.

($\mathbf{H}_1$) $\nu(\cdot) \in C^0([0,1))$, such that $0 < \nu_1 \le \nu(\cdot)$, and $\lim_{y \to 1-} \nu(s) = +\infty$.

($\mathbf{H}_2$) $k(\cdot) \in C^1(R)$ such that $|k(\cdot)| \le \rho$ and $|k'(\cdot)| \le \rho_1$ for $\rho, \rho_1 \in R^+$.

Theorem 1 *Assume that hypotheses* ($\mathbf{H}_0$), ($\mathbf{H}_1$), ($\mathbf{H}_2$) *hold. Suppose also that* $N = 2$, $(\varphi_0, c_0) \in (W^{3/2,4}(\Omega))^2$ *and* $u_0 \in H$. *Then there exists a solution* (φ, c, u, p) *of the problem* (5), (6), (10), (7), (12), (8) (14), *which satisfies* $(\varphi, c) \in (W_4^{2,1}(Q))^2$, $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $u_t \in L^2(0,T;V')$, $p \in L_2(Q)$.

Basic idea for the proofs of existence of weak solutions of the previous model with convection

We introduce a approximate problem by a regularization technique whose purpose is to allow the dealing with the Navier-Stokes equations in whole domain instead of in unknown regions. Thus, the problem will be adequately regularized with the help of a positive parameter, and the existence of solutions for this regularized problem will obtained by using the Leray-Schauder degree theory and other techniques. Then, as this parameter approaches zero, a sequence of regularized solutions is obtained. With the help of suitable estimates and

compactness arguments, a limit of a subsequence is then proved to exist and to be a solution of problem (1), (2), (3).

Theorem 1 is proved by using a regularization technique and semi-discrete Galerkin method together with a suitable maximum principle. The purpose such regularization of the equations is to deal with the Navier-Stokes equations in whole domain instead of unknown regions. Thus, the problem will be adequately regularized with the help of a positive parameter, and the existence of solutions for this regularized problem will obtained by using semi-discrete Galerkin method together a maximum principle. Then, as this parameter approaches zero, a sequence of regularized solutions is obtained. With the help of suitable estimates and compactness arguments, a limit of a subsequence is then proved to exist and to be a solution of problem (5), (6), (10), (7), (12), (8) (14).

Some of the works that follow were not referenced to earlier in the present work. However, we think that they form a collection of interesting papers and books that are relevant to the previous discussions and that any reader will profit from consulting them. So we decided to keep them in the list.

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SOBRE ECUACIONES DIFERENCIALES DIFUSAS

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Resumen

Varios trabajos relacionados con la existencia y unicidad de soluciones para ecuaciones diferenciales difusas son basados en que el problema de Cauchy, con la H-derivada, es equivalente a una ecuación integral, similar al caso clásico. En este trabajo estudiamos esta equivalencia considerando una ecuacion diferencial difusa con derivada generalizada.

Palabras clave: *Ecuaciones diferenciales difusas, derivadas generalizadas, representación integral*
Clasificación por materias AMS: *34A99 03E72 26E25*

1 Introducción

Existen varias maneras de extender la noción de derivada al contexto difuso. Una de las primeras generalizaciones es debida a Puri y Ralescu [14] y esta basada en la noción debida a Hukuhara en el contexto multívoca (H-derivada). Posteriormente, Kaleva [8] usa esta noción para desarrollar una teoría de ecuaciones diferenciales difusas (EDF).

Usando los conceptos de H-derivada muchos resultados sobre existencia y unicidad son obtenidos para el problema de Cauchy difuso:

$$\begin{cases} \dot{x}(t) = f(t, x(t)); \\ x(t_0) = x_0 \end{cases} \tag{1}$$

donde $f : T \times \mathcal{F}^n \to \mathcal{F}^n$ es continua y $\mathcal{F}^n$ es la familia de conjuntos difusos con niveles compactos y convexos. Estos resultados son basados en que el problema de Cauchy con la derivada de Hukuhara es equivalente a una ecuación integral del tipo Aumann, vea [8] (vea también [4, 11, 17, 15, 16, 17, 18]), similar al caso clásico.

La interpretación del problema (1) tiene algunas desventajas, ya que en muchos casos la solución obtenida tiene la propiedad que $diam(supp\ x(t))$ es ilimitado cuando $t \to \infty$, demostrando que esta interpretación no generaliza en una forma apropiada el caso clásico [1], [2], [5]. Así, es necesario la introducción

del concepto de derivadas generalizadas, la cual amplia la clase de funciones difusas diferenciables. Con este concepto surgen nuevas soluciones para una EDF, con otras propiedades.

En este trabajo, mostramos la equivalencia entre la ecuación diferencial difusa (1), considerando $x^{'}$ en el sentido generalizado [2], y una ecuación integral, donde la integral es en el sentido de Aumann [4, 8].

2 Conceptos básicos

Denotaremos por $\mathcal{K}^n$ la familia de todos los subconjuntos compactos no vacios de $\mathbb{R}^n$. Si $A, B \in \mathcal{K}^n$ y $\lambda \in \mathbb{R}$, entonces las operaciones de adición y multiplicación por un escalar son definidas como

$$A + B = \{a + b : a \in A,\, b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}.$$

Un conjunto difuso en un conjunto universo X es una función $u : X \to [0, 1]$. El valor $u(x) \in [0, 1]$ es el grado con que un elemento x pertenece a conjunto difuso u. Si u es un conjunto difuso en $\mathbb{R}^n$, definimos el nivel α de u como $[u]^\alpha = \{x \in \mathbb{R}^n : \underline{u(x) \geq \alpha\}}$ si $0 < \alpha \leq 1$ y llamamos soporte de u al conjunto $[u]^0 = supp(u) = \overline{\{x \in \mathbb{R}^n : u(x) > 0\}}$.

Un conjunto difuso es llamado compacto si $[u]^\alpha \in \mathcal{K}^n$, $\forall \in [0, 1]$. También, u es llamado convexo si $[u]^\alpha$ es un conjunto convexo $\forall \alpha \in [0, 1]$. Denotaremos por $\mathcal{F}^n$ el espacio de todos los conjuntos difusos convexos y compactos en $\mathbb{R}^n$.

Si $u \in \mathcal{F}$, entonces u es llamado intervalo difuso y el conjunto de nivel $[u]^\alpha$ es un intervalo compacto no vacio para todo $\alpha \in [0, 1]$.

La suma y multiplicación por un escalar en $\mathcal{F}^n$ son definidos como sigue:

$$(u \oplus v)(y) = \sup_{y_1 + y_2 = y} \min\{u(y_1), v(y_2)\}$$

y

$$(\lambda u)(y) = \begin{cases} u(\frac{y}{\lambda}) & \text{si} \quad \lambda \neq 0 \\ \chi_{\{0\}}(y) & \text{si} \quad \lambda = 0, \end{cases}$$

donde $\chi_{\{0\}}$ es la función carasterística de $\{0\}$. Es bien conocido que las siguientes operaciones son verdaderas para todos los α-niveles:

$$[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda \cdot u]^\alpha = \lambda [u]^\alpha, \quad \forall \alpha \in [0, 1]. \tag{2}$$

Podemos considerar la siguiente métrica en $\mathcal{F}^n$:

$$D(u, v) = \sup_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha),$$

para todo $u, v \in \mathcal{F}^n$, donde h es la métrica de Hausdorff usual.

3 Ecuaciones diferenciales difusas

Es conocido que la H-derivada (diferenciabilidad en el sentido de Hukuhara) para aplicaciones difusas fue inicialmente introducida por Puri y Ralescu [14] y esta está basada en la H-diferencia de conjuntos, como sigue.

Definición 1 *Sean $u, v \in \mathcal{F}^n$. Si existe $w \in \mathcal{F}^n$ tal que $u = v \oplus w$, entonces w es llamada la H-diferencia de u y v y lo denotamos por $u \ominus v$.*

Definición 2 *Sea $T = [a, b]$ un intervalo compacto y consideremos la aplicación difusa $F : T \to \mathcal{F}^n$. Diremos que F es H-diferenciable en $t_0 \in T$ si existe un elemento $F^{'}(t_0) \in \mathcal{F}^n$ tal que los límites*

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \quad \textit{and} \quad \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

existen y son iguales a $F^{'}(t_0)$.

Aquí los límites son tomados en $(\mathcal{F}^n, D)$. En los extremos del intervalo T consideramos sólo la derivada de un lado.

Sea $F : T \times \mathcal{F}^n \to \mathcal{F}^n$ continua y consideremos el siguiente problema de valor inicial:

$$x^{'}(t) = F(t, x(t)), \qquad x(a) = x_0, \tag{3}$$

donde $x^{'}$ es la H-derivada de x y $x_0 \in \mathcal{F}^n$.

El problema (3) fue estudiado por diversos autores tanto del punto de vista teorico [4, 8, 11, 15, 16, 17, 18] asi como de sus aplicaciones [6, 7]. Mas, esta interpretación no generaliza en una forma apropiada algunos casos clásicos como vemos en el siguiente ejemplo.

Ejemplo 1 *Consideremos el problema Malthusiano difuso*

$$\begin{cases} x'(t) = -\lambda x(t) \\ x(0) = X_0, \end{cases} \tag{4}$$

donde $\lambda > 0$ y, como en [4], la condición inicial X_0 es un triángulo difuso simétrico con soporte $[-a, a]$. Es decir,

$$[X_0]^\alpha = [-a(1-\alpha), a(1-\alpha)] = (1-\alpha)[-a, a].$$

Si $x'(t)$ es la H-derivada y denotamos por $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$, tenemos que resolver el siguiente sistema de ecuaciones diferenciales

$$\begin{cases} u_\alpha^{'}(t) = -\lambda v_\alpha(t), & u_\alpha(0) = -a(1-\alpha) \\ v_\alpha^{'}(t) = -\lambda u_\alpha(t), & v_\alpha(0) = a(1-\alpha). \end{cases}$$

La solución de este sistema es

$$u_\alpha(t) = -a(1-\alpha)e^{\lambda t} \quad y \quad v_\alpha(t) = a(1-\alpha)e^{\lambda t},$$

y podemos ver que $u_\alpha(t) \leq v_\alpha(t)$ *para todo* $t \geq 0$. *Por tanto, la función difusa* $x(t)$ *que resuelve el problema (4) tiene conjunto de niveles*

$$[x(t)]^\alpha = [-a(1-\alpha)e^{\lambda t}\ ,\ a(1-\alpha)e^{\lambda t}],$$

para todo $t \geq 0$.

Esta solución de (4), considerando la H-derivada, tiene la propiedad que $diam(supp\ x(t)) = 2ae^{\lambda t}$ *es ilimitado cuando* $t \to \infty$, *demostrando que esta interpretación no generaliza en una forma apropiada el caso clásico.*

Para mejorar esta situación, en [2] (ver también [1]) se introduce el concepto de derivada generalizada para la aplicación difusa $F : T \to \mathcal{F}^n$, ampliando la clase de aplicaciones difusas diferenciables. Esta generalización es definida como sigue.

Definición 3 *Sean* $F : T \to \mathcal{F}^n$ *y* $t_0 \in T$. *Diremos que* F *es diferenciable en* t_0 *si:*
(I) *existe un elemento* $F^{'}(t_0) \in \mathcal{F}^n$ *tal que para todo* $h > 0$ *suficientemente cerca a 0, existen* $F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ *y los límites (en la métrica D)*

$$\lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h} = F^{'}(t_0)$$

o
(II) *existe un elemento* $F^{'}(t_0) \in \mathcal{F}^n$ *tal que, para todo* $h < 0$ *suficientemente cerca a 0, existen* $F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ *y los límites*

$$\lim_{h\to 0^-} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^-} \frac{F(t_0) \ominus F(t_0-h)}{h} = F^{'}(t_0).$$

Note que la primera forma (I) de la Definición 3 coincide con la H-derivada. Si F es diferenciable en la primera forma (I) de la Definición 3, esta no es diferenciable en la segunda forma (II) de la Definición 3 y viciversa.

El siguiente Teorema nos guia como resolver una EDF.

Teorema 1 *[2] Sea* $F : T \to \mathcal{F}$ *una función y denotemos* $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, *para cada* $\alpha \in [0,1]$. *Entonces*

(i) *Si* F *es diferenciable en la primera forma (I), entonces* f_α *y* g_α *son diferenciables y*

$$\left[F^{'}(t)\right]^\alpha = \left[f_\alpha^{'}(t)\ ,\ g_\alpha^{'}(t)\right]. \tag{5}$$

(ii) *Si* F *es diferenciable en la segunda forma (II), entonces* f_α *y* g_α *son diferenciables y*

$$\left[F^{'}(t)\right]^\alpha = \left[g_\alpha^{'}(t)\ ,\ f_\alpha^{'}(t)\right]. \tag{6}$$

Ejemplo 2 *Volvamos al problema Malthusiano difuso*

$$\begin{cases} x'(t) = -\lambda x(t) \\ x(0) = X_0, \end{cases} \tag{7}$$

Ahora, consideramos $x'(t)$ en la segunda forma (II) de la Definición 3 y denotemos por $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$ los niveles de $x(t)$. Entonces, del Teorema 1, resolvemos el siguiente sistema diferencial

$$\begin{cases} u_\alpha^{'}(t) = -\lambda u_\alpha(t), & u_\alpha(0) = -a(1-\alpha) \\ v_\alpha^{'}(t) = -\lambda v_\alpha(t), & v_\alpha(0) = a(1-\alpha), \end{cases}$$

y la solución es

$$u_\alpha(t) = -a(1-\alpha)e^{-\lambda t} \quad and \quad v_\alpha(t) = a(1-\alpha)e^{-\lambda t},$$

y podemos ver que $u_\alpha(t) \leq v_\alpha(t)$ for all $t \geq 0$. Por lo tanto, la función difusa $x(t)$ que es solución de (7) en este caso tiene niveles

$$[x(t)]^\alpha = [-a(1-\alpha)e^{-\lambda t}\ ,\ a(1-\alpha)e^{-\lambda t}],$$

para todo $t \geq 0$.

Note que si consideramos en (7) la derivada $x^{'}$ en la segunda forma (II) de la Definición 3, entonces el resultado es mas intuitivo para (7) que usando la H-derivada, ya que ahora $diam(supp\, x(t)) = 2ae^{-\lambda t} \to 0$ cuando $t \to \infty$.

De los ejemplos anteriores podemos ver que la solución de una EDF depende de la elección de la derivada: en la primera forma o en la segunda forma. De esta manera, la solución puede ser elegida adecuadamente.

4 Ecuación integral

En esta sección, nosotros estudiamos el problema de valor inicial difuso

$$x^{'}(t) = F(t, x(t)), \qquad x(a) = x_0, \tag{8}$$

donde $x^{'}(t)$ es la derivada generalizada en el sentido de la Definición 3, $F : [a, b] \times \mathcal{F} \to \mathcal{F}$ es una función continua y x_0 es un intervalo difuso.

Si la derivada es considerada en la primera forma, es conocido los resultados sobre la existencia de soluciones y su equivalencia con una ecuación integral. A continuación presentamos un tipo de ecuación integral para la ecuacion diferencial difusa (8).

Denotemos por

$$[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$$

y

$$[F(t, x(t))]^\alpha = [f_\alpha(t, u_\alpha(t), v_\alpha(t))\ ,\ g_\alpha(t, u_\alpha(t), v_\alpha(t))].$$

Proposición 2 *Sea $F : T \to \mathcal{F}$ una función continua.*

(i) *Sea F diferenciable en la primera forma (I). Si $F^{'}$ es integrable, entonces para cada $t \in T$,*

$$F(t) = F(a) \oplus \int_a^t F^{'}(s)ds$$

(ii) *Sea F diferenciable en la segunda forma (II). Si $F^{'}$ es integrable, entonces para cada $t \in T$,*

$$F(t) = F(a) \ominus (-1) \int_a^t F^{'}(s)ds \tag{9}$$

Demostración. (i) Este resultado está demostrado en [8]. Nosotros damos una demostración para (ii). Para esto, tomamos $\alpha \in [0,1]$, entonces

$$\left[\int_a^t F^{'}(s)ds\right] = \left[\int_a^t g_{\alpha}^{'}(s)ds, \int_a^t f_{\alpha}^{'}(s)ds\right] = \left[g_{\alpha}(t) - g_{\alpha}(a), f_{\alpha}(t) - f_{\alpha}(a)\right].$$

Luego,

$$\begin{array}{rcl} [F(t)]^{\alpha} + (-1)\left[\int_a^t F^{'}(s)ds\right] & = & [f_{\alpha}(t), g_{\alpha}(t)] + [f_{\alpha}(a) - f_{\alpha}(t), g_{\alpha}(a) - g_{\alpha}(t)] \\ & = & [f_{\alpha}(a), g_{\alpha}(a)] = [F(a)]^{\alpha} \end{array}$$

Portanto, (9) se cumple. □

Teorema 3 (i) *Sea $x^{'}$ la derivada en la primera forma. Entonces, una aplicación $x : T \to \mathcal{F}$ es una solución de (8) si y sólo si esta es continua y satisface la siguiente ecuación integral*

$$x(t) = x_0 \oplus \int_a^t F(s, x(s))ds, \quad t \in T. \tag{10}$$

(ii) *Sea $x^{'}$ la derivada en la segunda forma. Entonces, una aplicación $x : T \to \mathcal{F}$ es una solución de (8) si y sólo si esta es continua y satisface la siguiente ecuación integral*

$$x(t) = x_0 \ominus (-1) \int_a^t F(s, x(s))ds, \quad t \in T. \tag{11}$$

Demostración. (i) este resultado es conocido, ver [8]. Mostraremos (ii). Si $x^{'}$ es una solución de (8), entonces

$$x(t) \oplus (-1) \int_a^t x^{'}(s)ds = x(t) \oplus (-1) \int_a^t F(s, x(s))ds, \quad t \in T.$$

Tomando en cuenta la Proposición 2(ii), tenemos

$$x(a) = x(t) \oplus (-1) \int_a^t F(s, x(s))ds, \quad t \in T.$$

Por tanto, (11) se verifica.

Reciprocamente, sea $x^{'}$ una función continua tal que (11) se verifica. Entonces, para cada $\alpha \in [0,1]$

$$[u_\alpha(t), v_\alpha(t)] + (-1)\left[\int_a^t f_\alpha(s, u_\alpha(s), v_\alpha(s))ds\ ,\ \int_a^t g_\alpha(s, u_\alpha(s), v_\alpha(s))ds\right] = [u_\alpha(a), v_\alpha(a)],$$

luego

$$\begin{aligned} u_\alpha(t) - \int_a^t g_\alpha(s, u_\alpha(s), v_\alpha(s))ds &= u_\alpha(a) \quad y \\ v_\alpha(t) - \int_a^t f_\alpha(s, u_\alpha(s), v_\alpha(s))ds &= v_\alpha(a) \end{aligned}$$

y

$$u_\alpha^{'}(t) = g_\alpha(t, u_\alpha(t), v_\alpha(t)) \quad y \quad v_\alpha^{'}(t) = f_\alpha(t, u_\alpha(t), v_\alpha(t)).$$

Entonces, $[x^{'}(t)]^\alpha = [v_\alpha^{'}(t), u_\alpha^{'}(t)] = [f_\alpha(t, u_\alpha(t), v_\alpha(t)), g_\alpha(t, u_\alpha(t), v_\alpha(t))] = [F(t, x(t))]^\alpha$. Portanto, x es una solución de (8). □

5 Conclusiones

En este trabajo hemos obtenido una equivalencia entre una ecuación diferencial difusa con derivada generalizada y una ecuación integral con integral en el sentido de Aumann.

Note que en [1], de manera diferente, se obtiene un resultado similar al Teorema 3 usando la inmersión de $\mathcal{F}$ en un espacio de Banach y la integral de Henstock difuso.

Por otro lado, cabe resaltar que en [2] los autores establecen un resultado de existencia de soluciones para una ecuación diferencial difusa con derivada generalizada, dando como condición suficiente la continuidad de F. Tal resultado no es cierto como fue mostrado en [3], en particular cuando se considera la derivada en la segunda forma. Ahora, con el resultado presentado en este artículo, en trabajos futuros buscaremos condiciones para la existencia de soluciones de una ecuación diferencial difusa con derivada generalizada.

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A REVIEW ON REPRODUCTIVITY AND TIME PERIODICITY FOR INCOMPRESSIBLE FLUIDS

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Abstract

In this article, our aims is to review some of the results that are currently available concerning the existence, uniqueness and regularity of reproductive and time periodic solutions of the Navier-Stokes equations and some variants. By the way, we present some open problems.

Key words: *Reproductive and time periodic solutions, Navier-Stokes type equations, regularity of solutions*
AMS subject classifications: *35B10 35Q35 76D03*

1 Introduction

We study some problems related with time periodic solutions for models of incompressible fluids.

We start recalling the main ideas to prove the existence of reproductive weak solutions (i.e. weak solutions defined in the time interval $(0,T)$ taking the same initial and final values in time) for the Navier-Stokes equations and some variants where these ideas are applicable, such as Boussinesq, micropolar and magneto-micropolar models. This proof relies on the obtention of time periodic Galerkin approximations via Leray-Schauder point fixed argument.

Moreover, in the case of $2D$ domains, using the uniqueness of weak solutions, the regularizing property of the system and the existence of global regular solutions when data are regular, one has that the periodic in time weak solutions defined as extension of reproductive solutions to the whole time interval $(0,+\infty)$ will be regular solutions. An extension of these results to the $3D$ case is possible imposing small enough external force, using the so called "weak/strong uniqueness" and the global strong solutions for small enough data (see Section 5).

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Also, we study in Section 4 some coupled models for velocity and pressure dynamic variables with another variable where the maximum principle holds, such as the generalized Boussinesq model (with temperature-dependent viscosity) and a nematic liquid crystal model with a Ginzburg-Landau penalization. In these cases one has, thanks to an adequate reformulation of the problem by truncation, existence of reproductive weak solutions as limit of time periodic Galerkin approximations. It is important to remark that Galerkin approximations do not verify the maximum principle but their limit does.

Finally, we will see that, for these models related with the maximum principle, the argument to prove regularity of reproductive solutions in the Navier-Stokes framework (see Section 5 below) are not valid in general. The particular case of generalized Boussinesq model with Neumann boundary condition for the temperature can be solved with other arguments, but the case of nematic liquid crystal model remains as an open problem.

2 Navier-Stokes equations

The modern theory of the Navier-Stokes equations began in the 1930s with Leray's pioneering work ([10]).

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) a bounded and regular enough domain filled by the fluid, and $[0, T]$ the time interval. We denote $Q = (0, T)\times\Omega$ and $\Sigma = (0, T)\times\partial\Omega$.

In the case where the fluid is subject to the action of a body force $\boldsymbol{f}$, the Navier-Stokes equations can be written as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{div } \boldsymbol{u} = 0\,, \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(x, t)$ is the velocity field evaluated at the point $\boldsymbol{x} \in \Omega$ and at time $t \in [0, T]$, $p = p(\boldsymbol{x}, t)$ is the pressure field and $\nu > 0$ is the coefficient of kinematical viscosity (which is taken constant). This system can be completed with several boundary conditions. For simplicity, we fix the following non-slip boundary conditions:

$$\boldsymbol{u}(t, \boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad t > 0 \tag{2}$$

Finally, supplementary conditions in time must be considered. The more classical is the initial condition:

$$\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{3}$$

Other possibility is to change this initial condition by the following time-periodic condition:

$$\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}(T, \boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{4}$$

Mathematical properties for system (1) have been deeply investigated over the years and are still the object of profound researches.

We introduce some space functions. Let $\mathcal{V}$ the vectorial space formed by all fields $\boldsymbol{v} \in C_0^\infty(\Omega)^d$ satisfying $\nabla \cdot \boldsymbol{v} = 0$. We consider the Hilbert spaces $\boldsymbol{H}$

(respectively $\boldsymbol{V}$) as the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). Furthermore, one has

$$\begin{aligned} \boldsymbol{H} &= \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ \boldsymbol{V} &= \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega\} \end{aligned}$$

We denote $L^2_0(\Omega) = \left\{ p \in L^2(\Omega) : \int_\Omega p\, dx = 0 \right\}$.

2.1 Main classical results for the initial-boundary problem

Definition 1 *Given $\boldsymbol{u}_0 \in H$ and $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, it will said that $\boldsymbol{u}$ is a weak solution of the problem (1),(2), (3) in $(0,T)$, if*

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}),$$

and verifies (3) and the variational formulation

$$\int_0^T \int_\Omega \Big\{ -\boldsymbol{u}(t)\boldsymbol{v}'(t) + \nabla \boldsymbol{u}(t) : \nabla \boldsymbol{u}(t) - (\boldsymbol{u}(t) \cdot \nabla)\boldsymbol{v}(t)\boldsymbol{u}(t) - \boldsymbol{f}(t)\boldsymbol{v}(t) \Big\} dxdt = 0,$$

for all $\boldsymbol{v} \in C^1([0,T];\boldsymbol{H}) \cap C([0,T];\boldsymbol{V})$, with compact support contained in $(0,T)$.

In addition, if $\boldsymbol{u}_0 \in \boldsymbol{V}$ and $f \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ any weak solution will be a strong solution if

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^2 \cap \boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{V}), \quad \boldsymbol{u}_t \in L^2(0,T;\boldsymbol{H}), \quad p \in L^2(0,T;H^1 \cap L^2_0(\Omega))$$

and verifies the system (1) pointwise a.e. in $(0,T) \times \Omega$.

Remark 1 *The previous definition can be extend to the case of final time $T = \infty$ changing the regularity $L^2(0,T)$ by $L^2_{\text{loc}}(0,+\infty)$.*

The following results hold.

Theorem 1 *[22] For any $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, the problem (1)-(2) has (at least) a weak solution. If $\Omega \subset \mathbb{R}^2$, one has uniqueness of weak solutions.*

Theorem 2 *[22] For any $\boldsymbol{u}_0 \in \boldsymbol{V}$ and $\boldsymbol{f} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$, the problem (1)-(2) has a unique strong solution $(\boldsymbol{u},p)$ local in time, defined in $(0,T^\star)$ with $T^\star > 0$ small enough. In fact, if a solution has the strong regularity, it coincides with any weak solution associated with the same data (this property is called weak/strong uniqueness). Moreover, this strong solution is global in time, defined in the whole time interval $(0,\infty)$ if either $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ and data $(\boldsymbol{u}_0, \boldsymbol{f})$ are small enough in their respective spaces $\boldsymbol{V} \times L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$.*

2.2 On the time-periodic weak solutions

Theorem 3 *[8] For any $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, there exists a weak solution of (1)-(2) and (4), (i.e. the weak solution $\boldsymbol{u}$ has the so-called reproductive property: $\boldsymbol{u}(0,x) = \boldsymbol{u}(T,x)$).*

Notice that the time periodic extension, $\widetilde{\boldsymbol{u}}$, of any weak reproductive solution $\boldsymbol{u}$ to the whole time interval $(0,+\infty)$ is a periodic weak solution of (1)-(2) corresponding to the data, $\widetilde{\boldsymbol{f}}$, defined as the time periodic extension of $\boldsymbol{f}$.

Main ideas of the proof of Theorem 3

Let $\boldsymbol{u}^k$ the unique approximate solution of the Galerkin initial-boundary problem of Navier-Stokes in the finite-dimensional subspace $\boldsymbol{V}^k$, spanned by the first k elements of the "spectral" basis of $\boldsymbol{V}$ (orthogonal in $\boldsymbol{V}$ and orthonormal in $\boldsymbol{H}$), associated to a initial discrete data $\boldsymbol{u}_0^k \in \boldsymbol{V}^k$.

Since $\boldsymbol{V} \hookrightarrow \boldsymbol{H}$, there exists a Poincaré constant $c_1 > 0$ such that

$$c_1 \|\boldsymbol{u}^k\|_{L^2}^2 \leq \|\nabla \boldsymbol{u}^k\|_{L^2}^2,$$

thus, from energy inequality, we have

$$\frac{d}{dt}\|\boldsymbol{u}^k\|_{L^2}^2 + c_1 \|\boldsymbol{u}^k\|_{L^2}^2 \leq C\,\|\boldsymbol{f}\|_{H^{-1}}^2, \tag{5}$$

or equivalently

$$\frac{d}{dt}(e^{c_1 t}\|\boldsymbol{u}^k\|_{L^2}^2) \leq C\, e^{c_1 t}\|\boldsymbol{f}\|_{H^{-1}}^2.$$

Integrating from 0 to T, we have

$$e^{c_1 T}\|\boldsymbol{u}^k(T)\|_{L^2}^2 \leq \|\boldsymbol{u}^k(0)\|_{L^2}^2 + C\int_0^T e^{c_1 t}\,\|\boldsymbol{f}(t)\|_{H^{-1}}^2 dt. \tag{6}$$

Now, we define the operator $L^k : [0,T] \to \mathbb{R}^k$ as follows

$$L^k(t) = (c_1^k(t), \ldots, c_k^k(t))$$

where $c_i^k(t),\ i = 1, \ldots, k$, are the coefficients of the expansion of $\boldsymbol{u}^k(t)$ in V^k.

Note that

$$\|L^k(t)\|_{\mathbb{R}^k} = \|\boldsymbol{u}^k\|_{L^2},$$

because we have choose the (orthonormal in L^2) spectral basis in $\boldsymbol{V}$.

We define the operator $\Phi^k : \mathbb{R}^k \to \mathbb{R}^k$ as follows: Given $L_0^k \in \mathbb{R}^k$, we define $\Phi^k(L_0^k) = L^k(T)$, where $L^k(t)$ are the coefficients of the Galerkin solution with initial value with coefficients L_0^k. It is easy to see that Φ^k is continuous and we want to prove that Φ^k has a fixed point.

For this, thanks to the Leray-Schauder Theorem, it suffices to show that for all $\lambda \in [0,1]$, the possible solutions of the equation

$$L_0^k(\lambda) = \lambda \Phi^k(L_0^k(\lambda)), \tag{7}$$

are bounded independently of λ.

Since $L_0^k(0) = 0$, it suffices to consider $\lambda \in (0,1]$. In this case, (7) is equivalent to $\Phi^k(L_0^k(\lambda)) = \frac{1}{\lambda} L_0^k(\lambda)$. Moreover, by the definition of Φ^k and (6), one obtains

$$e^{c_1 T} || \frac{1}{\lambda} L_0^k(\lambda)||^2_{\mathbb{R}^k} \leq \|L_0^k(\lambda)\|^2_{\mathbb{R}^k} + c \int_0^T e^{c_1 T} \|\boldsymbol{f}(t)\|^2_{H^{-1}} dt,$$

which implies

$$\|L_0^k(\lambda)\|^2_{\mathbb{R}^k} \leq \frac{c \int_0^T e^{c_1 T} \|\boldsymbol{f}(t)\|^2_{H^{-1}} dt}{e^{c_1 T} - 1} = M,$$

for each $\lambda \in (0,1]$. This bound is independent of $\lambda \in [0,1]$ and k. Consequently, Leray-Shauder Theorem implies the existence of at least one fixed point of Φ^k, that is the existence of reproductive Galerkin solution.

Thus, since previous estimates are independent of k, one has the same estimates for these reproductive Galerkin solutions.

Finally, the convergence of a subsequence to a reproductive solution of (1),(2), (4) hold.

2.3 Relation between weak periodic solutions and global solutions

Assume $\boldsymbol{f}\colon [0,+\infty) \to \boldsymbol{H}^{-1}(\Omega)$ and T-time periodic.

Navier-Stokes $2D$

One has (see Theorem 1) uniqueness of weak solution for the initial-boundary problem (associated to any initial data $\boldsymbol{u}_0$). Consequently, given a reproductive solution $\boldsymbol{u}$ associated to $\boldsymbol{u}(0) = \boldsymbol{u}(T) := \boldsymbol{u}_0$, then $\boldsymbol{u}$ is the (unique) solution of the initial-boundary problem associated to the initial data $\boldsymbol{u}_0$, which is defined for all time $t \in (0,\infty)$. Moreover, this solution is T-periodic, because in $(T,2T)$ must be equal to the reproductive solution defined as $\bar{\boldsymbol{u}}(t) = \boldsymbol{u}(t-T)$ (which verifies $\boldsymbol{u}(T) = \boldsymbol{u}(2T) = \boldsymbol{u}_0$) and so on.

Finally, using regularity of solution $\boldsymbol{u}$ for strictly positive times (see [5]), it is easy to prove that every periodic solution is regular.

Navier-Stokes $3D$

Since uniqueness of weak solution is not known, it is possible that the reproductive solution $\boldsymbol{u}$ and the global weak solution $\tilde{\boldsymbol{u}}$ associated to the initial data $\boldsymbol{u}_0 := \boldsymbol{u}(0) = \boldsymbol{u}(T)$ are different in $(0,T)$, although they coincide locally in time, near of the initial time $t = 0$.

2.4 Open problems

Navier-Stokes with large Reynolds number and a reaction term adding energy

Previous arguments of the proof of reproductive solutions are based on (exponential) decreasing of energy (thanks to dissipative terms). Naturally, the

same argument, is applicable to models with energy strictly decreasing in finite time. But this is not always possible. For instance, we consider the following Navier-Stokes system with large Reynolds number and a reaction term adding energy:

$$\begin{array}{l} \partial_t \boldsymbol{u} - \varepsilon\Delta\boldsymbol{u} - \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}(T), \quad \boldsymbol{u}_{|\Sigma} = 0. \end{array} \tag{8}$$

The energy inequality is

$$\partial_t \|\boldsymbol{u}\|_{L^2}^2 + \varepsilon\|\nabla\boldsymbol{u}\|_{L^2}^2 \leq C(\|\boldsymbol{f}\|_{H^{-1}}^2 + \|\boldsymbol{u}\|_{L^2}^2).$$

Assuming ε small enough such that $\|\boldsymbol{u}\|_{L^2}^2 \not\leq \varepsilon\|\nabla\boldsymbol{u}\|_{L^2}^2$, the strictly decreasing in time of $\|\boldsymbol{u}\|_{L^2}^2$ is not clear. Consequently, the existence of time-periodic weak solutions of (8) remains as an open problem.

Exterior domains

Assume Ω is an exterior domain where the Poincaré inequality is not true. Then, to show the existence of reproductive solutions one could use the "embedding domain technique" together with the Galerkin Method, obtaining reproductive solutions in a sequence of (bounded) truncated domains, see for instance [6, 18, 17]. However, since Poincaré imbedding is not applicable, it is not clear the controll to the pass to the limit from truncated domains to the whole domain.

Some partial results are known. For example, the existence of strong periodic solutions for the Navier-Stokes equations in the following unbounded domains, either Ω is the whole space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$ has been investigated by Kozono and Nakao [9] and Taniuchi [21] using the semigroup approach. By using potential theory, Maremonti [15] proved the existence of a unique time periodic solution on the whole space $\mathbb{R}^3$ for small external force. The problem, in the half-space $\mathbb{R}^3_+$, was considered in [16]. Kozono and Nakao [9], making use of $\mathbf{L}^p - \mathbf{L}^r$ estimates for the semigroup generated by the Stokes operator, constructed time-periodic solutions for small time-periodic forces and the stability of these solutions was considered in [21]. Yamazaki [23] analyzed the same problem of [9] in Morrey spaces.

3 Some variants of Navier-Stokes equations

We can apply the argument to find reproductive solutions done for Navier-Stokes in the precedent Section, for some variants:

3.1 Boussinesq equations

The Boussinesq system of hydrodynamics equations (see Joseph [7]) arise from zero order approximation to the coupling between the Navier-Stokes equation and the thermodynamic equation. Such a mathematical model reads:

Find the field $\boldsymbol{u} : Q \to \mathbb{R}^3$, the scalar functions $(\theta, p) : Q \to \mathbb{R}^2$ which satisfy the system of equations:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \alpha\theta\boldsymbol{g} + \boldsymbol{f} \quad \text{in } Q, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } Q, \\ \frac{\partial\theta}{\partial t} - \chi\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta &= 0 \quad \text{in } Q. \end{aligned} \tag{9}$$

with $\partial_n\theta = 0$ on $\partial\Omega$ and $\int_\Omega \theta = 0$

Here $\boldsymbol{u}, p, \theta$ denote the velocity, the pressure and the temperature, respectively. $\boldsymbol{g}$ denotes the gravitational field, $\alpha > 0$ is a constant associated to the coefficient of volume expansion and $\boldsymbol{f}$ is a field of external forces. Again, $\nu > 0$ is the viscosity coefficient. Finally, $\chi > 0$ is the thermal conductivity coefficient.

This system is completed with the boundary conditions (for instance) $\boldsymbol{u}_{|\Sigma} = 0$, $\theta_{|\Sigma} = 0$ and the time-periodic conditions $\boldsymbol{u}(0) = \boldsymbol{u}(T)$, $\theta(0) = \theta(T)$ in Ω.

By taking $\boldsymbol{u}$ and θ as test function in the $\boldsymbol{u}$-system and θ-equation of (9) respectively, adding the resulting equalities considering an adequate balance (in order to eliminate the term that contains $\boldsymbol{g}$), we obtain

$$\frac{d}{dt}\|\boldsymbol{u}\|_{L^2}^2 + \beta\frac{d}{dt}\|\theta\|_{L^2}^2 + \nu\|\nabla\boldsymbol{u}\|_{L^2}^2 + \beta\chi\|\nabla\theta\|_{L^2}^2 \le \|\boldsymbol{f}\|_{H^{-1}}^2, \tag{10}$$

where β is a big enough number depending on α and $\|\boldsymbol{g}\|_{L^\infty}$. This together with the Poincaré inequality gives an inequality of type (5). Indeed, it suffices to consider a Galerkin approximation for both variables, velocity and temperature, and to follow the proof of Theorem 2.3, changing $\boldsymbol{u}^k$ by $(\boldsymbol{u}^k, \theta^k)$.

Another boundary conditions are possible: Neumann, mixed, etc, whenever an inequality for $(\boldsymbol{u}, \theta)$ similar to (5) holds.

3.2 Micropolar equations

The equations that describes the motion of a incompressible viscous and micropolar fluids in Q are given by (see [12])

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\nu+\nu_r)\Delta\boldsymbol{u} + \nabla p = 2\nu_r \text{ rot } \mathbf{w} + \boldsymbol{f}, \\ &\text{div } \boldsymbol{u} = 0, \\ &\frac{\partial \mathbf{w}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\mathbf{w} - (c_a+c_d)\Delta\mathbf{w} - (c_0+c_d-c_a)\nabla \text{ div } \mathbf{w} + 4\nu_r\mathbf{w} \\ &\qquad = 2\nu_r \text{ rot } \boldsymbol{u} + \mathbf{g}. \end{aligned} \tag{11}$$

The functions $\boldsymbol{u} : Q \to \mathbb{R}^3$, $\boldsymbol{w} : Q \to \mathbb{R}^3$ and $p : Q \to \mathbb{R}$ denote the liner velocity, the angular velocity (of rotation of particles) and the pressure of the fluid, respectively. The functions $\boldsymbol{f} : Q \to \mathbb{R}^3$ and $\boldsymbol{g} : Q \to \mathbb{R}^3$ denote external

sources of linear and angular momentum, respectively. The positive constants ν, ν_r, c_0, c_a and c_d are viscosities, such that $c_0 + c_d > c_a$.

This system is completed with the boundary conditions $\boldsymbol{u}_{|\Sigma} = 0$, $\boldsymbol{w}_{|\Sigma} = 0$ (for instance) and the time-periodic conditions $\boldsymbol{u}(0) = \boldsymbol{u}(T)$, $\boldsymbol{w}(0) = \boldsymbol{w}(T)$ in Ω.

By taking $\boldsymbol{u}$ and $\boldsymbol{w}$ as test function in the $\boldsymbol{u}$-system and $\boldsymbol{w}$-system of (11) respectively, adding the resulting equalities, taking into account that $2\nu_r(\text{ rot } \boldsymbol{w}, \boldsymbol{u}) + 2\nu_r(\text{ rot } \boldsymbol{u}, \boldsymbol{w}) = 4\nu_r(\text{ rot } \boldsymbol{u}, \boldsymbol{w})$ and $|\nabla \boldsymbol{u}|^2 = |\text{ rot} \boldsymbol{u}|^2$, we obtain

$$\frac{d}{dt}\|\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{w}\|_{L^2}^2 + \nu\|\nabla \boldsymbol{u}\|_{L^2}^2 + (c_a + c_d)\|\nabla \boldsymbol{w}\|_{L^2}^2 + (c_0 + c_d - c_a)\|\text{ div } \boldsymbol{w}\|_{L^2}^2 \leq C(\|\boldsymbol{f}\|_{H^{-1}}^2 + \|\boldsymbol{g}\|_{H^{-1}}^2).$$

Starting from this inequality, the argument follows as in previous section.

3.3 Other models

Other fluid models where one has existence of reproductive solutions are: magnetohydrodynamic model [14], Magneto-micropolar fluid motion [20], a convection-diffusion model describing binary alloy solidification processes [3], etc.

4 Reproductivity and maximum principle

Given $\boldsymbol{u} : Q \to \mathbb{R}^3$ such that $\nabla \cdot \boldsymbol{u} = 0$ in Q and $\boldsymbol{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we consider the (reproductive) diffusion-advection problem for the unknown $c : Q \to \mathbb{R}$ (a concentration):

$$\partial_t c - \Delta c + \boldsymbol{u} \cdot \nabla c = 0, \quad c_{|\Sigma} = c_\Sigma, \quad c(0) = c(T),$$

where $0 < \underline{c} \leq c_\Sigma \leq \overline{c}$ on Σ, for some constants $\underline{c}$ and $\overline{c}$. In particular,

$$\partial_t(c - \overline{c}) - \Delta(c - \overline{c}) + (\boldsymbol{u} \cdot \nabla)(c - \overline{c}) = 0 \quad \text{in } Q.$$

Multiplying by $(c - \overline{c})_+$ and integrating in Ω (notice that $(c - \overline{c})_+ = 0$ on Σ), one has

$$\frac{d}{dt}\int_\Omega |(c - \overline{c})_+|^2 + \int_\Omega |\nabla(c - \overline{c})_+|^2 \leq 0.$$

Integrating in $t \in (0, T)$ and using the periodic condition $c(0) = c(T)$, one arrives at

$$\int_0^T \|\nabla(c - \overline{c})_+\|_{L^2}^2 = 0.$$

Hence $c \leq \overline{c}$ in Q hold. Similarly $c \geq \underline{c}$ in Q hold.

Therefore, one has the following conclusion: The reproductive solution conserve the maximum principle.

In the following models, the maximum principle has an important role.

4.1 Generalized Boussinesq system, with diffusion depending on temperature

When the viscosity and heat conductivity are temperature dependent funtions in the Boussinesq system, one has the following system:

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla \cdot (\nu(\theta)\nabla \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \alpha\theta\mathbf{g} + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \partial_t \theta - \nabla \cdot (k(\theta)\nabla\theta) + (\boldsymbol{u}\cdot\nabla)\theta = 0, \end{cases} \tag{12}$$

where $\nu : \mathbb{R} \to \mathbb{R}^+$ and $k : \mathbb{R} \to \mathbb{R}^+$ are strictly positive continuous functions (the kinematic viscosity and the thermal conductivity respectively).

The problem is to find a regular solution $\{\boldsymbol{u}, \theta, p\}$ of (12) in $\Omega \times [0,T]$, together the following boundary Dirichlet data:

$$\boldsymbol{u} = 0, \qquad \theta = \theta_{\partial\Omega} \qquad \text{on } \partial\Omega \times [0,T), \tag{13}$$

and time-periodic conditions:

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \theta(0) = \theta(T) \qquad \text{in } \Omega. \tag{14}$$

We define

$$\theta_{\min} = \min \theta_{\partial\Omega} \qquad \theta_{\max} = \max \theta_{\partial\Omega}.$$

Thanks to the maximum principle, one has $\theta_{\min} \le \theta \le \theta_{\max}$ in Q. Then, there exists $\nu_{\min} > 0$, $k_{\min} > 0$, $\nu_{\max} > 0$ and $k_{\max} > 0$ such that

$$\nu_{\min} \le \nu(s) \le \nu_{\max} \quad \text{and} \quad k_{\min} \le k(s) \le k_{\max}, \quad \forall\, s \in [\theta_{\min}, \theta_{\max}].$$

One can proves the existence of reproductive solution in the same way that in the classical Boussinesq case (see Section 3.1), considering the equivalent problem that result changing ν by $\widetilde{\nu}$ and k by $\widetilde{k}$, where

$$\widetilde{\nu}(\theta) = \begin{cases} \nu(\theta_{\min}) & \text{if} \quad \theta < \theta_{\min}, \\ \nu(\theta) & \text{if} \quad \theta_{\min} \le \theta \le \theta_{\max}, \\ \nu(\theta_{\max}) & \text{if} \quad \theta > \theta_{\max}, \end{cases}$$

$$\widetilde{k}(\theta) = \begin{cases} k(\theta_{\min}) & \text{if} \quad \theta < \theta_{\min}, \\ k(\theta) & \text{if} \quad \theta_{\min} \le \theta \le \theta_{\max}, \\ k(\theta_{\max}) & \text{if} \quad \theta > \theta_{\max}. \end{cases}$$

4.2 Penalized Nematic liquid crystal model

We assume the following nematic liquid crystal model in $(0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^N$ for $N = 2$ or 3 is an open bounded domain:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p = -\lambda\nabla\cdot(\nabla \boldsymbol{d}^t \nabla \boldsymbol{d}), \quad \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t \boldsymbol{d} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d} = \gamma(\Delta\boldsymbol{d} - \boldsymbol{f}_\varepsilon(\boldsymbol{d})). \end{cases} \tag{15}$$

The positive constants ν, λ and γ, are the fluid viscosity, the elasticity constant and the relaxation time, respectively.

In this penalized model, the constraint $|\boldsymbol{d}| = 1$ (where $|\cdot|$ is the punctual euclidean norm) is partially conserved to $|\boldsymbol{d}| \leq 1$ as consequence of the maximum principle for the Ginzburg-Landau equation considering the penalization function

$$\boldsymbol{f}_\varepsilon(\boldsymbol{d}) = \varepsilon^{-2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}$$

where $\varepsilon > 0$ is the penalization parameter. There exists a potential function

$$\boldsymbol{F}_\varepsilon(\boldsymbol{d}) = \frac{1}{4\varepsilon^2}(|\boldsymbol{d}|^2 - 1)^2$$

such that $\boldsymbol{f}_\varepsilon(\boldsymbol{d}) = \nabla_{\boldsymbol{d}}(\boldsymbol{F}_\varepsilon(\boldsymbol{d}))$ for each $\boldsymbol{d} \in \mathbb{R}^N$.

The problem (15) is completed with the (Dirichlet) boundary conditions

$$\boldsymbol{u} = 0, \qquad \boldsymbol{d} = \boldsymbol{h} \qquad \text{on } \partial\Omega \times (0,T) \tag{16}$$

and the time-periodic conditions:

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \boldsymbol{d}(0) = \boldsymbol{d}(T) \qquad \text{in } \Omega. \tag{17}$$

It is important to remark that reproductive solution with the following boundary data independent of time $\boldsymbol{d}(x,t)_{|\partial\Omega\times(0,T)} = \boldsymbol{d}_0(x)$ has the trivial stationary (static) solution:

$$\begin{aligned}
&\boldsymbol{u} \equiv 0,\\
&\boldsymbol{d} \quad \text{solution of the elliptic problem:} \quad -\Delta\boldsymbol{d} + \boldsymbol{f}_\varepsilon(\boldsymbol{d}) = 0 \quad \text{in } \Omega, \quad \boldsymbol{d}_{|\partial\Omega} = \boldsymbol{d}_0,\\
&p = -\lambda\left(\frac{|\nabla\boldsymbol{d}|^2}{2} + \boldsymbol{F}_\varepsilon(\boldsymbol{d})\right).
\end{aligned}$$

The expression of p is due to the momentum equation reduces to

$$\nabla p = -\lambda\nabla\cdot(\nabla\boldsymbol{d}^t\nabla\boldsymbol{d}) = -\lambda\nabla\left(\frac{|\nabla\boldsymbol{d}|^2}{2} + \boldsymbol{F}_\varepsilon(\boldsymbol{d})\right) + \lambda\nabla\boldsymbol{d}^t(\boldsymbol{f}_\varepsilon(\boldsymbol{d}) - \Delta\boldsymbol{d}).$$

Therefore, since $-\Delta\boldsymbol{d} + \boldsymbol{f}_\varepsilon(\boldsymbol{d}) = 0$, one has $\nabla p = -\lambda\nabla\left(\dfrac{|\nabla\boldsymbol{d}|^2}{2} + \boldsymbol{F}_\varepsilon(\boldsymbol{d})\right)$.

Therefore, in this work will be fundamental assume time-dependent boundary data for $\boldsymbol{d}$.

In order to obtain the maximum principle for $|\boldsymbol{d}|^2$, we multiply the $\boldsymbol{d}$-system by $\boldsymbol{d}$ getting

$$\frac{1}{2}\partial_t|\boldsymbol{d}|^2 + \frac{1}{2}\boldsymbol{u}\cdot\nabla|\boldsymbol{d}|^2 - \gamma\Delta|\boldsymbol{d}|^2 + \gamma|\nabla\boldsymbol{d}|^2 + \gamma\boldsymbol{f}_\varepsilon(\boldsymbol{d})\cdot\boldsymbol{d} = 0,$$

whence the following differential inequality holds for $c = |\boldsymbol{d}|^2$:

$$\partial_t c + \boldsymbol{u}\cdot\nabla c - 2\gamma\Delta c + 2\gamma\frac{1}{\epsilon^2}(c-1)c \leq 0.$$

Notice that, if $c = |\boldsymbol{d}|^2 \geq 1$ then $\frac{1}{\epsilon^2}(c-1)c = \boldsymbol{f}_\varepsilon(\boldsymbol{d}) \cdot \boldsymbol{d} \geq 0$. Therefore, assuming $|\boldsymbol{h}| \leq 1$, we obtain $c \leq 1$ in $\partial\Omega$. Then, we can apply the maximum principle argument obtaining $c \leq 1$ in Ω, i.e. $|\boldsymbol{d}| \leq 1$ in Ω.

This maximum principle is fundamental in order to obtain solution of the (15)-(17) problem because we can consider a equivalent problem changing $\boldsymbol{f}_\varepsilon$ by $\widetilde{\boldsymbol{f}}_\varepsilon$, the auxiliary function

$$\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d}) = \begin{cases} \boldsymbol{f}_\varepsilon(\boldsymbol{d}) & \text{if} \quad |\boldsymbol{d}| \leq 1, \\ 0 & \text{if} \quad |\boldsymbol{d}| > 1. \end{cases}$$

Indeed, if $(\boldsymbol{u}, p, \boldsymbol{d})$ is a solution of (15)-(17) with $\widetilde{\boldsymbol{f}}_\varepsilon$, in particular $|\boldsymbol{d}| \leq 1$ (because the maximum principle is also verified, since $\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d}) \cdot \boldsymbol{d} \geq 0$ as $|\boldsymbol{d}| > 1$), then $(\boldsymbol{u}, p, \boldsymbol{d})$ is also a solution of (15)-(17) with $\boldsymbol{f}_\varepsilon$. The inverse statement is easy to verify.

Now, the key is that $|\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d})| \leq \frac{1}{\varepsilon^2} \ \forall \boldsymbol{d} \in \mathbb{R}^3$. Then, existence of weak reproductive solution of this model is proved in [1]. The main steps of the proof are to prove existence and uniqueness of solution for a Galerkin initial-boundary problem, to obtain the reproductivity of approximate solution with the argument of Theorem 3 and to pass to the limit. More concretely, using the lifting function $\widetilde{\boldsymbol{d}}(t)$ as the solution of Laplace-Dirichlet problem

$$\begin{cases} -\Delta\widetilde{\boldsymbol{d}} = 0 & \text{in } \Omega, \\ \widetilde{\boldsymbol{d}}|_{\partial\Omega} = \boldsymbol{h}(t) & \text{on } \partial\Omega, \end{cases},$$

defining $\widehat{\boldsymbol{d}}(t) = \boldsymbol{d}(t) - \widetilde{\boldsymbol{d}}(t)$ and taking $\boldsymbol{u}$ and $-\lambda\Delta\widehat{\boldsymbol{d}} = -\lambda\Delta\boldsymbol{d}$ as test functions in the equations for $\boldsymbol{u}$ and $\widehat{\boldsymbol{d}}$ of (15) respectively, one has the energy inequality:

$$\begin{aligned} \frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2 + \lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right) + 2\mu\|\nabla\boldsymbol{u}\|_{L^2}^2 + \lambda\gamma\|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2 \\ \leq C\left(\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2 + \|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right), \end{aligned} \tag{18}$$

where the right hand side is bounded in $L^1(0,T)$ if $\boldsymbol{f}_\varepsilon$, $\partial_t\widetilde{\boldsymbol{d}} \in L^2(\boldsymbol{L}^2)$.

Remark 2 *An interesting open problem in this context is the asymptotic behavior as $\varepsilon \to 0$ of the reproductive solutions of this liquid crystal model (15)-(17). For the initial-boundary problem, this asymptotic behavior is studied in [4], for time independent boundary data.*

5 Regularity of periodic solutions via regularity of reproductive solutions

We consider the time-periodic boundary problem associated to 3D Navier Stokes model with data $\boldsymbol{f}$.

Let $\boldsymbol{u}$ be a reproductive solution in $[0,T]$ (given in Theorem 3). The problem is to obtain regularity for this solution. A possible argument is to prove that there exists at least one time $t_\star \in [0,T]$ such that $\|\boldsymbol{u}(t_\star)\|_{H^1}$ is small enough. In fact, we can find that $t_\star$ exists, integrating in $(0,T)$ the energy inequality

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|_{L^2}^2 + \nu\|\nabla\boldsymbol{u}(t)\|_{L^2}^2 \leq \frac{1}{\nu}\|\boldsymbol{f}\|_{H^{-1}}^2$$

and applying the reproductive condition $u(0) = u(T)$, arriving at

$$\nu\int_0^T \|\nabla\boldsymbol{u}(t)\|_{L^2}^2 \leq \frac{1}{\nu}\int_0^T ||\boldsymbol{f}(t)||_{H^{-1}}^2.$$

Assuming external forces $\boldsymbol{f}$ small enough in the $L^\infty(0,\infty;\mathbf{L}^2)$ norm, in particular $\frac{1}{\nu^2}\int_0^T ||\boldsymbol{f}(t)||_{H^{-1}}^2 \leq \varepsilon T$ for some ε small enough, hence $\int_0^T \|\nabla\boldsymbol{u}(t)\|_{L^2}^2 \leq \varepsilon T$. From integral mean value theorem, there exists $t_\star \in [0,T]$ such that $\|\nabla\boldsymbol{u}(t_\star)\|_{L^2}^2 \leq \varepsilon$.

On the other hand, let $\overline{\boldsymbol{u}}$ be the unique regular strong solution (see Theorem 2) with initial data $\boldsymbol{u}(t_\star)$ and the same force $\boldsymbol{f}$. Moreover, following the proof of this type of global in time results with small data (see for instance ([22]), one has $\|\nabla\overline{\boldsymbol{u}}(t)\|_{L^2}^2 \leq 2\,\varepsilon$ for each $t \geq t_\star$ (here $\boldsymbol{f}$ small enough in the $L^\infty(0,\infty;\mathbf{L}^2)$ norm is necessary).

By uniqueness of weak-strong solution (Theorem 2), one has $\overline{\boldsymbol{u}} \equiv \boldsymbol{u}$ in $[t_\star,T]$ and therefore $\boldsymbol{u}$ is regular in $[t_\star,T]$. In particular, $\|\nabla\boldsymbol{u}(T)\|_{L^2}^2 = \|\nabla\overline{\boldsymbol{u}}(T)\|_{L^2}^2 \leq 2\,\varepsilon$. Therefore $\|\nabla\boldsymbol{u}(0)\|_{L^2}^2 \leq 2\,\varepsilon$, hence $\boldsymbol{u}$ is a strong solution in $[0,T]$. Finally, in $[T,2\,T]$, $\boldsymbol{u}(t-T) \equiv \overline{\boldsymbol{u}}(t)$ and so on. The precedent argument is used, for instance in [13].

Therefore, we arrive at the following conclusion: The periodic extension of a reproductive solution $\boldsymbol{u}$ is a regular solution in $[0,+\infty)$ assuming small enough external forces $\boldsymbol{f}$. This conclusion is also valid for the models presented in Section 3.

Previous argument is based on to obtain $\int_0^T \|\boldsymbol{u}(t)\|_{H^1}^2 dt$ small enough, only assuming force $\boldsymbol{f}$ small enough (in particular, $\|\boldsymbol{u}(t_\star)\|_{H^1}^2$ is small for some $t_\star \in [0,T]$). But, there are some fluids models, where this is not always possible to obtain. For example, we will see below the $3D$ penalized nematic liquid crystal (in Subsection 5.1) and the generalized Boussinesq model (in Subsection 5.2).

5.1 $3D$ penalized nematic liquid crystal (15)-(17)

Considering an adequate lifting function $\widetilde{\boldsymbol{d}}$ (the solution of problem: $-\Delta\widetilde{\boldsymbol{d}} = 0$ in Ω, $\widetilde{\boldsymbol{d}} = \boldsymbol{h}$ on $\partial\Omega$), and denoting $\widehat{\boldsymbol{d}} = \boldsymbol{d} - \widetilde{\boldsymbol{d}}$, testing the $\boldsymbol{u}$-system by $\boldsymbol{u}$ and the

$\boldsymbol{d}$-system by $-\Delta\widehat{\boldsymbol{d}}$, one has the energy inequality

$$\begin{aligned}\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2+\lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right)+2\mu\|\nabla\boldsymbol{u}\|_{L^2}^2+\lambda\gamma\|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2\\ \leq C\left(\lambda\gamma\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2+\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right).\end{aligned}\tag{19}$$

Notice that for $2D$ domains, with similar arguments as for Navier-Stokes case in $2D$ domains, applying now the existence and uniqueness of weak solutions of (15)-(17) ([11]), one has that the extension by periodicity of $(\boldsymbol{u},\boldsymbol{d})$ is a global solution defined in $[0,+\infty)$ and it is regular (if boundary data $\boldsymbol{h}$ is regular).

But, for $3D$ domains, if we intend to apply small data argument done before, smallness of the right hand-side of energy inequality (19) cannot be assured, because of the term $\|\boldsymbol{f}(\boldsymbol{d})\|_{L^2}^2$ (which is bounded by not small). Indeed, integrating (19) in $(0,T)$ and using the reproductivity, one has

$$\lambda\gamma\int_0^T\|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2\leq C\int_0^T\left(\lambda\gamma\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2+\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right)\leq\bar{\varepsilon}+C\lambda\gamma\int_0^T\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2$$

but this bound is not necessary small. It is only small for the penalty parameter ε big enough, which is not a physical interesting case.

Another possibility is to start from the energy equality:

$$\begin{aligned}\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2+\lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2+2\,\lambda\int_\Omega F_\varepsilon(\boldsymbol{d})\right)+2\,\mu\|\nabla\boldsymbol{u}\|_{L^2}^2+\lambda\gamma\|\Delta\widehat{\boldsymbol{d}}-\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2\\ \leq\frac{\lambda}{\gamma}\int_0^T\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2+\frac{2\lambda}{\varepsilon^2}\int_0^T\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^1}\end{aligned}\tag{20}$$

where $F_\varepsilon(\boldsymbol{d})=\dfrac{1}{4\varepsilon^2}(|\boldsymbol{d}|^2-1)^2$. Indeed, testing the $\boldsymbol{u}$-system by $\boldsymbol{u}$ and the $\boldsymbol{d}$-system by $\lambda(-\Delta\widehat{\boldsymbol{d}}+\boldsymbol{f}_\varepsilon(\boldsymbol{d}))$, one has

$$\begin{aligned}\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2+\lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right)+2\lambda\int_\Omega\partial_t\widehat{\boldsymbol{d}}\cdot\boldsymbol{f}_\varepsilon(\boldsymbol{d})+2\mu\|\nabla\boldsymbol{u}\|_{L^2}^2\\ +\lambda\gamma\|\Delta\widehat{\boldsymbol{d}}-\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2\leq C\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2,\end{aligned}$$

Then, using $\nabla_{\boldsymbol{d}}F_\varepsilon(\boldsymbol{d})=\boldsymbol{f}_\varepsilon(\boldsymbol{d})$, one has

$$\int_\Omega\partial_t\widehat{\boldsymbol{d}}\cdot\boldsymbol{f}_\varepsilon(\boldsymbol{d})=\frac{d}{dt}\int_\Omega F_\varepsilon(\boldsymbol{d})+\int_\Omega\partial_t\widetilde{\boldsymbol{d}}\cdot\boldsymbol{f}_\varepsilon(\boldsymbol{d})$$

Therefore, if we bound the last term by $\dfrac{1}{\varepsilon^2}\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^1}$, since $\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^\infty}\leq\dfrac{1}{\varepsilon^2}$, inequality (20) is proven.

In this case, for $\varepsilon>0$ fixed, smallness for $\int_0^T\|\Delta\widehat{\boldsymbol{d}}-\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2$ is obtained (using that $\int_0^T\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2$ and $\int_0^T\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^1}$ are small), but this does not give sufficient information to prove the smallness in the H^2 norm of $\widehat{\boldsymbol{d}}$ (again the term $\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2$ appears).

In conclusion, the regularity of the reproductive solutions for the $3D$ penalized nematic liquid crystal model (15)-(17) is an open problem.

5.2 Generalized Boussinesq model with Neumann boundary conditions for temperature

If we apply the same argument done in this section to prove regularity of the time-periodic solution for the 3D Navier Stokes model, now we obtain from energy inequality (10) that

$$\int_0^T \left\{ \|\boldsymbol{u}(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 \right\} \leq C \int_0^T \|\boldsymbol{f}(t)\|_{H^{-1}}^2$$

Hence assuming $\boldsymbol{f}$ small enough in the $L^\infty(0,+\infty;\mathbf{H}^{-1})$-norm, then there exists $t_\star \in [0,T]$ such that $\|\boldsymbol{u}(t_\star)\|_{H^1}^2 + \|\theta(t_\star)\|_{H^2}^2$ is small. To continue the argument, due to the highly nonlinear second order terms $\nabla\cdot(\nu(\theta)\nabla\boldsymbol{u})$ and $\nabla\cdot(k(\theta)\nabla\theta)$, also smallness in $\|\theta(t_\star)\|_{H^2}^2$ must be assured, in order to prove global and small regular solution $(\boldsymbol{u},\theta)$, but to obtain smallness of $\|\theta(t_*)\|_{H^2}$ for some t_*, is not clear.

Nevertheless a more direct argument could be considered. For instance, this argument works when the involved equations are (12), together with the Dirichlet-Neumann boundary conditions:

$$\boldsymbol{u} = 0, \qquad \partial_n \theta = 0 \qquad \text{on } [0,\infty)\times\partial\Omega, \tag{21}$$

and the time reproductive condition (14) as is proved in [2]. Indeed, assuming $\boldsymbol{f}$ small enough (but no the function $\boldsymbol{g}$ depending on the gravity), $H^2(\Omega)$ regularity for velocity and $H^3(\Omega)$ regularity for the temperature can be obtained ([2]) and consequently, a regular (and small) reproductive solution of (12), (14), (21) in $(0,T)$ exists (which a reproductive condition for time derivative of temperature also holds, i.e. $\partial_t\theta(0) = \partial_t\theta(T)$). The main ideas in the proof are, to obtain some differential inequalities in regular norms ($H^2(\Omega)$ for velocity and $H^3(\Omega)$ for temperature) and to use an argument of global solution for small data jointly with the argument of regular time periodic solution (see [2]).

Notice that, the uniqueness of regular time periodic solutions remains open, because higher regularity for the velocity (for instance of the H^3 type) is necessary in order to get uniqueness of the model (12), (14), (21). To obtain H^3 regularity for the velocity is not clear because the argument made in order to get H^3 regularity for θ is based in the Neumann condition, but for the velocity we have Dirichlet condition.

Remark 3 *When Dirichlet boundary conditions for* $\boldsymbol{u}$ *and* θ *are assumed, it is not clear how to obtain appropriate differential inequalities in* H^2 *for velocity and* H^3 *for temperature. In conclusion, the regularity of the time-periodic solution for the model* (12), (13), (14) *is an open problem.*

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WELLPOSEDNESS FOR THE SYSTEM MODELLING THE MOTION OF A RIGID BODY OF ARBITRARY FORM IN AN INCOMPRESSIBLE VISCOUS FLUID

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Abstract

In this paper (see [2] for the full version), we consider the interaction between a rigid body and an incompressible, homogeneous, viscous fluid. This fluid-solid system is assumed to fill the whole space $\mathbb{R}^d$, $d = 2$ or 3. The equations for the fluid are the classical Navier-Stokes equations whereas the motion of the rigid body is governed by the standard conservation laws of linear and angular momentum. The time variation of the fluid domain (due to the motion of the rigid body) is not known *a priori*, so we deal with a free boundary value problem. We improve the known results by proving a complete wellposedness result: our main result yields a local in time existence and uniqueness result of strong solutions for $d = 2$ or 3. Moreover, we prove that the solution is global in time for $d = 2$ and also for $d = 3$ if the data are small enough.

Key words: *Fluid-solid interaction, Navier-Stokes equations, incompressible viscous fluid, PDE-ODE coupled system*
AMS subject classifications: *74F10 76D03 76D05 35Q30*

1 Introduction and statement of the main results

The aim of this work is to prove a result on wellposedness for a coupled system of nonlinear partial and ordinary differential equations modelling the motion of a rigid body immersed into a viscous incompressible fluid. The fluid flow is governed by the classical Navier-Stokes system, whereas the motion of the rigid body is governed by the balance equations for linear and angular momentum (Newton's laws).

For $d = 2$ or $d = 3$, denote by $\mathcal{O}(t) \subset \mathbb{R}^d$, the domain occupied by the rigid body and set $\mathcal{F}(t) = \mathbb{R}^d \setminus \mathcal{O}(t)$ the exterior domain occupied by the fluid at the instant t. For the sake of simplicity, we assume that the fluid is homogeneous and of density one. Moreover, we assume that the rigid body is also homogeneous.

By choosing a frame of coordinates whose origin initially coincides with the mass center of the rigid body, the domain occupied by this last one at instant t is given by

$$\mathcal{O}(t) = \{Q(t)y + h(t),\ y \in \mathcal{O}(0)\}, \tag{1}$$

where $h(t)$ is the position of the mass center of the rigid body, and where $Q(t)$ is a rotation matrix associated to the angular velocity $\omega(t)$ of the rigid body. The matrix $Q(t)$ is the solution of the initial value problem

$$\begin{aligned} &Q'(t)Q^*(t)y = \omega(t) \times y \quad \forall y \in \mathbb{R}^d, \\ &Q(0) = \mathrm{Id}, \end{aligned} \tag{2}$$

where, for all matrix A, we have denoted by A^* the transpose matrix of A and by Id the identity matrix.

For planar motion (i.e. $d = 2$) we can assume that

$$\omega(t) = \widetilde{\omega}(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and in that case,

$$Q(t) = \begin{pmatrix} \widetilde{Q}(t) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

with

$$\widetilde{Q}(t) = \begin{pmatrix} \cos\widetilde{\theta}(t) & -\sin\widetilde{\theta}(t) \\ \sin\widetilde{\theta}(t) & \cos\widetilde{\theta}(t) \end{pmatrix}, \tag{3}$$

and $\widetilde{\theta}(t) = \displaystyle\int_0^t \widetilde{\omega}(s)\,\mathrm{d}s$. The important quantities for $d = 2$ are $\widetilde{\omega}$, $\widetilde{Q}$ and $\widetilde{\theta}$ and for simplicity of notation, we will omit the tilde in all these quantities. Therefore, for $d = 2$, ω will be a *scalar* function and $Q(t)$ a rotation matrix of *order 2* and of angle $\theta(t)$ (see (3)).

The system of equations modelling the motion of the fluid and of the rigid body can be written as

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)\, u + \nabla p = f, \qquad x \in \mathcal{F}(t), \tag{4}$$

$$\operatorname{div} u = 0, \qquad x \in \mathcal{F}(t), \tag{5}$$

$$u(x,t) = h'(t) + \omega(t) \times [x - h(t)], \qquad x \in \partial\mathcal{O}(t), \tag{6}$$

$$Mh''(t) = -\int_{\partial\mathcal{O}(t)} \sigma(u,p)n\,\mathrm{d}\Gamma + \int_{\mathcal{O}(t)} \rho f(x,t)\,\mathrm{d}x, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(J\omega\right)(t) = -\int_{\partial\mathcal{O}(t)} [x - h(t)] \times \sigma(u,p)n\,\mathrm{d}\Gamma + \int_{\mathcal{O}(t)} [x - h(t)] \times \rho f(x,t)\,\mathrm{d}x, \tag{8}$$

$$u(x,0) = u_0(x), \quad x \in \mathcal{F}(0), \tag{9}$$

$$h(0) = 0 \in \mathbb{R}^d,\ h'(0) = h^{(1)} \in \mathbb{R}^d,\ \omega(0) = \omega^{(0)}, \tag{10}$$

In the above system, $t \in (0,T)$, the unknowns are $u(x,t)$ (the Eulerian velocity field of the fluid), $p(x,t)$ (the pressure field of the fluid), $h(t)$ (the position of the mass center of the rigid body) and $\omega(t)$ (the angular velocity of the rigid body). For $d = 3$, we have denoted by $a \times b$ the classical cross product for $a, b \in \mathbb{R}^3$ whereas for $d = 2$, for $a, b \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have denoted

$$a \times b = a_1 b_2 - a_2 b_1 \quad \text{and} \quad \alpha \times b = \alpha \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix}.$$

The boundary of the rigid body at instant t is denoted by $\partial\mathcal{O}(t)$ and the normal unit vector directed to the interior of the rigid body is denoted by $n(x,t)$. We have also denoted by $f(x,t)$ the applied body forces (per unit mass). The positive constant ν stands for the viscosity of the fluid. Furthermore, we have denoted by M (respectively, by ρ) the mass (respectively, the density of the rigid body) and by J the inertia moment related to the mass center of the rigid body.

The formulae for M and J are

$$M = \int_{\mathcal{O}(t)} \rho \mathrm{d}x = \int_{\mathcal{O}(0)} \rho \mathrm{d}y,$$

$$J = \int_{\mathcal{O}(t)} \rho |x - h(t)|^2 \mathrm{d}x = \int_{\mathcal{O}(0)} \rho |y|^2 \mathrm{d}y \quad \text{if } d = 2,$$

and

$$J(t)_{kl} = \int_{\mathcal{O}(t)} \rho \left[|x - h(t)|^2 \delta_{kl} - (x - h(t))_k (x - h(t))_l \right] \mathrm{d}x$$

for $k,\ l \in \{1, \ldots, d\}$ if $d = 3$, where δ_{kl} is the Kronecker's symbol.

Moreover, the notation $x \cdot y$ stands for the inner product of x and y and the notation $|x|$ stands for the corresponding norm. Finally, we have denoted by $\sigma(u,p)$ the Cauchy stress tensor field in the fluid defined by

$$\sigma_{kl}(u,p) = -p\delta_{kl} + \nu \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right), \quad \text{for } k,\ l \in \{1, \ldots, d\}. \tag{11}$$

The problem of interaction between a viscous incompressible fluid and a rigid body has been studied intensively in the recent years (see References [1], [4], [5], [6], [10], [12], [13], [14], [15], [16], etc.). However, as far as we know, only few results concerning the existence and uniqueness of strong solutions for the problem (1), (2), (4)-(10) are available in the case where the system fills the whole space. In that case, we can mention the results of Takahashi and Tucsnak [17], and of Galdi and Silvestre [8]. In [17], the authors show the global in time existence and uniqueness of strong solutions in two spatial dimensions in the particular case where the rigid body is a disk. In [8], the authors prove the

existence of local in time strong solutions for a rigid body having an arbitrary regular shape. Nevertheless, their result does not yield neither the uniqueness of solutions nor the global in time existence (even for small data).

On the other hand, due to the complexity of the problem, another related problem simpler than (1), (2), (4)-(10), in which the motion of the rigid body is prescribed as a constant rotation has also been investigated. In particular, a local in time existence and uniqueness result of *mild* solutions has been proved by Hishida [11], and recently a local in time existence result of strong solutions has been proved by Galdi and Silvestre [9]. Moreover, the authors prove that the solution is global in time, provided that the initial velocity u_0, in an appropriate norm, and the magnitude of ω do not exceed a certain constant depending only on the viscosity and on the regularity of $\mathcal{F}(0)$. However, the authors do not make any reference to uniqueness properties of the solution. Both works mentioned before deal with the problem by writing the equations of motion of the fluid-rigid body system in a frame attached to the rigid body. Furthermore, a local in time existence and uniqueness result of strong solutions has been very recently proved by Cumsille and Tucsnak [3]. There, the authors proved that the solution is global in time in two spatial dimensions provided that the velocity performs suitable *a priori* estimates. We remark that the work previously cited deals with the problem by making a new change of variables, instead of writing the equations of motion in a frame attached to the rigid body. We use here a similar idea to work with the problem (1), (2), (4)-(10).

In order to make the region occupied by the fluid time independent, it is quite natural to refer the equations of motion of the fluid-rigid body system in a frame attached to the rigid body, with origin in the center of mass of this last one, and coinciding with an inertial frame at time $t = 0$ (see [7] for details). More precisely, let us denote

$$\overline{u}(y,t) = Q^*(t)u(Q(t)y + h(t), t); \quad \overline{p}(y,t) = p(Q(t)y + h(t), t);$$
$$\overline{h}(t) = \int_0^t Q^*(s)h'(s)\,\mathrm{d}s;$$

$$\overline{J} = J(0); \quad \overline{\omega}(t) = \begin{cases} \omega(t) & \text{for } d = 2, \\ Q^*(t)\omega(t) & \text{for } d = 3; \end{cases}$$

$$\overline{f}_M(t) = -M\overline{\omega}(t) \times \overline{h}'(t); \quad \overline{f}_J(t) = \begin{cases} 0 & \text{for } d = 2, \\ \left(\overline{J}\overline{\omega}(t)\right) \times \overline{\omega}(t) & \text{for } d = 3; \end{cases}$$

$$\overline{f}(y,t) = Q^*(t)f(Q(t)y + h(t), t).$$

In this case, the equivalent system to the original one can be written as

$$\frac{\partial \overline{u}}{\partial t} - \nu\Delta\overline{u} + \left[\left(\overline{u} - \overline{h}' - \overline{\omega} \times y\right) \cdot \nabla\right] \overline{u} + \nabla\overline{p} + \overline{\omega} \times \overline{u} = \overline{f}, \quad (y,t) \in Q_{\mathcal{F}}, \quad (12)$$

$$\operatorname{div} \overline{u} = 0, \quad (y,t) \in Q_{\mathcal{F}} \quad (13)$$

$$\overline{u}(y,t) = \overline{h}'(t) + \overline{\omega}(t) \times y, \quad (y,t) \in \Sigma_{\mathcal{F}}, \quad (14)$$

$$M\overline{h}''(t) = -\int_{\partial\mathcal{O}(0)} \sigma(\overline{u},\overline{p})n\,\mathrm{d}\Gamma + \int_{\mathcal{O}(0)} \rho\overline{f}(y,t)\,\mathrm{d}y + \overline{f}_M(t), \tag{15}$$

$$\overline{J}\overline{\omega}'(t) = -\int_{\partial\mathcal{O}(0)} y\times(\sigma(\overline{u},\overline{p})n)\,\mathrm{d}\Gamma + \int_{\mathcal{O}(0)} y\times\rho\overline{f}(y,t)\,\mathrm{d}y + \overline{f}_J(t), \tag{16}$$

$$\overline{u}(y,0) = u_0(y), \quad y\in\mathcal{F}(0), \tag{17}$$

$$\overline{h}(0) = 0,\ \overline{h}'(0) = h^{(1)},\ \overline{\omega}(0) = \omega^{(0)}, \tag{18}$$

where $Q_{\mathcal{F}} = \mathcal{F}(0)\times(0,T)$, $\Sigma_{\mathcal{F}} = \mathcal{O}(0)\times(0,T)$ and $t\in(0,T)$. One of the main difficulties comes from the term $[(\overline{\omega}\times y)\cdot\nabla]\overline{u}$, whose coefficient becomes unbounded at large spatial distances. In order to overcome this difficulty, we use another change of variables which coincides with $Q(t)y+h(t)$ in a neighborhood of the rigid body and is equal to the identity far from the rigid body. By using this change of variables, we obtain a system of equations whose coefficients are bounded at large spatial distances, instead of the term $[(\overline{\omega}\times y)\cdot\nabla]\overline{u}$. This feature of our method allows us to improve the results of [8], in the sense that we get the uniqueness as well as the global character (in time) of the solution.

In the rest of this work, we denote $\mathcal{F} = \mathcal{F}(0)$ and $\mathcal{O} = \mathcal{O}(0)$. As usual, for $m\in\mathbb{N}$ and $\alpha\in[1,\infty]$ we denote by $W^{m,\alpha}(\mathcal{F})$ the Sobolev spaces formed by the functions in $L^\alpha(\mathcal{F})$ which have distributional derivatives, up to the order m, in $L^\alpha(\mathcal{F})$, and by $H^m(\mathcal{F}) = W^{m,2}(\mathcal{F})$. We also denote by $\widehat{H}^1(\mathcal{F})$ the homogeneous Sobolev space

$$\widehat{H}^1(\mathcal{F}) = \{q\in L^2_{loc}(\overline{\mathcal{F}}) \mid \nabla q\in[L^2(\mathcal{F})]^d\},$$

where $q\in L^2_{loc}(\overline{\mathcal{F}})$ means that $q\in L^2(\mathcal{F}\cap B_0)$ for all open balls $B_0\subset\mathbb{R}^d$ with $B_0\cap\mathcal{F}\neq\emptyset$. We identify two functions of $\widehat{H}^1(\mathcal{F})$ if they differ by a constant.

Moreover, we set

$$\mathcal{H}^m(\mathcal{F}(t)) = [H^m(\mathcal{F}(t))]^d, \qquad \mathcal{H}^m(\mathbb{R}^d) = [H^m(\mathbb{R}^d)]^d$$
$$\mathcal{L}^\alpha(\mathcal{F}(t)) = [L^\alpha(\mathcal{F}(t))]^d, \qquad \mathcal{L}^\alpha(\mathbb{R}^d) = [L^\alpha(\mathbb{R}^d)]^d.$$

We denote

$$\mathcal{F}_T = \{(x,t)\in\mathbb{R}^d\times[0,T]\ ;\ x\in\mathcal{F}(t)\}.$$

Consider a smooth mapping $X:\mathcal{F}\times[0,T]\to\mathbb{R}^d$ such that for all $t\in[0,T]$, $X(\cdot,t)$ is a C^∞-diffeomorphism from $\mathcal{F}$ onto $\mathcal{F}(t)$. Moreover, suppose that the mappings

$$(y,t)\mapsto D_tD_y^\alpha X(y,t), \quad \alpha\in\mathbb{N}^d,$$

exist, are continuous and compactly supported in $\mathcal{F}$ (such a mapping will be given in Section 2). For any $g:\mathcal{F}_T\to\mathbb{R}^d$, we denote by $g_X:\mathcal{F}\times[0,T]\to\mathbb{R}^d$ the mapping $g_X(y,t) = g(X(y,t),t)$, for all $t\geq 0$ and for all $y\in\mathcal{F}$. In order to analyze the problem (1), (2), (4)-(10), we need to introduce the following

functions spaces in variable domain:

$$\begin{aligned} L^2(0,T;\mathcal{H}^2(\mathcal{F}(t))) &= \{u \,:\, u_X \in L^2(0,T;\mathcal{H}^2(\mathcal{F}))\},\\ H^1(0,T;\mathcal{L}^2(\mathcal{F}(t))) &= \{u \,:\, u_X \in H^1(0,T;\mathcal{L}^2(\mathcal{F}))\},\\ C([0,T],\mathcal{H}^1(\mathcal{F}(t))) &= \{u \,:\, u_X \in C([0,T],\mathcal{H}^1(\mathcal{F}))\},\\ L^2(0,T;\widehat{H}^1(\mathcal{F}(t))) &= \{p \,:\, p_X \in L^2(0,T;\widehat{H}^1(\mathcal{F}))\}. \end{aligned}$$

Finally, let us denote by $\mathcal{U}(\mathcal{F}_T)$ the space of strong solutions for the velocity, defined by

$$\mathcal{U}(\mathcal{F}_T) = L^2(0,T;\mathcal{H}^2(\mathcal{F}(t))) \cap C([0,T],\mathcal{H}^1(\mathcal{F}(t))) \cap H^1(0,T;\mathcal{L}^2(\mathcal{F}(t))). \quad (19)$$

We also define

$$\mathcal{U}_T = L^2(0,T;\mathcal{H}^2(\mathcal{F})) \cap C([0,T],\mathcal{H}^1(\mathcal{F})) \cap H^1(0,T;\mathcal{L}^2(\mathcal{F})). \quad (20)$$

The main results of this paper are the three following theorems.

Theorem 1 *Suppose that $\mathcal{O}$ is a bounded open connected subset of $\mathbb{R}^d$ with boundary $\partial\mathcal{O}$ of class C^3. Assume that*

$$f \in L^2_{loc}(0,\infty;[W^{1,\infty}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)]^d)$$

and that $u_0 \in \mathcal{H}^1(\mathcal{F})$ with

$$\begin{cases} \operatorname{div} u_0 = 0 & \text{in } \mathcal{F},\\ u_0(x) = h^{(1)} + \omega^{(0)} \times x & \text{on } \partial\mathcal{O}. \end{cases}$$

Then, there exists $T_0 > 0$ such that the problem (1), (2), (4)-(10) admits a unique strong solution (u, p, h, ω) defined in $[0, T_0)$, that is

$$u \in \mathcal{U}(\mathcal{F}_T), \quad p \in L^2(0,T;\widehat{H}^1(\mathcal{F}(t))), \quad h \in H^2(0,T;\mathbb{R}^d), \quad \text{and}$$
$$\begin{cases} \omega \in H^1(0,T;\mathbb{R}) & \text{if } d = 2,\\ \omega \in H^1(0,T;\mathbb{R}^3) & \text{if } d = 3, \end{cases}$$

for all $T \in (0, T_0)$.
Moreover, we can choose T_0 such that one the following alternatives holds true:

(a) $T_0 = +\infty$,

(b) *the function $t \mapsto \|u(t)\|_{\mathcal{H}^1(\mathcal{F}(t))}$ is not bounded in $[0, T_0)$.*

Theorem 2 *Assume that the hypotheses in Theorem* 1 *hold true and suppose that $d = 2$. Then, assertion* (a) *in Theorem* 1 *holds true, that is, the strong solution of the problem* (1), (2), (4)-(10) *is global in time.*

Theorem 3 *Assume that the hypotheses in Theorem* 1 *hold true and suppose that* $d = 3$. *Moreover, suppose that*

$$f \in L^2(0,\infty,\mathcal{L}^2(\mathbb{R}^3)) \cap L^1(0,\infty,\mathcal{L}^2(\mathbb{R}^3)).$$

There exists a positive constant $c = c(\mathcal{O},\nu,\rho)$ *such that if*

$$\|u_0\|^2_{\mathcal{L}^2(\mathbb{R}^3)} + \|D(u_0)\|^2_{[L^2(\mathbb{R}^3)]^9} + \|f\|^2_{L^2(0,\infty;\mathcal{L}^2(\mathbb{R}^3))} + \|f\|^2_{L^1(0,\infty;\mathcal{L}^2(\mathbb{R}^3))} < c, \quad (21)$$

then assertion (a) *in Theorem* 1 *holds true, that is, the strong solution of the problem* (1), (2), (4)-(10) *is global in time.*

As a direct consequence of Theorem 1, we recover the result [8, Theorem 4.1]:

Corollary 1 *Assume that the hypotheses in Theorem* 1 *hold true and suppose that* $d = 3$. *Let us denote by* B_R *the open ball centered at 0 and of radius* R *in* $\mathbb{R}^3$, *by* $\mathrm{diam}(\mathcal{O})$ *the diameter of* $\mathcal{O}$ *and* $\mathcal{F}_R = \mathcal{F} \cap B_R$ *for* $R > \mathrm{diam}(\mathcal{O})$. *Then there exists* T^* *and* $\overline{u} = \overline{u}(y,t)$, $\overline{p} = \overline{p}(y,t)$, $\overline{h} = \overline{h}(t)$, $\overline{\omega} = \overline{\omega}(t)$ *and* $Q = Q(t)$ *satisfying* (12)-(18) *and* (2) *almost everywhere in* $\mathcal{F} \times (0,T^*)$ *such that*

$$\overline{u}_i,\ \frac{\partial \overline{u}_i}{\partial y_j} \in L^\infty(0,T;L^2(\mathcal{F})), \quad \frac{\partial \overline{p}}{\partial y_i},\ \frac{\partial^2 \overline{u}_i}{\partial y_j \partial y_k} \in L^2(0,T;L^2(\mathcal{F})),$$
$$\overline{h}_i \in H^2(0,T), \quad \overline{\omega}_i \in H^1(0,T), \quad Q_{ij} \in H^2(0,T)$$
$$\frac{\partial \overline{u}_i}{\partial t} \in L^2(0,T;L^2(\mathcal{F}_R)) \quad \forall R > \mathrm{diam}(\mathcal{O}),$$

for any $T \in (0,T^*)$. *Moreover,*

$$\overline{h}_i \in C^1([0,T]), \quad \overline{\omega}_i,\ Q_{ij} \in C([0,T]), \quad \overline{h}(0) = 0, \quad \overline{h}'(0) = h^{(1)},$$
$$\overline{\omega}(0) = \omega^{(0)}, \quad Q(0) = \mathrm{Id},$$

$$\overline{u} \in C([0,T],H^1(\mathcal{F})) \quad \textit{with } \overline{u}(\cdot,0) = u_0(\cdot),$$

for any $T \in (0,T^*)$.

Remark 1 *The existence of solutions for the problem* (1), (2), (4)-(10), *with initial data satisfying the same assumptions as in Theorem 1 has already been investigated in [17] assuming that the rigid body is an infinite cylinder of circular cross-section. Moreover, a similar problem was studied in [8], where the difference with our problem is that in [8], the authors suppose that there are prescribed external forces and torques acting on the rigid body and assume that only conservative forces act on the fluid. The novelty of our results consists in the fact that we obtain an existence and uniqueness result for strong solutions in the case of a rigid body of arbitrary and regular shape. On the other hand, we obtain a solution which is* unique and global in time *(assuming that the data are small enough if* $d = 3$*).*

2 A brief outline of the proof

In this section, we briefly describe the steps of the proof of our main results. These steps can be summarized as follow:

Step 1. (Reduction to a fixed domain). As already mentioned, the key of our approach is to perform another change of variables (instead of to simply refer the problem (1), (2), (4)-(10) to a frame of coordinates attached to the rigid body). This change of variables coincides with $Q(t)y + h(t)$ in a neighborhood of the rigid body and it is equal to the identity mapping far from the last one. With this change of variables we get a system written in the fixed (cylindrical) domain $\mathcal{F} \times (0, T)$.

Step 2. (Analysis of the problem after change of variables). Once the problem is transformed to a one written in fixed domain, we have to show the wellposedness to this last one. To do this, we write our transformed problem as a Stokes problem with right-hand side depending on the solution (i.e. this term contains, among others, the non-linear part of the Navier-Stokes equation). We note that the transformed problem takes into account the rigid structure of the solution (in the solid domain). Then, we write the linearized Stokes problem as an abstract ordinary differential equation on an appropriate Hilbert space. The involved operator is as the classical Stokes operator in the fluid domain and is a rigid motion in the solid domain. Therefore, by using semigroup theory we are able to show the wellposedness as well as continuity with respect to the data of this problem. Finally, by means of an appropriate fixed point procedure, we get a (unique) solution to our problem written in fixed domain.

Step 3. (*A priori* estimates for the original problem). The last step is to show that our solution is global in time. To do this, once obtained the (local) solution to our transformed problem, we return to our original problem (written in the time-dependent domain). This is due to the fact that it is easier to perform *a priori* estimates in the original problem than in the transformed problem (because there, the terms in the main equation are more complicated). We note that, because of the fact that the domain is time-dependent, special care must be taken in picking the right test function (which has to multiply to the Navier-Stokes equation). The idea is to find the right test function, in such a way that allows us to appropriately bound all of the terms appearing in the Navier-Stokes equation, and which, at the same time, allows us to take advantage of the solid equations.

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ANÁLISIS MATEMÁTICO DE ALGUNOS PROBLEMAS DE CONTACTO

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Dedicado a la memoria de mi padre: José Ramón Fernández Lazcano (1946–2006)

Resumen

En este artículo se realiza una revisión de los resultados obtenidos por el autor en el estudio de diferentes problemas de contacto en elasticidad, elasto-piezoelectricidad, viscoelasticidad o elasto-viscoplasticidad. Para cada problema se consideró diversos aspectos como la existencia y unicidad de solución débil y su regularidad, el análisis numérico de aproximaciones obtenidas mediante el método de los elementos finitos y, cuando fuera necesario, el esquema de Euler, y, por último, su implementación en ordenador.

Palabras clave: *Viscoelasticidad, elasto-viscoplasticidad, respuesta normal, Signorini, rozamiento, elementos finitos, estimaciones del error, resultados numéricos.*

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1 Introducción

Desde el punto de vista de la Ingeniería hace mucho tiempo que se consideran problemas de contacto de materiales, siendo numerosas las referencias que trataron, fundamentalmente, la modelación o la resolución numérica de estos problemas (véanse, por ejemplo, [34, 35, 46, 49, 50, 58] y las referencias allí citadas). En la industria aparecen con frecuencia situaciones que se pueden modelar de esta forma: por ejemplo, el contacto de los sistemas de frenado

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con las ruedas o de estas con la carretera, y diferentes procesos de creación de metales o de su manufacturación.

Durante los últimos años, el estudio matemático de los problemas de contacto en sus diferentes campos (existencia y unicidad de solución débil y su regularidad, el análisis numérico de sus aproximaciones y su implementación en ordenador, etc) ha experimentado un creciente interés. Desde uno de los primeros artículos debido a Signorini ([52]) en el que se planteó, por primera vez, un problema de contacto en elasticidad con una base rígida, pasando por su análisis matemático en [33] y el desarrollo más profundo, incluyendo más materiales y otras condiciones, realizado en la clásica referencia [21], hasta la actualidad mucho ha cambiado y es por ello que podemos hablar de una *Teoría Matemática de la Mecánica del Contacto*. En este trabajo se recogen los resultados obtenidos por el autor en esta línea de investigación, fundamentalmente en su vertiente del análisis numérico, y por otros investigadores en problemas relacionados.

El artículo se divide en cuatro partes principales. En la Sección 2 se presentan las diferentes leyes constitutivas (elasticidad o elasto-piezoelectricidad, viscoelasticidad y elasto-viscoplasticidad) y las condiciones de contacto que dan lugar a los problemas que se considerarán en el resto del trabajo, así como una breve descripción de dos fenómenos mecánicos que suelen presentarse en los procesos de contacto: el daño del material y la adherencia de la superficie de contacto. En la Sección 3 se describen los principales problemas que se han estudiado en el campo de la elasticidad, con mención especial a las aplicaciones biomédicas, y de la elasto-piezoelectricidad. Algunos problemas de contacto en viscoelasticidad se introducen en la Sección 4, incluyendo el daño del material, la adherencia de la superficie o los efectos de inercia, y, finalmente, en elasto-viscoplasticidad y en el caso cuasiestático, en la Sección 5.

2 Modelos mecánicos: leyes constitutivas y condiciones de contacto

2.1 Leyes constitutivas

La relación entre las tensiones en el cuerpo y las deformaciones resultantes caracteriza el tipo de material que compone el cuerpo y se describe mediante una ley constitutiva. De esta forma, determina las deformaciones del cuerpo como consecuencia de la acción de fuerzas volúmicas o superficiales. En esta sección, y en el resto de la memoria, únicamente consideraremos el contexto de *pequeñas deformaciones*.

Denotemos por Ω un dominio de $\mathbb{R}^d$ $(d = 1, 2, 3)$, representando la configuración de referencia de un cuerpo deformable que puede entrar en contacto con un obstáculo. Sea Γ su frontera exterior, dividida en tres partes disjuntas y medibles que denotamos por Γ_D (frontera Dirichlet), Γ_N (frontera Neumann) y Γ_C (frontera de contacto). Además, suponemos que en el dominio Ω actúan fuerzas volúmicas de densidad $\boldsymbol{f}_B$ y que en Γ_N se prescriben fuerzas superficiales de densidad $\boldsymbol{f}_N$.

Sean $\boldsymbol{\sigma}$ el tensor de tensiones, $\boldsymbol{u}$ el campo de desplazamientos y $\boldsymbol{\varepsilon}(\boldsymbol{u})$ el

tensor de deformaciones linealizado dado por

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^{d}, \quad \varepsilon_{ij}(\boldsymbol{u}) = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

A continuación describimos brevemente los cuatro tipos de materiales considerados (es decir, las cuatro leyes constitutivas que los definen):

- Recordemos, en primer lugar, la clásica ley constitutiva que define un material elástico lineal (también conocida como ley de Hooke, véase [21]):

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),$$

  donde $\mathcal{C}$ es un tensor de cuarto orden acotado, simétrico y definido positivo.

- Si suponemos que los efectos debidos al campo eléctrico son importantes, entonces el material se supone elasto-piezoeléctrico y la correspondiente ley constitutiva se escribe como ([21, 53]):

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - Estudiante^{*}\mathbf{E}(\varphi),$$

  donde $\mathbf{E}(\varphi)$ representa el campo eléctrico (φ es el potencial eléctrico) dado por

$$E_i(\varphi) = -\frac{\partial \varphi}{\partial x_i}, \quad i = 1, \ldots, d,$$

  y $Estudiante^{*} = (e^{*}_{ijk})_{i,j,k=1}^{d}$ denota el traspuesto del tensor piezoeléctrico de tercer orden $Estudiante = (e_{ijk})_{i,j,k=1}^{d}$. Recordemos que

$$e^{*}_{ijk} = e_{kij}, \quad \text{para todo} \quad i, j, k = 1, \ldots, d.$$

- Si el material se supone viscoelástico, la ley constitutiva se expresa mediante la siguiente relación (véanse [21, 39]):

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{C}(\boldsymbol{\varepsilon}(\boldsymbol{u})),$$

  donde $\mathcal{A}$ es un tensor de cuarto orden acotado, simétrico y definido positivo que contiene las propiedades de viscosidad del material y $\mathcal{C}$ es una función Lipschitziana. Si suponemos que los operadores $\mathcal{A}$ y $\mathcal{C}$ son lineales, entonces la relación anterior se conoce como ley de Kelvin-Voigt ([21]):

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}(\boldsymbol{u}) + a_{ijkl}\varepsilon_{kl}(\dot{\boldsymbol{u}}),$$

  donde c_{ijkl}, a_{ijkl} son los coeficientes de elasticidad y viscosidad, respectivamente. También podemos incluir los efectos eléctricos y se hablaría, entonces, de un material electro-viscoelástico.

  Además, si suponemos que la "memoria del material" es importante (es decir, las deformaciones en instantes anteriores modifican el estado

actual), entonces el término de viscosidad se reemplaza por (véanse [21, 47]):

$$\int_0^t \mathcal{A}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds.$$

- En [20] se describe mecánicamente, desde un punto de vista experimental, el comportamiento de un tipo de materiales, llamados elasto-viscoplásticos, que se pueden definir a partir de la siguiente ecuación diferencial ordinaria:

$$\dot{\boldsymbol{\sigma}} = \mathcal{C}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})),$$

donde $\mathcal{G}$ es una función constitutiva no lineal que contiene las propiedades viscoplásticas del material.

Leyes constitutivas dependientes del tiempo de esta forma han sido utilizadas para la modelación de materiales como caucho, metales, pasta de papel o rocas (véanse [20, 40] y las referencias allí citadas). Un ejemplo clásico de función viscoplástica $\mathcal{G}$ es la ley de Perzyna introducida en [21].

2.2 Condiciones de contacto

Además del tipo de material (o ley constitutiva), la otra cuestión que determina la naturaleza de los problemas de contacto es el tipo de contacto que se puede establecer con el obstáculo. Denotamos por $\boldsymbol{\nu}$ el vector unitario normal y exterior a la frontera Γ_C, $\boldsymbol{\tau}$ el conjunto de vectores unitarios tangentes a Γ_C, y definimos la función g que representa la distancia que separa el cuerpo del obstáculo, medida en la dirección del vector $\boldsymbol{\nu}$, el desplazamiento normal $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, la tensión normal $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$, el desplazamiento tangencial $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$ y la tensión tangencial $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$. Además, en el caso bidimensional, denotamos por u_τ y σ_τ las proyecciones en la dirección del vector tangente; es decir, $u_\tau = \boldsymbol{u}\cdot\boldsymbol{\tau}$ y $\sigma_\tau = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\tau}$.

Las siguientes condiciones de contacto se pueden definir en la parte de la frontera Γ_C:

- *Contacto bilateral*: se supone que el cuerpo y el obstáculo están siempre pegados (es decir, $g = 0$ y $u_\nu = 0$).

- *Condición de contacto con respuesta normal*: el obstáculo se supone deformable. De acuerdo a [42, 48], se supone que existe una relación semilineal entre la tensión normal y la penetración $u_\nu - g$ en la forma:

$$-\sigma_\nu = p(u_\nu - g),$$

donde p es una función que mide la penetración en el obstáculo. Como una variación de esta condición, podemos considerar la llamada *condición de respuesta normal amortiguada*, que se escribe de la forma $-\sigma_\nu = p(\dot{u}_\nu)$. Esta condición se puede utilizar, por ejemplo, para modelar situaciones como la excavación de pozos en zonas rocosas.

- *Condiciones de Signorini*: el obstáculo se supone ahora rígido. Se escriben en la forma $u_\nu \leq g$ (no hay penetración), $\sigma_\nu \leq 0$ (sólo hay reacción) y la condición complementaria $\sigma_\nu(u_\nu - g) = 0$ (sólo hay reacción si hay contacto y viceversa).

- *Rozamiento o fricción*: es una fuerza que, al producirse el contacto, se opone al movimiento tangencial. En las siguientes secciones, en el contexto bidimensional, este se supondrá de la forma:

$$|\sigma_\tau| \leq h(t).$$
$$\text{Si } |\sigma_\tau| = h(t) \Rightarrow \exists \lambda \geq 0\,;\, \dot{u}_\tau = -\lambda \sigma_\tau.$$
$$\text{Si } |\sigma_\tau| < h(t) \Rightarrow \dot{u}_\tau = 0.$$

  Debemos observar que cuando $h(t) = \textit{constante}$, las condiciones anteriores se conocen como *ley de rozamiento de Tresca* (véase [21]), y cuando $h(t) = \mu\, p_T(u_\nu - g)$ ($\mu > 0$ constante) es un caso particular de la ley de rozamiento de Coulomb. Además, si suponemos que la *fuerza de rozamiento es despreciable* esta condición se escribe como $\boldsymbol{\sigma}_\tau = \mathbf{0}$, y el problema de contacto se dice sin rozamiento.

2.3 Daño del material

En los últimos años existe un creciente interés por determinar la variación de las propiedades mecánicas del material al someterlo a un proceso mecánico repetitivo. En muchos materiales como el *cemento*, se ha observado un decrecimiento de su capacidad de carga como consecuencia de la aparición de microfracturas internas. Por supuesto, este es un tema de gran importancia ya que afecta al funcionamiento, a la calidad o a la duración de algunas máquinas o de sus componentes.

Desde el punto de vista de la ingeniería este tema ha sido muy abordado (véase, por ejemplo, [35] y las numerosas referencias allí citadas). Sin embargo, sólo recientemente se han estudiado matemáticamente diversos modelos de daño que intervienen en problemas de contacto. En nuestro caso, los que hemos considerado están basados en el introducido por Frémond y Nedjar ([36]). La idea principal es la definición de una función de daño, ζ, que en el caso unidimensional elástico coincide con el cociente entre el "módulo de Young efectivo" (es decir, el módulo de Young que tiene el material en cada instante) y el módulo de Young del material sin estar sometido a ningún esfuerzo. De esta definición se deduce que esta función está en el intervalo $[0, 1]$, verificando que cuando $\zeta = 1$ el material no está dañado, cuando $\zeta = 0$ está completamente dañado y cuando $0 < \zeta < 1$ hay un daño parcial. De acuerdo a [36], el modelo mecánico del daño se formula mediante la siguiente inclusión parabólica diferencial y no lineal:

$$\dot{\zeta} - \kappa \triangle \zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta),$$

donde $I_{[0,1]}$ denota la función indicatriz del intervalo $[0, 1]$, κ es la difusión de las microfracturas (por simplicidad, suponemos que es constante) y ϕ representa

la función que mide la generación de daño. Finalmente, se ha demostrado en [13, 44] que se puede sustituir la inclusión precedente por una ecuación diferencial en derivadas parciales suponiendo ciertas hipótesis sobre la función ϕ. Se utiliza un teorema de comparación que permite, bajo estas condiciones, demostrar que la función del daño pertenece al intervalo $[0, 1]$ y, por tanto, la función indicatriz no es necesaria.

2.4 Adherencia de la superficie

Los procesos de adherencia, o adhesión, son muy importantes en muchos procesos industriales donde partes de su maquinaria, normalmente no metálicas, están pegadas. Existen muchos artículos desarrollados en el campo de la Ingeniería que tratan problemas de este tipo. En nuestro caso, seguimos los trabajos de Frémond (véase, para más detalles, [35]). La idea principal es la definición de una variable superficial, β, llamada *función de adherencia* que describe la densidad puntual de las uniones activas en la zona de contacto. Esta función toma valores en el intervalo $[0, 1]$, de tal forma que cuando $\beta = 0$ no hay uniones activas (es decir, no hay adherencia), cuando $\beta = 1$ las uniones están todas activas (la adherencia es total) y cuando $0 < \beta < 1$, sólo una parte de las uniones está actuando y la adherencia es parcial.

En la literatura hay muchos modelos que representan la evolución de esta función de adherencia (véase la reciente monografía [51] para más detalles). Por ejemplo, en [1] la evolución de esta función estaba regida por la siguiente ecuación diferencial:

$$\dot{\beta} = -\gamma_\nu \beta((-R(u_\nu))_+)^2,$$

donde R es una función de truncamiento y $\gamma_\nu > 0$ es un coeficiente de adherencia. En este caso, describe un proceso irreversible donde sólo el "despegue" de la superficie tiene lugar (la función β es siempre decreciente). En [29] se consideró que el pegamento también puede recuperarse (la función también puede crecer) y se modificó, de forma adecuada, la anterior ecuación diferencial.

3 Problemas de contacto en elasticidad y elasto-piezoelectricidad

Los problemas de contacto en elasticidad ya fueron estudiados en los años 70-80 (la existencia y unicidad de solución, así como su regularidad, pueden verse en [21], y el análisis numérico y su implementación en ordenador en [55]; véase [41] para un análisis detallado). Nuestra contribución se centró en la aplicación de estos tipos de problemas en la simulación de los siguientes procesos en Ortodoncia:

1. El cálculo de las líneas de máxima tensión para la localización de fracturas de mandíbula tras recibir un impacto ([38, 56]). Este fenómeno se modeló como un problema dinámico de Signorini en elasticidad.

2. La reducción de fracturas mandibulares mediante miniplacas corticales ([25]). Este proceso se representó como un problema de contacto entre tres cuerpos elásticos: las dos partes de la mandíbula fracturada y las miniplacas. En la Figura 1 se muestran, como ejemplo, los resultados obtenidos tras un golpe lateral en una mandíbula fracturada en la zona sinfisaria (parte media) y reducida con una o dos miniplacas. Los resultados obtenidos sugieren que dos miniplacas conllevan una mayor estabilidad estructural y una reducción de los esfuerzos internos (producto de impactos externos).

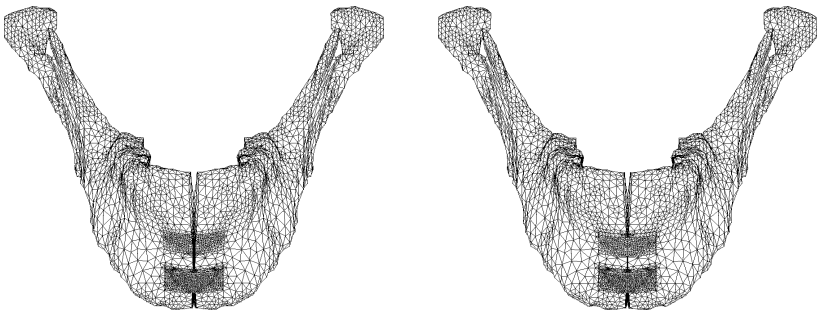

Figura 1: Deformaciones tras un golpe en la zona sinfisaria en una mandíbula reducida con una o dos miniplacas.

3. El comportamiento de un implante dental en la mandíbula al estar sometido a una serie de fuerzas como, por ejemplo, una mordida ([37]). Este problema se modeló como un problema de contacto con condiciones de Signorini en elasticidad.

Esta contribución se describió con detalle en [57] y debemos destacar que fue galardonada con el Premio S$\vec{e}$MA de Divulgación Matemática 2005.

Desde el punto de vista matemático, es muy importante la introducción del daño en el estudio de problemas elásticos ya que presenta una serie de dificultades que constituyen todavía problemas abiertos (incluso si no hay contacto). Así, por ejemplo, en [23] se consideró un problema unidimensional donde el desplazamiento se puede despejar en función de la variable daño y se estudió su comportamiento dependiendo de varios parámetros. Recientemente, en colaboración con el profesor Kenneth L. Kuttler (Brigham Young University, EEUU) se han aplicado nuevas técnicas de operadores pseudomonótonos ([43]) en el estudio de problemas elásticos con daño (véanse, por ejemplo, [13, 44]), sin considerar, por el momento, el contacto con un obstáculo.

Durante los últimos años se ha comenzado a estudiar un tipo especial de materiales en los que el efecto eléctrico tiene una gran importancia. Son los llamados materiales elasto-piezoeléctricos. La piezoelectricidad es la capacidad de ciertos cristales, como el cuarzo, algunos materiales cerámicos ($BaTiO_3$, $LiNbO_3$, PZT-5A, etc) o incluso el hueso humano, para producir una corriente eléctrica cuando están sometidos a esfuerzos internos. El efecto piezoeléctrico

está caracterizado por el acoplamiento entre las propiedades mecánicas y eléctricas del material: se ha observado que la aparición de cargas eléctricas en algunos cristales estaba motivada por la acción de fuerzas, volúmicas o superficiales, y recíprocamente, la acción de este campo eléctrico generaba deformaciones o tensiones internas en el cuerpo. Este tipo de materiales aparece comunmente en la industria como interruptores en electroacústica o equipos de medida. En [5], y como continuación a [54], se estudiaron numéricamente dos problemas de contacto de un cuerpo elasto-piezoeléctrico suponiendo que el obstáculo era rígido (condiciones de Signorini) o deformable (condición de respuesta normal).

4 Problemas de contacto en viscoelasticidad

El análisis matemático de los problemas de contacto cuasiestáticos (es decir, con efectos de inercia despreciables), y que incluían materiales viscoelásticos se inició en [21]. El comportamiento se supone lineal y se estudian problemas asociados a diferentes condiciones (con y sin rozamiento, sólido rígido o deformable, etc). En la reciente monografía [39] se retomaron estos problemas para extenderlos al caso no lineal y se proporcionaron, además, su análisis numérico y se mostraron resultados numéricos que confirmaron la precisión de los algoritmos que los resolvían. En este tipo de problemas, y suponiendo que no hay rozamiento, hemos estudiado el contacto de un obstáculo deformable con una condición de respuesta normal amortiguada ([30]), con un obstáculo deformable analizando el efecto de la adherencia en la superficie de contacto ([19]), entre dos cuerpos viscoelásticos con daño y adherencia ([28]) y con un obstáculo deformable incluyendo los efectos térmicos ([7]). En [4] se presenta un caso abstracto que incluye el problema de contacto con un sólido deformable o un sólido rígido. Recientemente, en [16] hemos introducido un algoritmo que permite, en el caso bidimensional, la resolución de problemas de contacto con rozamiento, aplicándolo al caso de un problema de contacto viscoelástico con la ley de Tresca.

Sin embargo, dado que, en cierta medida, estos resultados son extensiones o generalizaciones de otros estudios previos, una mayor novedad consiste en suponer que los problemas son dinámicos. Es decir, que los efectos de inercia no son despreciables. Como ejemplo, podemos considerar el problema de contacto dinámico con respuesta normal en viscoelasticidad y con daño. Denotando por $\mathbb{S}^d$ el espacio de tensores simétricos de orden 2 en $\mathbb{R}^d$, este se escribe de la forma siguiente (véase [8]):

Encontrar el campo de desplazamientos $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, el campo de tensiones* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, y la función daño* $\zeta : \Omega \times [0,T] \to [\zeta_*, 1]$ *tales*

que:

$$\rho\,\ddot{\boldsymbol{u}} = \mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_B \quad \text{en} \quad \Omega \times (0,T), \tag{1}$$
$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\zeta) \quad \text{en} \quad \Omega \times (0,T), \tag{2}$$
$$\dot{\zeta} - \kappa\,\triangle\zeta + \partial I_{[\zeta_*,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}),\zeta) \quad \text{en} \quad \Omega \times (0,T), \tag{3}$$
$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{en} \quad \Gamma \times (0,T), \tag{4}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{en} \quad \Gamma_D \times (0,T), \tag{5}$$
$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{en} \quad \Gamma_N \times (0,T), \tag{6}$$
$$-\sigma_\nu = p(u_\nu - g) \quad \text{en} \quad \Gamma_C \times (0,T), \tag{7}$$
$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{en} \quad \Gamma_C \times (0,T), \tag{8}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \zeta(0) = \zeta_0 \quad \text{en} \quad \Omega. \tag{9}$$

Como podemos ver, la ecuación de equilibrio (1) contiene el término de inercia (ρ representa la densidad del material) ya que el problema se supone dinámico. Además, se postula la existencia de un valor inferior $\zeta_* > 0$ para la función daño ya que, por debajo de este, la densidad de microfracturas es demasiado grande y no tiene sentido considerar el material como elástico.

Sea $\boldsymbol{v}$ el campo de velocidades dado por $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$ y definimos los siguientes espacios variacionales:

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \ ; \ \boldsymbol{v} = \boldsymbol{0} \quad \text{en} \quad \Gamma_D\},$$
$$Q = \{\, \boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \ ; \ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\,\},$$
$$\mathcal{K} = \{\xi \in H^1(\Omega) \ ; \ \ \zeta_* \le \xi \le 1 \quad \text{c.p.d. en} \quad \Omega\}.$$

Aplicando la fórmula de Green el problema mecánico anterior se escribe de la forma siguiente:

Encontrar un campo de velocidades $\boldsymbol{v} : [0,T] \to V$ *y una función daño* $\zeta : [0,T] \to \mathcal{K}$ *tales que* $\boldsymbol{v}(0) = \boldsymbol{v}_0$, $\zeta(0) = \zeta_0$ *y para casi todo* $t \in (0,T)$*:*

$$\begin{aligned}
&\langle \dot{\boldsymbol{v}}(t), \boldsymbol{w}\rangle_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\mathcal{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\
&\qquad + j(\boldsymbol{u}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} \quad \forall \boldsymbol{w} \in V, \\
&(\dot{\zeta}(t), \xi - \zeta(t))_{L^2(\Omega)} + a(\zeta(t), \xi - \zeta(t)) \\
&\qquad \ge (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_{L^2(\Omega)} \quad \forall \xi \in \mathcal{K},
\end{aligned}$$

donde los funcionales $\boldsymbol{f} : [0,T] \to V'$, $j : V \times V \to \mathbb{R}$ y $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ se definen de la forma siguiente:

$$\langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} = (\boldsymbol{f}_B(t), \boldsymbol{w})_{[L^2(\Omega)]^d} + (\boldsymbol{f}_N(t), \boldsymbol{w})_{[L^2(\Gamma_N)]^d}\, \forall \boldsymbol{w} \in V, \ t \in [0,T],$$
$$j(\boldsymbol{u}, \boldsymbol{w}) = \int_{\Gamma_C} p(u_\nu - g)\, w_\nu \, da \quad \forall \boldsymbol{u}, \boldsymbol{w} \in V,$$
$$a(\xi, \psi) = \kappa \int_\Omega \nabla\xi \cdot \nabla\psi \, d\boldsymbol{x} \quad \forall\, \xi,\, \psi \in H^1(\Omega).$$

El campo de desplazamientos $\boldsymbol{u}(t)$ se define como:

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s) ds + \boldsymbol{u}_0, \quad t \in [0,T].$$

Bajo ciertas hipótesis sobre los datos, en [8] se demostró que el problema variacional anterior tiene una única solución con la regularidad:

$$\begin{aligned}
&\boldsymbol{u} \in W^{1,2}(0,T;V) \cap C^1([0,T];[L^2(\Omega)]^d), \quad \ddot{\boldsymbol{u}} \in L^2(0,T;V'),\\
&\boldsymbol{\sigma} \in L^2(0,T;Q), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in L^2(0,T;V'),\\
&\zeta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).
\end{aligned}$$

La demostración de este resultado se basa en la aplicación del teorema del punto fijo de Banach y resultados conocidos de inecuaciones variacionales parabólicas y ecuaciones variacionales no lineales.

Veamos como se construye un esquema numérico completamente discretizado de este problema variacional. Primero, debemos aproximar la variable espacial. Para ello, definimos dos espacios de elementos finitos $V^h \subset V$ y $B^h \subset H^1(\Omega)$ que aproximen los espacios V y $H^1(\Omega)$, respectivamente (h es el parámetro de discretización espacial). Denotamos por $\mathcal{K}^h = B^h \cap \mathcal{K}$ un subconjunto convexo de B^h. En segundo lugar, discretizamos las derivadas temporales (de primer orden). Sea $0 = t_0 < t_1 < \ldots < t_N = T$ una partición del intervalo temporal $[0,T]$ y denotamos por $k = T/N$ el paso de tiempo (parámetro de discretización temporal). Finalmente, como notación, para una función continua $f(t)$ sea $f_n = f(t_n)$ y, para una sucesión $\{w_n\}_{n=0}^N$, denotamos por $\delta w_n = (w_n - w_{n-1})/k$ las diferencias divididas.

Sean $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$ y ζ_0^h aproximaciones adecuadas de las condiciones iniciales $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ y ζ_0, respectivamente. Un esquema numérico completamente discretizado del problema variacional anterior es el siguiente (véase [8]):

Encontrar un campo de velocidades discreto $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ *y una función de daño discreta* $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$ *tales que* $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$, $\zeta_0^{hk} = \zeta_0^h$ *y para* $n = 1, 2, \ldots, N$*:*

$$\begin{aligned}
&(\rho\delta\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_{[L^2(\Omega)]^d} + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + (\mathcal{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q\\
&\qquad + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}^h) = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \quad \forall \boldsymbol{w}^h \in V^h, \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&(\delta\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_{L^2(\Omega)} + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk})\\
&\qquad \geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_{L^2(\Omega)} \quad \forall \xi^h \in \mathcal{K}^h, \qquad (11)
\end{aligned}$$

donde el campo de desplazamientos discreto $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ *se obtiene como:*

$$\boldsymbol{u}_n^{hk} = \sum_{j=1}^{n} k\boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h, \quad n = 1, \ldots N. \qquad (12)$$

Bajo las condiciones de regularidad adicionales:

$$\boldsymbol{u} \in C^1([0,T];V), \quad \ddot{\boldsymbol{u}} \in C([0,T];V'), \quad \zeta \in H^2(0,T;L^2(\Omega)),$$

en [8] se demostró la siguiente estimación del error para todos $\{\boldsymbol{w}_j^h\}_{j=1}^N \subset V^h$

y $\{\xi_j^h\}_{j=1}^N \subset \mathcal{K}^h$:

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|^2_{[L^2(\Omega)]^d}+\|\zeta_n-\zeta_n^{hk}\|^2_{L^2(\Omega)}\Big\}\\
&\qquad +k\sum_{j=1}^N(\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+\|\nabla(\zeta_j-\zeta_j^{hk})\|^2_{[L^2(\Omega)]^d})\\
&\le c\Big\{\sum_{j=1}^N k\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|^2_{V'}+I_j^2+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2\Big)\\
&\quad +\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|^2_{[L^2(\Omega)]^d}+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|^2_{[L^2(\Omega)]^d}\\
&\quad +\|\zeta_0-\zeta_0^h\|^2_{L^2(\Omega)}+\|\zeta_1-\xi_1^h\|^2_{L^2(\Omega)}+\max_{0\le n\le N}\|\zeta_n-\xi_n^h\|^2_{L^2(\Omega)}+k^2\\
&\quad +\frac{1}{k}\sum_{j=1}^{N-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|^2_{L^2(\Omega)}+k\sum_{j=1}^N\|\delta\zeta_j-\dot{\zeta}_j\|^2_{L^2(\Omega)}\\
&\quad +k\sum_{j=1}^N\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_{L^2(\Omega)}\cdot\|\zeta_j-\xi_j^h\|_{L^2(\Omega)}\\
&\quad +\max_{0\le n\le N}\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|^2_{[L^2(\Omega)]^d}+k\sum_{j=1}^N\|\zeta_j-\xi_j^h\|^2_{H^1(\Omega)}\\
&\quad +\frac{1}{k}\sum_{j=1}^{N-1}\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|^2_{[L^2(\Omega)]^d}\Big\},
\end{aligned}$$

donde I_j es el error de integración numérica dado por:

$$I_j=\Big\|\int_0^{t_{j-1}}\boldsymbol{v}(s)ds-\sum_{l=1}^{j-1}k\boldsymbol{v}_l\Big\|_V.$$

La estimación anterior es la base para obtener el orden de convergencia del algoritmo. Como un ejemplo de aplicación directa, supongamos que la solución tiene la siguiente regularidad:

$$\begin{aligned}
&\boldsymbol{u}\in H^2(0,T;V)\cap C^1([0,T];[H^2(\Omega)]^d),\quad \ddot{\boldsymbol{u}}\in L^2(0,T;V'),\\
&\zeta\in C([0,T];H^2(\Omega))\cap H^2(0,T;L^2(\Omega))\cap H^1(0,T;H^1(\Omega)).
\end{aligned}$$

Sean los espacios de elementos finitos V^h y B^h dados por:

$$\begin{aligned}
V^h&=\{\boldsymbol{v}^h\in[C(\overline{\Omega})]^d\ ;\ \boldsymbol{v}^h|_{\mathcal{T}}\in[P_1(\mathcal{T})]^d,\quad \mathcal{T}\in\mathcal{T}^h,\quad \boldsymbol{v}^h=\boldsymbol{0}\text{ en }\Gamma_D\},\\
B^h&=\{\xi^h\in C(\overline{\Omega})\ ;\ \xi^h|_{\mathcal{T}}\in P_1(\mathcal{T}),\quad \mathcal{T}\in\mathcal{T}^h\},
\end{aligned}$$

donde $\mathcal{T}^h$ es una familia regular de triangulaciones tipo elementos finitos de $\overline{\Omega}$ compatible con la partición de la frontera $\Gamma=\partial\Omega$ en Γ_D, Γ_N y Γ_C, y definamos las condiciones iniciales discretas como:

$$\boldsymbol{u}_0^h=\Pi^h\boldsymbol{u}_0,\quad \boldsymbol{v}_0^h=\Pi^h\boldsymbol{v}_0,\quad \zeta_0^h=\Pi^h\zeta_0,$$

donde Π^h es el operador de interpolación de elementos finitos usual. Entonces el esquema numérico es linealmente convergente; es decir, existe $c > 0$, independiente de h y k, tal que:

$$\max_{0\le n\le N}\left\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{[L^2(\Omega)]^d}+\|\zeta_n-\zeta_n^{hk}\|_{L^2(\Omega)}\right\}\le c(h+k).$$

En [8] se estudió con detalle el problema discreto anterior. Su resolución numérica se realiza en tres pasos para cada $n = 1, \ldots, N$. En primer lugar, se calcula la función daño reescribiendo (11) en la forma:

$$\begin{aligned}&(\zeta_n^{hk},\xi^h-\zeta_n^{hk})_{L^2(\Omega)}+k\,a(\zeta_n^{hk},\xi^h-\zeta_n^{hk})\\&\ge k(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\xi^h-\zeta_n^{hk})_{L^2(\Omega)}+(\zeta_{n-1}^{hk},\xi^h-\zeta_n^{hk})_{L^2(\Omega)}\quad\forall\xi^h\in\mathcal{K}^h.\end{aligned}$$

Esta es una inecuación variacional de primera clase discreta que se puede resolver aplicando un algoritmo de tipo penalización-dualidad similar al descrito en [55]. En segundo lugar, se obtiene el campo de velocidades discreto a partir de (10), teniendo en cuenta que esta ecuación variacional se escribe como

$$\begin{aligned}&(\rho\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_{[L^2(\Omega)]^d}+k(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q+k(\mathcal{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q\\&+kj(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{w}^h)=k(\boldsymbol{f}_n,\boldsymbol{w}^h)_V+(\rho\boldsymbol{v}_{n-1}^{hk},\boldsymbol{w}^h)_{[L^2(\Omega)]^d}\quad\forall\boldsymbol{w}^h\in V^h.\end{aligned}$$

Por tanto, esta da lugar a un sistema lineal que se puede resolver con el método de Cholesky. Finalmente, el campo de desplazamientos se actualiza por la expresión (12). Este algoritmo se implementó en un ordenador utilizando FORTRAN 77 y se obtuvieron distintos resultados numéricos. En uno de los ejemplos descritos en [8] el objetivo era observar la influencia del daño en el comportamiento del material. En la Figura 2 se muestra el campo de deformaciones en el tiempo final, para unos determinados datos, considerando que el material nunca está dañado (izquierda) y que está dañado (derecha). Como podemos ver, si se produce daño del material, este se vuelve más blando y las deformaciones son mayores.

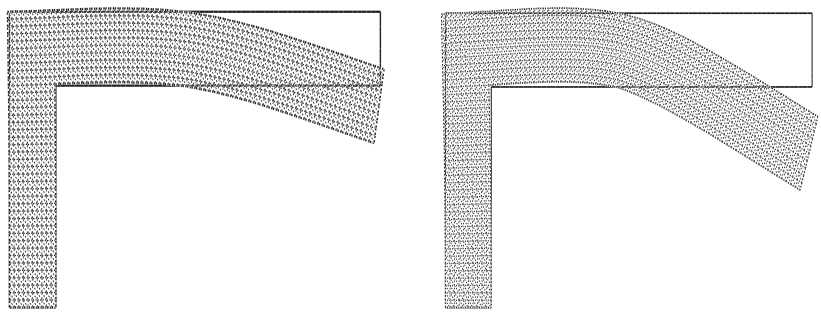

Figura 2: Deformaciones en el instante final sin daño (izquierda) y con daño (derecha).

En la ley constitutiva (2), el daño sólo afecta a la parte elástica, $\mathcal{C}$. Sin embargo, parece natural que este también pueda afectar, en mayor o menor

medida, a la parte viscosa. Por tanto, esta ley se reescribiría en la forma:

$$\boldsymbol{\sigma} = \zeta \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{en} \quad \Omega \times (0,T),$$

y, debido a que cuando $\zeta = 0$ se tiene que $\boldsymbol{\sigma} = \mathbf{0}$, la condición de respuesta normal (7) se reemplazaría por:

$$-\sigma_\nu = \zeta p(u_\nu - g) \quad \text{en} \quad \Gamma_C \times (0,T).$$

Estas dos condiciones vuelven el problema mucho más difícil. En [45] se demostró, usando la teoría de los operadores pseudomonótonos, que este problema, en el caso más general de incluir también la adherencia del material, tiene solución y, si además, $|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})| \in L^\infty(0,T;L^\infty(\Omega))$, esta es única. El análisis numérico de este problema se realizó en [14], incluyendo resultados numéricos que demuestran la convergencia y eficacia del algoritmo, así como la influencia de diferentes parámetros en las leyes constitutivas.

Problemas dinámicos que trataban la adherencia de la superficie de contacto y que incluían materiales viscoelásticos se estudiaron en [29] (se propuso un esquema de resolución numérica y se mostraron algunos resultados numéricos) y [18], donde se utilizó una condición de respuesta normal regularizada.

Suponiendo que los modelos se pueden reducir a problemas unidimensionales, se estudiaron los siguientes casos: en [2] se trató el problema de la vibración de una viga en contacto con un cuerpo rígido, que siempre permanece unido a esta, e incluyendo los efectos térmicos, en [3] se consideró el problema de contacto de una barra con un obstáculo deformable teniendo en cuenta la influencia del daño del material y los efectos térmicos, y en [15] se estudió el problema de contacto de una viga con un obstáculo deformable incluyendo el rozamiento (este se planteó utilizando bien la ley de Tresca o la ley de Coulomb). Finalmente, en [6] se consideró el problema de contacto entre un sólido electro-viscoelástico y un obstáculo deformable y en [24] se estudió el comportamiento de un "thermistor" viscoelástico (mecanismo que previene a algunos dispositivos de sufrir cortocircuitos).

5 Problemas de contacto en elasto-viscoplasticidad

En [39] se describen algunos de los problemas de contacto en elasto-viscoplasticidad que hemos estudiado. Los primeros resultados sobre problemas de contacto entre un cuerpo elasto-viscoplástico y un obstáculo deformable ([32]) o entre dos cuerpos elasto-viscoplásticos ([27]) se extendieron para incluir variables genéricas de estado internas ([26]), el endurecimiento del material ([22]) o el daño ([9, 17, 31]).

Nuevamente, aquí la contribución fundamental se centra en los problemas en los que interviene el daño del material dado que introduce las mayores novedades. Sin embargo, en los trabajos antes referenciados, debemos observar que este se supone que sólo afecta a la parte plástica, la función $\mathcal{G}$. Aunque se puede justificar mecánicamente (véase el libro [47]), parece natural que siguiendo las ideas de Frémond este también afecte, y de hecho es la parte principal, al

término de elasticidad. Como en la sección anterior, este problema introduce dificultades matemáticas adicionales de gran interés. En el reciente trabajo [11] hemos considerado el problema de contacto de un cuerpo elasto-viscoplástico con un obstáculo deformable incluyendo el daño del material. Este problema se escribe de la forma siguiente:

Encontrar el campo de desplazamientos $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, el campo de tensiones* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, y la función daño* $\zeta : \Omega \times [0,T] \to \mathbb{R}$ *tales que:*

$$-\mathrm{Div}\,\boldsymbol{\sigma} = \boldsymbol{f}_B \quad en \quad \Omega \times [0,T], \tag{13}$$

$$\dot{\boldsymbol{\sigma}} = \mathcal{C}(\dot{\eta_*(\zeta)\boldsymbol{\varepsilon}(\boldsymbol{u})}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad en \quad \Omega \times [0,T], \tag{14}$$

$$\dot{\zeta} - \kappa \triangle \zeta = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad en \quad \Omega \times [0,T], \tag{15}$$

$$\zeta = 1 \quad en \quad \Gamma \times (0,T), \tag{16}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad en \quad \Gamma_D \times (0,T), \tag{17}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{en} \quad \Gamma_N \times (0,T), \tag{18}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \quad -\sigma_\nu = p(u_\nu - g) \quad en \quad \Gamma_C \times (0,T), \tag{19}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \zeta(0) = \zeta_0 \quad en \quad \Omega, \tag{20}$$

donde $\eta_* : \mathbb{R} \to \mathbb{R}$ es un operador de truncamiento dado por:

$$\eta_*(r) = \begin{cases} r & \text{si} \quad \zeta_* < r < 1, \\ 1 & \text{si} \quad r \geq 1, \\ \zeta_* & \text{si} \quad r \leq \zeta_*. \end{cases}$$

En la ecuación de equilibrio (13) se ha despreciado el término de inercia y, por tanto, el problema es cuasiestático. Además, en la ley constitutiva (14) hemos incluido el daño en la parte elástica (de hecho, este está truncado ya que sino el problema podría degenerar en el caso $\zeta = 0$), y la ley del daño (15) está escrita como una ecuación diferencial en derivadas parciales en lugar de una inclusión subdiferencial. Esto se puede hacer (véanse [13, 44]), suponiendo ciertas condiciones sobre la función fuente de daño ϕ.

Sean $E = H^1(\Omega)$, V y Q los siguientes espacios variacionales:

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \ ; \ \boldsymbol{v} = \boldsymbol{0} \quad \text{en} \quad \Gamma_D\},$$
$$Q = \{\boldsymbol{\tau} \in [L^2(\Omega)]^{d\times d} \ ; \ \tau_{ij} = \tau_{ji}, \quad 1 \leq i,j \leq d\},$$

y denotemos por $\boldsymbol{f}(\cdot)$, $a(\cdot,\cdot)$ y $j(\cdot,\cdot)$ los funcionales definidos en la sección anterior. Finalmente, sea K el siguiente subconjunto de E:

$$K = \{\xi \in E \ ; \ \xi = 1 \quad \text{en} \quad \Gamma\}.$$

La formulación variacional del problema mecánico anterior se escribe de la siguiente forma:

Encontrar el campo de desplazamientos $\boldsymbol{u} : [0,T] \to V$*, el campo de tensiones* $\boldsymbol{\sigma} : [0,T] \to Q$*, y la función daño* $\zeta : [0,T] \to K$ *tales que* $\boldsymbol{u}(0) = \boldsymbol{u}_0$*,* $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$*,*

$\zeta(0) = \zeta_0$ *y para casi todo* $t \in [0,T]$*:*

$$\begin{aligned}
&\dot{\boldsymbol{\sigma}}(t) = \mathcal{C}\overbrace{(\eta_*(\zeta(t))\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))}^{\cdot} + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \eta_*(\zeta(t))),\\
&(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}(t), \boldsymbol{w}) = (\boldsymbol{f}(t), \boldsymbol{w})_V \quad \forall \boldsymbol{w} \in V,\\
&(\dot{\zeta}(t), \xi)_{L^2(\Omega)} + a(\zeta(t), \xi) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \eta_*(\zeta(t))), \xi)_{L^2(\Omega)} \quad \forall \xi \in E.
\end{aligned}$$

Bajo ciertas hipótesis sobre los datos, en [12] se demostró que el problema variacional anterior admite una única solución con la siguiente regularidad:

$$\begin{aligned}
&\zeta \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)), \quad \zeta(t,\boldsymbol{x}) \in [\zeta_*, 1] \text{ p.c.t. } \boldsymbol{x} \in \Omega,\\
&\boldsymbol{u} \in L^\infty(0,T; V).
\end{aligned}$$

En particular, esta regularidad implica que en el problema variacional se puede eliminar el operador de truncamiento η_*.

La aproximación numérica de este problema se realiza de la siguiente forma. En primer lugar, definimos tres espacios de elementos finitos $V^h \subset V$, $Q^h \subset Q$ y $E^h \subset E$, aproximando los espacios V, Q y E, respectivamente, donde $h > 0$ denota el parámetro de discretización espacial. Sea $K^h = K \cap E^h$.

Denotemos por $\mathcal{P}_{Q^h} : Q \to Q^h$ el operador de proyección ortogonal sobre Q^h dado por

$$(\mathcal{P}_{Q^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q = (\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q \quad \forall \boldsymbol{\tau} \in Q,\, \boldsymbol{\tau}^h \in Q^h.$$

Este operador verifica que

$$\|\mathcal{P}_{Q^h}\boldsymbol{\tau}\|_Q \leq \|\boldsymbol{\tau}\|_Q \quad \forall \boldsymbol{\tau} \in Q.$$

Para discretizar las derivadas temporales, utilizamos una partición uniforme del intervalo temporal que denotamos por $0 = t_0 < t_1 < \ldots < t_N = T$, y sea $k = T/N$ el paso de tiempo. Para una función continua $f(t)$ denotamos nuevamente por $f_n = f(t_n)$, y para una sucesión de valores $\{w_n\}_{n=0}^N$ sean $\delta w_n = (w_n - w_{n-1})/k$ sus diferencias divididas.

Sean $\boldsymbol{u}_0^h, \boldsymbol{\sigma}_0^h$ y ζ_0^h aproximaciones apropiadas de las condiciones iniciales. Utilizando el esquema de Euler se obtiene el siguiente problema completamente discretizado:

Encontrar el campo de desplazamientos discreto $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$*, el campo de tensiones discreto* $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N \subset Q^h$ *y la función de daño discreta* $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset K^h$*, tales que* $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h$*,* $\boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h$*,* $\zeta_0^{hk} = \zeta_0^h$*, y para* $n = 1, 2, \ldots, N$*,*

$$\begin{aligned}
&(\delta\zeta_n^{hk}, \xi^h)_{L^2(\Omega)} + a(\zeta_n^{hk}, \xi^h) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \eta_*(\zeta_{n-1}^{hk})), \xi^h)_{L^2(\Omega)} \quad \forall \xi^h \in E^h,\\
&\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{Q^h}(\eta_*(\zeta_n^{hk})\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})) + \boldsymbol{\sigma}_0^h - \mathcal{P}_{Q^h}(\eta_*(\zeta_0^h)\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h))\\
&\qquad\qquad + \sum_{j=1}^{n} k\mathcal{P}_{Q^h}\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \eta_*(\zeta_{j-1}^{hk})),\\
&(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}_n^{hk}, \boldsymbol{w}^h) = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \quad \forall \boldsymbol{w}^h \in V^h.
\end{aligned}$$

Utilizando resultados conocidos de ecuaciones variacionales no lineales se puede demostrar fácilmente que el problema anterior admite una única solución.

Supongamos la siguiente regularidad de la solución continua:

$$\begin{aligned}&\boldsymbol{u} \in W^{1,\infty}(0,T;V) \cap L^\infty(0,T;[W^{1,\infty}(\Omega)]^d),\\ &\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q),\\ &\zeta \in C([0,T];E) \cap C^1([0,T];L^2(\Omega)).\end{aligned}$$

La siguiente estimación del error se obtuvo en [11] para todos $\{\boldsymbol{w}_j^h\}_{j=1}^N \subset V^h$ y $\{\xi_j^h\}_{j=1}^N \subset E^h$:

$$\begin{aligned}&\max_{0\le n\le N}\Big\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q^2+\|\zeta_n-\zeta_n^{hk}\|_{L^2(\Omega)}^2\Big\}+k\sum_{j=1}^N\|\zeta_j-\zeta_j^{hk}\|_E^2\\ &\quad\le c\Big(e_0+\max_{0\le n\le N}\|\boldsymbol{u}_n-\boldsymbol{w}_n^h\|_V^2+\max_{0\le n\le N}\|I_n\|_Q^2+\max_{0\le n\le N}\|\zeta_n-\xi_n^h\|_{L^2(\Omega)}^2\\ &\quad+k^2+\sum_{j=1}^N k\|(I-\mathcal{P}_{Q^h})\mathcal{G}(\boldsymbol{\sigma}_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\eta_*(\zeta_{j-1}))\|_Q^2+\sum_{j=1}^N k\|\dot\zeta_j-\delta\zeta_j\|_{L^2(\Omega)}^2\\ &\quad+\sum_{j=1}^N k\|\zeta_j-\xi_j^h\|_E^2+\frac{1}{k}\sum_{j=1}^{N-1}\|\zeta_j-\xi_j^h-(\zeta_{j+1}-\xi_{j+1}^h)\|_{L^2(\Omega)}^2\Big),\end{aligned}$$

donde I_n es el siguiente error de integración:

$$I_n=\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\eta_*(\zeta(s)))\,ds-\sum_{j=1}^n k\mathcal{G}(\boldsymbol{\sigma}_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\eta_*(\zeta_{j-1})),$$

y e_0 es el error cometido en la aproximación de las condiciones iniciales, es decir:

$$e_0=\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q^2+\|\zeta_0-\zeta_0^h\|_{L^2(\Omega)}^2.$$

La estimación precedente es el punto de partida para obtener el orden de convergencia del algoritmo. Definamos los siguientes espacios de elementos finitos:

$$\begin{aligned}V^h&=\{\boldsymbol{w}^h\in V\cap[C(\overline{\Omega})]^d\ ;\ \boldsymbol{w}^h_{|_{Tr}}\in[P_1(Tr)]^d,\quad Tr\in\mathcal{T}^h\},\\ Q^h&=\{\boldsymbol{\tau}^h\in Q\ ;\ \boldsymbol{\tau}^h_{|_{Tr}}\in[P_0(Tr)]^{d\times d},\quad Tr\in\mathcal{T}^h\},\\ E^h&=\{\xi^h\in C(\overline{\Omega})\ ;\ \xi^h_{|_{Tr}}\in P_1(Tr),\quad Tr\in\mathcal{T}^h\},\\ K^h&=\{\xi^h\in E^h\ ;\ \xi^h=1\quad\text{en}\quad\Gamma\},\end{aligned}$$

donde Ω se supone que es un dominio poliédrico, $\mathcal{T}^h$ denota una triangulación tipo elementos finitos regular de $\overline{\Omega}$, y $P_q(Tr)$, para $q=0,1$, representa el espacio de polinomios de grado global menor o igual que q en Tr.

Supongamos que las condiciones iniciales discretas están definidas de la forma siguiente:

$$\boldsymbol{u}_0^h=\Pi^h\boldsymbol{u}_0,\quad \boldsymbol{\sigma}_0^h=\mathcal{P}_{Q^h}\boldsymbol{\sigma}_0,\quad \zeta_0^h=\pi^h\zeta_0,$$

donde $\pi^h : C(\overline{\Omega}) \to E^h$ es el operador de interpolación de elementos finitos usual y $\Pi^h = (\pi_i^h)_{i=1}^d : [C(\overline{\Omega})]^d \to V^h$.

Se puede entonces demostrar, de la estimación anterior y bajo condiciones de regularidad adicionales, que el algoritmo es linealmente convergente respecto de los parámetros de discretización h y k, es decir, existe $c > 0$, independiente de h y k, tal que:

$$\max_{0\le n\le N}\{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q + \|\zeta_n - \zeta_n^{hk}\|_{L^2(\Omega)}\} \le c(h+k).$$

Finalmente, veamos un ejemplo numérico que demuestra la eficacia del algoritmo de resolución. En la Figura 3 se representan la norma de von Mises del campo de tensiones y la función daño en el instante final y sobre la configuración deformada. Las zonas de máxima tensión se localizan en donde el cuerpo se flexiona y en el extremo izquierdo, que supusimos fijado, y, como podíamos esperar, las zonas de mayor daño (esto es, donde la función daño es menor) coinciden con las zonas de máxima tensión.

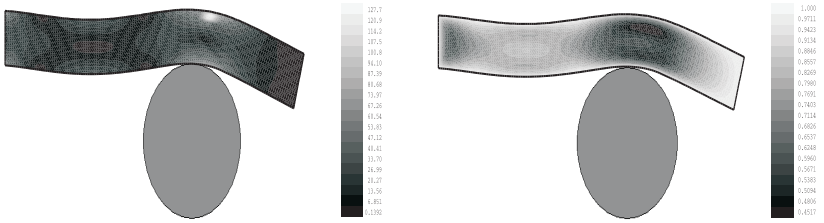

Figura 3: Norma von Mises de las tensiones y función daño en el instante final y sobre la configuración deformada.

En la Figura 4 se muestra la evolución, a lo largo del tiempo, del desplazamiento del punto $\boldsymbol{x} = (9, 0)$ (nodo inferior derecho). Como podemos ver, aunque se supone que la fuerza es cíclica y siempre de la misma intensidad, este va aumentando como consecuencia de que el material resulta dañado.

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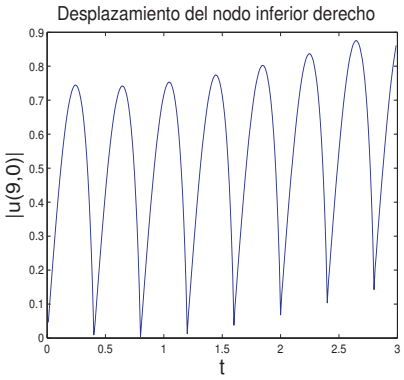

Figura 4: Evolución del desplazamiento del nodo inferior derecho.

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Sus principales áreas de interés son el Análisis Numérico de las inecuaciones variacionales y sus aplicaciones en particular a problemas con origen en Mecánica del contacto, Elasto-Viscoplasticidad, Viscoelasticidad, Piezoelectricidad, etc.

Es autor de cerca de 50 artículos en revistas y ha participado en más de 40 Congresos de diusión internacional. Ha realizado estancias de investigación en las Universidades de Iowa y Oakland (EEUU) y Savoie y Perpignan (Francia), entre otras. Ha dirigido y codirigido varios trabajos de investigación a nivel predoctoral y doctoral.

Título:	Esquemas entrópicos para la resolución de sistemas de tipo Saint Venant.
Doctorando:	Tomás Morales de Luna.
Director/es:	François Bouchut - Carlos Parés.
Defensa:	21 de septiembre de 2007, Universidad de Málaga.
Calificación:	Sobresaliente Cum Laude por unanimidad.

Resumen:

El objetivo de la tesis es el de contribuir al estudio numérico de leyes de conservación hiperbólicas y, en particular, al concepto de desigualdades de entropía. Se introducen en primer lugar los conceptos básicos necesarios para el desarrollo de los distintos temas tratados.

El segundo capítulo se centra en el sistema de Saint Venant para aguas someras con fondo no plano. Se introducen los esquemas equilibrados, esto es, que preservan los equilibrios (well-balanced schemes). Se define un esquema numérico que preserva todos los equilibrios de tipo subsónico, garantiza la no-negatividad de la altura de agua y que verifica una desigualdad de entropía semi-discreta.

En el tercer capítulo, se estudia el sistema bicapa para aguas someras. Este modelo puede verse como dos sistemas de Saint Venant acoplados. La resolución de cada capa de forma independiente tiene la ventaja de evitar calcular los autovalores del sistema en cada iteración. Sin embargo, esto lleva en general a esquemas inestables. Se pretende controlar las posibles inestabilidades mediante una desigualdad de entropía. Se introducen dos esquemas distintos que permiten resolver cada capa de forma independiente: un esquema que verifica una desigualdad de entropía completamente discreta y otro que verifica una desigualdad de entropía semi-discreta.

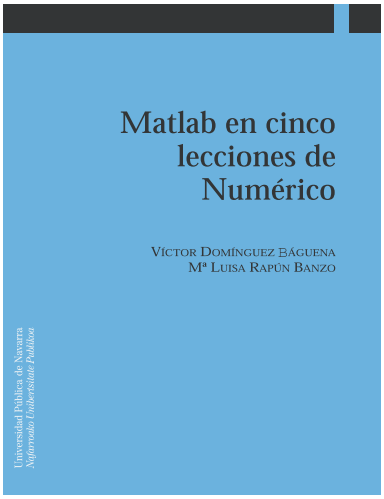
Algunas de las simulaciones numéricas utilizadas en los primeros capítulos se basan sobre el resolvedor simple de Suliciu. Este resolvedor se estudia más detenidamente en el capítulo cuarto. La definición de este resolvedor depende en particular de la elección de ciertos parámetros. Estudios recientes muestran una elección particular de estos parámetros que permite asegurar que el esquema verifica una desigualdad de entropía discreta. Pero esta elección hace que los choques no sean resueltos de forma exacta. Se propone adaptar el resolvedor de Suliciu con el fin de capturar los choques del sistema de Saint Venant de forma exacta y verificar una desigualdad de entropía semi-discreta.

Finalmente, se presenta el modelo de Savage-Hutter que es un sistema de tipo Saint Venant con un término fuente modificado y admite una desigualdad de entropía. La deducción del modelo parte de las ecuaciones de Euler

incompresibles. Se propone una generalización de este modelo en el quinto capítulo, donde se considerarán dos regiones: una capa superior compresible y una capa inferior incompresible. Se obtiene así un modelo de tipo Saint Venant que depende de unos ciertos parámetros y que admite también una desigualdad de entropía.

Matlab en cinco lecciones de Numérico.
Víctor Domínguez Báguena y María Luisa Rapún Banzo
Universidad Pública de Navarra, Pamplona.
ISBN: 978–84–9769–195–6 (354+xi páginas) – 2007

Por Salim Meddahi



Matlab es un lenguaje de alto nivel para Cálculo Científico. En poco tiempo se ha convertido en un estándar en muchas universidades y centros de investigación. Las razones de su popularidad son varias. En primer lugar, es un lenguaje sencillo, flexible e interactivo. Esto permite al usuario experimentar, testear y depurar programas con relativa rapidez. En segundo lugar, posee un entorno gráfico bastante poderoso que facilita el postproceso de cualquier programa de Cálculo Científico. Por último, existe un gran número de rutinas y funciones de Matlab desarrolladas para distintas aplicaciones en forma de 'toolboxes'.

En el libro *Matlab en cinco lecciones de Numérico* se presenta una introducción a los fundamentos del lenguaje de programación Matlab. Los autores muestran las posibilidades de este lenguaje mediante la resolución de algunos problemas de Cálculo Numérico. La presentación del material es atractiva, clara y de fácil lectura; después de una breve introducción se proponen cinco bloques, como el propio título indica. Cada bloque engloba dos capítulos del libro: en el primer capítulo se presentan comandos y estructuras de programación de Matlab que se aplican en el segundo capítulo para implementar algunos algoritmos del Cálculo Numérico.

En el primer bloque se introducen algunas nociones básicas de Matlab que se utilizan luego para programar varios métodos directos de resolución de sistemas de ecuaciones lineales. Desde el principio se hace un énfasis especial

en una característica sobresaliente de Matlab: la vectorización. En el segundo bloque se tratan aspectos más avanzados del cálculo matricial como, por ejemplo, la manipulación de matrices 'sparse'. Los métodos iterativos para sistemas lineales constituyen un banco de pruebas idóneo para trabajar con los comandos presentados. Las funciones como argumentos de otras funciones y la recursividad son el objetivo del tercer bloque. Aquí los autores presentan como aplicaciones la integración numérica y la Transformada Rápida de Fourier. Algo tremendamente útil en cualquier programa de Cálculo Científico es el cálculo simbólico. Aunque en este aspecto existen alternativas superiores a Matlab, el bloque cuarto de este texto proporciona información que en muchos casos puede resultar suficiente. Finalmente, en el quinto bloque se tratan los comandos de procesamiento gráfico. El texto se cierra con un índice de figuras y otro de comandos de Matlab.

Un problema común a todo aquel que se inicia en Matlab es el elevado número de comandos que el usuario puede utilizar. Este texto hace una selección de herramientas muy útiles para programar algoritmos de cálculo numérico con Matlab. Desde las primeras páginas se aprecia que los autores han dado prioridad a la interacción con el lector. El enfoque del libro es original, se dirige a un público con conocimientos rudimentarios en programación científica y con interés en el Análisis Numérico. La redacción ágil, clara y bien estructurada, además de permitir consultas rápidas, transmite confianza al lector y fomenta el autoaprendizaje.

En la página web

`http://www.unavarra.es/personal/victor_dominguez/libroMatlab.htm`

los autores se han propuesto actualizar el contenido del libro. Entre el material que se proporciona se incluyen varios de los programas en Matlab empleados en el texto, soluciones a problemas planteados en los ejercicios y varias GUI (Graphical User Interface) ilustrativas del funcionamiento de algunos de los métodos numéricos.

Sólo queda animar a los autores a redactar una segunda parte de este libro, donde plasmen su opinión y experiencia en el campo donde son especialistas y han desarrollado su investigación: los métodos numéricos para ecuaciones en derivadas parciales.

IX PREMIO S$\vec{e}$MA A LA "DIVULGACIÓN DE LA MATEMÁTICA APLICADA"

SOCIEDAD ESPAÑOLA DE MATEMÁTICA APLICADA

(PATROCINADO POR IBERDROLA)

PREÁMBULO

La Sociedad Española de Matemática Aplicada (S$\vec{e}$MA), en cumplimiento de su objetivo de contribuir al desarrollo en nuestro país de las Matemáticas y sus aplicaciones, consciente del notable desarrollo que las Matemáticas están experimentando, del incremento de su influencia sobre todos los aspectos de la vida en las sociedades desarrolladas, de la conveniencia de promover el interés de los investigadores por este punto de vista sin por ello ocultar sus peligros o dilemas, de la necesidad no menos acuciante de estimular el interés del público por la cultura científica y, finalmente continuando con una tradición honrosa y habitual tanto en las Artes como en las Ciencias, convoca el "IX Premio S$\vec{e}$MA de Divulgación de la Matemática aplicada", según las bases que se adjuntan.

S$\vec{e}$MA busca ante todo promover la divulgación de las Matemáticas, su relevancia y su eficacia. Dada la enorme variedad de intereses aplicados de las Matemáticas, las Bases del concurso pretenden dar preferencia a los temas que tradicionalmente han estado ligados a S$\vec{e}$MA de una u otra manera. Muy en especial, deben ser mencionados el análisis teórico y numérico, el control y los aspectos computacionales de sistemas que permiten modelizar fenómenos con origen en otras Ciencias.

BASES DE LA CONVOCATORIA

1. La Sociedad Española de Matemática aplicada (S$\vec{e}$MA) convoca el "Premio S$\vec{e}$MA a la Divulgación de la Matemática aplicada", que se concederá anualmente.

2. Son posibles candidatos todos los ciudadanos del mundo que sometan un texto de acuerdo con los puntos 4 y 9 de estas Bases.

3. El Premio está destinado a promover los valores de la belleza, relevancia y eficacia de las Matemáticas como instrumento indispensable del funcionamiento de la sociedad y cultura modernas. El Premio tomará en especial consideración los temas que incidan en la realidad de la Matemática aplicada en la sociedad española.

4. Los candidatos habrán de presentar dentro del plazo fijado en el punto 10 un texto original de una longitud mínima de 20 páginas mecanografiadas a un espacio y con el formato que juzguen conveniente. Los requisitos básicos son que el texto contribuya a la divulgación de algún aspecto relevante de la Matemática Aplicada y que su contenido esté pensado para un público **no exclusivamente formado por profesionales de las Matemáticas**. El trabajo será presentados bajo un seudónimo, incluyendo con el mismo un sobre cerrado en el que figuren el nombre y dirección del autor. El autor no podrá formar parte del Comité Científico que habrá de juzgar los trabajos por lo que en caso de ser propuesto para el mismo, deberá indicar al Presidente su incompatibilidad.

5. Los méritos serán juzgados por un Comité Científico de cinco miembros nombrados por el Comité Ejecutivo de la Sociedad, personalidades de probado prestigio en la Ciencia Matemática y la cultura científica. Este Comité tendrá su propio reglamento de funcionamiento, pudiendo quedar desierto el Premio. En todo caso, el Comité será presidido por el Presidente de la Sociedad u otro miembro del Comité Ejecutivo en quien delegue, no pudiendo ser miembros del Comité Científico más de dos miembros del Comité Ejecutivo.

6. El galardonado con el Premio recibirá de la Sociedad un Diploma acreditativo y una cuantía de 1500€. Además quedará eximido del pago de las cuotas como socio de S$\vec{e}$MA correspondientes a los años 2009 y 2010. En caso de no ser miembro de S$\vec{e}$MA, pasaría automáticamente a serlo.

7. El fallo del concurso es irrevocable. El Comité acompañará la concesión del Premio de una exposición de los méritos hallados en el candidato galardonado.

8. La Sociedad publicará la obra premiada en su Boletín.

9. Si el texto original no estuviera escrito en castellano, el jurado podrá solicitar al autor su traducción si así lo estimase necesario.

10. La fecha limite de presentación de originales es el 16 de mayo de 2008.

11. La documentación, compuesta del texto por quintuplicado, su traducción si ha lugar, así como los datos identificativos, debe ser dirigida a

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    VIII Premio S$\vec{e}$MA a la Divulgación de la Matemática Aplicada
    Departamento de Matemáticas
    Facultad de Informática
    Campus de Elviña s/n
    Universidad de La Coruña
    15071 – A Coruña

12. El Premio será fallado antes del 31 de agosto del año 2008 y será entregado con ocasión de la Asamblea Anual de la Sociedad, en el marco de la "XIII Escuela Jacques-Louis Lions Hispano-Francesa sobre Simulación Numérica en Física e Ingeniería" que se celebrará en Valladolid del 15 al 19 de Septiembre de 2008.

XI PREMIO S$\vec{e}$MA AL JOVEN INVESTIGADOR

SOCIEDAD ESPAÑOLA DE MATEMÁTICA APLICADA

PREÁMBULO

La Sociedad Española de Matemática Aplicada (S$\vec{e}$MA), en cumplimiento de su objetivo de contribuir al desarrollo en nuestro país de las Matemáticas y sus aplicaciones y, más en concreto, de promover y estimular la investigación y procurar medios para efectuarla, consciente del notable desarrollo que las Matemáticas están experimentando y de la necesidad de promover el interés de las jóvenes generaciones por la tarea de la creación científica, convencida del papel positivo que el aprecio de la comunidad juega en la vida científica de los investigadores y siguiendo con una tradición honrosa y habitual tanto en las Artes como en las Ciencias, convoca el "XI Premio S$\vec{e}$MA al Joven Investigador", según las bases que se adjuntan.

BASES GENERALES

1. La Sociedad Española de Matemática aplicada (S$\vec{e}$MA) convoca el "Premio S$\vec{e}$MA al Joven Investigador", que se concederá anualmente.

2. Son posibles candidatos todos los investigadores españoles que, a la fecha del límite de presentación de candidaturas, no rebasen la edad de 33 años. También pueden serlo aquellos investigadores de otras nacionalidades que tengan un puesto de trabajo permanente en una Universidad o Centro de investigación español y cumplan la condición de edad. No pueden concurrir al Premio candidatos galardonados en convocatorias precedentes.

3. El Premio está destinado a promover la excelencia en el trabajo matemático original en todas las ramas de las Matemáticas que tienen una componente aplicada. Su objetivo es premiar la contribución personal del candidato. El limite de edad fijado pretende señalar candidatos que hayan tenido tiempo de desarrollar su creatividad matemática independiente tras la etapa formativa correspondiente a la Tesis Doctoral. El Premio tiene así por objetivo abrirles el camino de su periodo de madurez y reconocer al mismo tiempo sus capacidades demostradas.

4. Los méritos serán juzgados por un Comité Científico de cinco miembros, nombrado por el Consejo Ejecutivo de la Sociedad entre investigadores de probado prestigio. Este Comité tendrá su propio reglamento de funcionamiento. En todo caso, será presidido por el Presidente de la Sociedad u otro miembro del Consejo Ejecutivo en quien delegue, no pudiendo ser miembros del Comité Científico más de dos miembros del Consejo Ejecutivo.

5. Los candidatos habrán de presentar, dentro del plazo que se cite, una Memoria exponiendo la trayectoria vital y los méritos que concurren, un currículum normalizado, así como otros documentos que puedan ser pertinentes para acreditar sus contribuciones originales a las Matemáticas y sus aplicaciones. Las candidaturas pueden ser presentadas también por otros investigadores. El Comité se reserva el derecho de recabar la información complementaria necesaria del candidato o de quien le haya presentado.

6. El galardonado con el Premio recibirá de la Sociedad un Diploma acreditativo y una cuantía que será establecida en cada convocatoria por la Sociedad.

7. La Sociedad requerirá al candidato galardonado un resumen de su trabajo de investigación escrito en estilo divulgativo, con una extensión a convenir entre las 6 y las 20 páginas, para su publicación en el Boletín de la Sociedad. Este resumen puede formar parte de la Memoria mencionada en el punto 5.

8. El fallo del concurso es irrevocable. El Comité acompañará la concesión del Premio de una exposición de los méritos hallados en el candidato galardonado. Por lo demás, las deliberaciones y resoluciones del Comité serán regidas por su reglamento.

BASES PARTICULARES DE LA CONVOCATORIA DE 2008

9. La fecha límite de presentación de candidaturas es el 30 de abril de 2008. Podrán concursar por tanto las personas que hayan nacido después del 30 de abril de 1974.

10. La documentación presentada constará de la Memoria y el currículo citados, así como copia de las cinco contribuciones más importantes del investigador a las Matemáticas y sus aplicaciones, todo ello por quintuplicado.

11. Se recomienda a los candidatos que presenten su propia candidatura y que la Memoria se adecúe, o en su caso contenga el resumen del trabajo de investigación referido en el apartado 7.

12. La documentación debe ser dirigida a:

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    Departamento de Matemáticas
    Facultad de Informática
    Campus de Elviña s/n
    Universidad de La Coruña
    15071 – A Coruña

13. La cuantía del Premio es de 1500€. El Premio es indivisible. Además, el candidato galardonado quedará eximido del pago de las cuotas como socio de S$\vec{e}$MA correspondientes a los años 2009 y 2010. En caso de no ser miembro de S$\vec{e}$MA, pasaría automáticamente a serlo.

14. El Premio será fallado antes del 31 de agosto de 2008 y será entregado con ocasión de la Asamblea Anual de la Sociedad, en el marco de la "XIII Escuela Jacques-Louis Lions Hispano-Francesa sobre Simulación Numérica en Física e Ingeniería" que se celebrará en Valladolid del 15 al 19 de Septiembre de 2008.

    A Coruña, a 1 de diciembre de 2007

Tipo de evento:	Workshop
Nombre:	Workshop on Harmonic Analysis and Partial Differential Equations (WHAPDE)
Lugar:	Mérida, Yucatán, MÉXICO
Fecha:	del 4 al 8 de febrero de 2008
Organiza:	Belén Gamboa, Gerardo Emilio García, Ramón Peniche, Luz del Carmen Torres (Universidad Autónoma de Yucaán); Magali Folch-Gabayet, Salvador Pérez-Esteva, Luz de Teresa (UNAM)
WWW:	`www.matem.unam.mx/whapde08`

Tipo de evento:	Cursos
Nombre:	Topics in PDEs and applications
Lugar:	Granada y Barcelona, SPAIN
Fecha:	del 7 al 11 de abril de 2008 en Granada; del 5 al 9 de mayo de 2008 en Bellaterra, Barcelona
Organiza:	Xavier Cabré (ICREA y Universitat Politècnica de Catalunya) y Juan Soler (Universidad de Granada)
E-mail:	`kinetic@ugr.es, pde_2008@crm.cat`
WWW:	`www.ugr.es/~kinetic/PDE` `www.crm.cat/PDE_2008`

Tipo de evento:	Congreso
Nombre:	Congress of The European Consortium For Mathematics In Industry 2008 (ECMI 2008)
Lugar:	University College London
Fecha:	del 30 de junio al 4 de julio de 2008
Organiza:	The Institute of Mathematics and its Applications (England, UK)
Información:	Miss Lucy Nye (*Conference Officer*), +44 (0)1702 356104
E-mail:	`lucy.nye@ima.org.uk`
WWW:	`www.ecmi2008.org`

Tipo de evento:	Congreso
Nombre:	ECCOMAS 2008
Lugar:	Venecia, ITALIA
Fecha:	del 30 de junio al 4 de julio de 2008
Organiza:	International Association for Computational Mechanics (IACM), European Community on Computational Methods in Applied Sciences (ECCOMAS). Local Organizers: Department of Structural and Transport Engineering Faculty of Engineering, University of Padua, and Department of Structural Engineering, Politecnico di Milano
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Tipo de evento:	Congreso
Nombre:	8th CONFERENCE ON REAL NUMBERS AND COMPUTERS (RNC8)
Lugar:	Santiago de Compostela, SPAIN
Fecha:	del 7 al 9 de julio de 2008
Organiza:	Elisardo Antelo, Santiago de Compostela, Spain (Local arrangements). Javier D. Bruguera, Santiago de Compostela, Spain (Program co-chair). Marc Daumas, Perpignan, France (Program co-chair)
E-mail:	elisardo@usc.es, elbrugue@usc.es
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Tipo de evento:	Congreso
Nombre:	The Tenth International Conference on Integral Methods in Science and Engineering (IMSE 2008)
Lugar:	Universidad de Cantabria, Santander, SPAIN
Fecha:	del 7 al 10 de julio de 2008
Organiza:	M. E. Pérez, Universidad de Cantabria (*chairperson*); M. Lobo, Universidad de Cantabria; D. Gómez, Universidad de Cantabria
E-mail:	imse08@unican.es, meperez@unican.es
WWW:	www.imse08.unican.es

Tipo de evento:	Congreso
Nombre:	5-TH EUROPEAN CONGRESS OF MATHEMATICS
Lugar:	Amsterdam RAI Center, Amsterdam (Holanda)
Fecha:	del 14 al 18 de julio de 2008
Organiza:	Stichting 5ECM, founded by VU Vrije Universiteit, CWI Centrum voor Wiskunde en Informatica, and UvA Universiteit van Amsterdam
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