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La imagen de la portada corresponde a una superficie de sección de Poincaré en el plano $(\rho, P)$ para el Hamiltoniano

$$\mathcal{H} = \frac{1}{2}(X^2 + P^2 + \frac{\Lambda^2}{\rho^2}) - \log\left(\frac{u(\rho,x)+1}{u(\rho,x)-1}\right)$$

donde $u(\rho,x) = \sqrt{4\rho^2 + (2x-1)^2} + \sqrt{4\rho^2 + (2x+1)^2}$, $\mathcal{H} = -0.5$, $\Lambda = 0.5$.

# sumario



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# POLYNOMIAL INTERPOLATION IN SEVERAL VARIABLES

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**Abstract**

This paper is a very subjective survey over computational and theoretical aspects of multivariate polynomial interpolation and tries to present and explain some more recent developments and results in this respect. The main emphasis is put on the Newton representation of the interpolation polynomial, the remainder formula derived from that and on minimal degree interpolation which is a purely multivariate phenomenon and can be connected to Gröbner bases with respect to arbitrary term orders.

## 1 Introduction

Polynomial interpolation is a classical topic of Numerical Analysis which is useful in various fields of applied mathematics. This stems from the fact that polynomials are among the mathematical objects which can be handled most easily in practice: they can be represented by finite information (storing the coefficients with respect to a monomial representation, for example) and they can be easily integrated and differentiated symbolically. Therefore, there is a wide field of applications for polynomial interpolation in several variables which ranges from surface reconstruction to cubature, finite elements and even optimization. For the latter one see, for example [9].

The most classical problem of polynomial interpolation is, of course, the univariate *Lagrange interpolation* problem which may be stated as follows: for

a set of knots, $x_0, \ldots, x_n \in \mathbb{R}$, and given data $f_0, \ldots, f_n$ one wants to find a polynomial $L_n(f;\cdot)$ of degree $n$ such that

$$L_n(f;x_k) = f_k, \qquad k = 0, \ldots, n.$$

It is usually a matter of first grade Numerical Analysis to show that this problem has a unique solution for all data $f_0, \ldots, f_n$ if and only if the knots $x_0, \ldots, x_n$ are pairwise disjoint. However, it is also possible to deal with repetitions of knots by interpolating *successive* derivatives at these points. This is known as the *Hermite interpolation problem*. To be more specific: suppose that $x_k = \cdots = x_{k+m}$ and all the other knots $x_0, \ldots, x_{k-1}$ and $x_{k+m+1}, \ldots, x_n$ are different from $x_k$, then the associated interpolation problem is

$$L_n^{(j)}(f;x_k) = f_{k+j}, \qquad j = 0, \ldots, m.$$

(Un)fortunately things change drastically when one considers the same problem in $d \geq 2$ variables: first, the number of points has to match the dimension of the space of all polynomials of a certain degree $n$ which is $\binom{n+d}{d}$ and even then the interpolation problem does not have to have a solution. Consider, for example, the case that all points are located on a straight line or the case of more than five points on the unit circle in $\mathbb{R}^2$. And things are becoming even more complicated when one considers extensions of Hermite interpolation: besides the fact that it is not clear what the "right" generalization of Hermite interpolation for several variables is, there are interpolation problems which are only unsolvable due to the selection of interpolation points only and there are interpolation problems which are *generically* unsolvable. The simplest case of the latter is interpolation of point evaluation and first partial derivatives at two points in $\mathbb{R}^2$ by bivariate quadratic polynomials. Although these are exactly six interpolation conditions which matches the dimension of the underlying polynomial space, this interpolation problem is unsolvable for any choice of the two points. For recent results on multivariate Hermite interpolation the reader is referred to [18].

Consequently, investigating polynomial interpolation in two and more variables leads to new questions and problems which are not encountered in the univariate situation:

1. How to construct sets of points which admit *unique* interpolation by polynomials of total degree at most $n$ in $d$ variables? Or, more general, how to construct points *and* appropriate polynomial subspaces which admit a unique interpolation polynomial? This question has been considered, e.g.

in [13, 8], but the probably most advanced work in this respect are the *interpolation systems* introduced by Gasca and Maeztu [12], which includes the aforementioned constructions.

2. If one is tied to the set of knots, for example when they come from measurements of physical quantities, then how to find a proper interpolation space of polynomials with some reasonable properties? This question has been considered by de Boor and Ron [3] who introduced the subspace of *least interpolation* as one important example of of *minimal degree interpolation*, see [4, 17]. This approach will be described in detail later.

3. Is there a (stable and efficient) method to decide algorithmically if the interpolation problem is uniquely solvable or not and to compute the interpolation polynomials in practice?

In order to comment on some of these problems, we have to fix some notation for multivariate polynomials. For that purpose, we let $d$ denote the number of variables and write any $x \in \mathbb{R}^d$ as $x = (\xi_1, \ldots, \xi_d)$. To represent polynomials, we use standard multiindex notation which means that for $\alpha \in \mathbb{N}_0^d$ we write

$$|\alpha| = \sum_{j=1}^{d} \alpha_j, \qquad x^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$$

for the *length* of the multiindex and the associated monomial of (total) degree $|\alpha|$, respectively, as well as

$$\Pi_n^d = \left\{ \sum_{|\alpha| \le n} c_\alpha x^\alpha : c_\alpha \in \mathbb{R}, |\alpha| \le n \right\}$$

for the vector space of all polynomials of degree at most $n$. In addition, $\Pi^d = \mathbb{K}[\xi_1, \ldots, \xi_d]$ will denote the ring of polynomials over the underlying field $\mathbb{K}$ which will not be specialized. We will also consider the data to be of the form $f_j = f(x_j)$ where usually $f$ only has to be defined on the set of interpolation points.

## 2 The Newton approach

The Newton approach is based on the idea to solve the interpolation problem by beginning with a very simple, or even the simplest, subproblem and then successively add more and more points to the interpolation problem and increasing

the degree of the polynomial at the same time. Recalling the univariate case $x_0, \ldots, x_n \in \mathbb{R}$ once more, this corresponds to starting with

$$L_0(f;x) = f(x_0) \tag{1}$$

and then extending it to

$$L_k(f;x) = L_{k-1}(f;x) + \frac{f(x_k) - L_{k-1}(f;x_k)}{(x_k - x_0)\cdots(x_k - x_{k-1})}(x - x_0)\cdots(x - x_{k-1}), \tag{2}$$

for $k = 1, \ldots, n$. In other words, one is just interpolating the properly normalized *error of interpolation* at the next step. This process may also be understood as extending the degree of the interpolation polynomial and "learning" an additional point at the same time. Using the notion of *divided differences* $f[x_0, \ldots, x_n]$ the Newton form of the interpolation polynomial is

$$L_n(f;x) = \sum_{j=0}^{n} f[x_0, \ldots, x_j]\ (x - x_0)\cdots(x - x_{j-1}). \tag{3}$$

Let us now consider a Lagrange interpolation problem based on a set of knots $\mathcal{X} = \{x_0, \ldots, x_N\} \subset \mathbb{R}^d$ and assume that the interpolation problem with respect to these knots is uniquely solvable, or *poised*, for short, in $\Pi_n^d$. Clearly, this implies $N = \binom{n+d}{d}$. Now, if we try to extend the idea of Newton interpolation to the multivariate case, we have to options: we may either add one point at any step or increase the degree of the interpolation polynomial from, say, $k-1$ to $k$ immediately which corresponds to adding not one but $\binom{k+d-1}{d-1}$ points at a time. This latter strategy, called *blockwise Newton interpolation* has been introduced and investigated in [19]. It makes use of the observation that the points in $\mathcal{X}$ can be re–indexed as $\{x_\alpha : |\alpha| \le n\}$, in such a way that all the interpolation problems based on the nested subsets

$$\mathcal{X}_k = \{x_\alpha : |\alpha| \le k\}$$

are poised in $\Pi_k^d$, respectively, $k = 0, 1, \ldots, n$. From this it immediately follows that there exist *Newton fundamental polynomials* $p_\beta \in \Pi_{|\beta|}^d$, $|\beta| \le n$, such that

$$p_\beta(x_\alpha) = \delta_{\alpha,\beta}, \qquad |\alpha| \le |\beta| \le n.$$

With these polynomials at hand one can introduce the *finite differences* $\lambda_{k+1}[\mathcal{X}_k, x]\, f$, $k = -1, \ldots, n$, $x \in \mathbb{R}^d$, inductively as

$$\begin{aligned} \lambda_0[x]f &= f(x) \\ \lambda_{k+1}[\mathcal{X}_k, x]\, f &= \lambda_k[\mathcal{X}_{k-1}, x]\, f + \sum_{|\alpha|=k} \lambda_k[\mathcal{X}_{k-1}, x_\alpha]\, f \cdot p_\alpha(x). \end{aligned}$$

These differences enable us to extend (3) to a multivariate Newton formula for the interpolation polynomial. Precisely, we have the following result.

**Theorem 1** *Let the Lagrange interpolation problem with respect to $\mathcal{X}$ be poised. Then*

$$L_n(f;x) = \sum_{|\alpha|\leq k} \lambda_{|\alpha|}\left[\mathcal{X}_{|\alpha|-1}, x_\alpha\right] f \cdot p_\alpha(x) \tag{4}$$

*and*

$$f(x) - L_n(f;x) = \lambda_{n+1}\left[\mathcal{X}_n, x\right] f. \tag{5}$$

It is worthwhile to mention here that the formula (4) can be turned into an algorithmic method for the effective computation of the interpolation polynomial. This process, described in [16], first generates the Newton fundamental polynomials by a Gram–Schmidt orthogonalization process and then computes the finite differences by a triangular scheme similar to the one used for the computation of univariate divided differences. Since the algorithm only uses evaluation of polynomials and vector space operations on $\Pi_n^d$, we also remark that the coefficients of the interpolation polynomial must belong to the same field as the components of the points. In particular, if the points are given in $\mathbb{Q}^d$, then it is even possible to decide *exactly* (by using exact rational arithmetics) if the interpolation problem is poised or not. There is also a C++ implementation of these algorithms which can be freely downloaded from my homepage.

Another advantage of the Newton formula (4) is that it can be used for the derivation of a *remainder formula* for sufficiently smooth functions which depends mainly of $(n+1)$st directional derivatives of the function. This formula, introduced in [19] requires some more notation. Let

$$\Lambda_n = \left\{(\mu_0, \ldots, \mu_m) : \mu_j \in \mathbb{N}_0^d,\ |\mu_j| = j,\ j = 0, \ldots, n\right\}$$

and define, for any such "path" $\mu \in \Lambda_n$ the quantities

$$\begin{aligned} \mathcal{X}_\mu &= \left\{x_{\mu_0}, \ldots, x_{\mu_n}\right\}, \\ \pi_\mu &= \prod_{j=0}^{n-1} p_{\mu_j}\left(x_{\mu_{j+1}}\right), \\ D_\mu^n &= D_{x_{\mu_n} - x_{\mu_{n-1}}} \cdots D_{x_{\mu_1} - x_{\mu_0}}, \end{aligned}$$

the latter one being a homogeneous differential operator of order $n$ built by the directional derivatives with respective to two successive points. Then we can state the following result.

**Theorem 2** *Let $\Omega \subset \mathbb{R}^d$ be a convex set, let $f \in C^{n+1}(\Omega)$ and assume that $\mathcal{X} \subset \Omega$. Then, for any $x \in \Omega$*

$$f(x) - L_n(f;x) = \sum_{\mu \in \Lambda_n} p_{\mu_n}(x) \pi_\mu \int\limits_{[\mathcal{X}_\mu, x]} D_{x - x_{\mu_n}} D_\mu^n f. \tag{6}$$

Here we recall the *simplex spline integral*

$$\int\limits_{[y_0,\ldots,y_n]} f = \int_{\Delta^n} f(u_0 y_0 + \cdots u_n y_n)\, du,$$

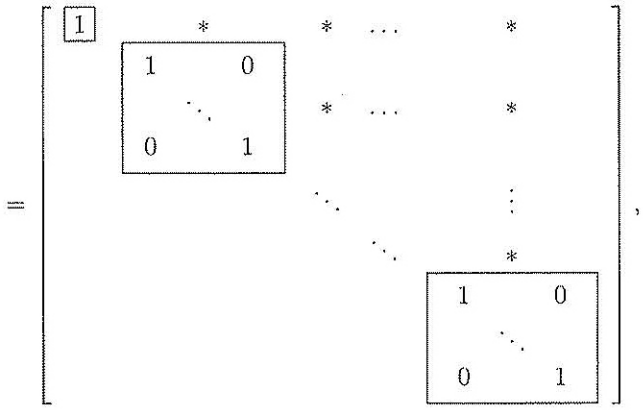
where

$$\Delta^n = \left\{ u = (u_0, \ldots, u_n) : u_j \geq 0,\, j = 0, \ldots, n, \sum_{j=0}^{n} u_j = 1 \right\},$$

cf. [15, 14]. A geometric interpretation of the terms in (6) is given in [19], for a different proof of (6) see [2]. Also, the main parts of this formula can be connected to another notion of a *multivariate divided difference* as pointed out in [1]. It is also possible to extend many of these ideas to a certain, quite general type of multivariate Hermite interpolation schemes, which was done in [18]. In addition to the introduction of a Newton approach and the derivation of a remainder formula of a similar structure as (6), [18] also discusses some of the general problems of multivariate Hermite interpolation like the (almost) poisedness of interpolation problems.

To clarify the difference from the "point-by-point" Newton approach, let us have a look at the Vandermonde matrix obtained from using the Newton fundamental polynomials $p_\alpha$, $|\alpha| \leq n$, as a basis of $\Pi_n^d$. Indeed, we get that

$$\left[p_\alpha(x_\beta)\right]_{|\alpha|,|\beta| \leq n} = \begin{bmatrix} I_0 & * & \ldots & * \\ & I_1 & \ldots & * \\ & & \ddots & \vdots \\ & & & I_n \end{bmatrix} =$$



where $I_k$ is the $\binom{k+d}{d} \times \binom{k+d}{d}$ unit matrix. In other words, the Vandermonde matrix is not only an upper triangular matrix but a *block upper triangular* matrix where the size of the blocks on the main diagonal increases. In contrast to that approach we can also consider "point–by–point" Newton schemes, as one more special case of the recursive expansions described in generality by Gasca and López-Carmona in [11]. Let us briefly sketch the idea in a wider generality and derive the finite difference associated to it.

Let $V$ be any linear space, let $V_N \subset V$ be a linear subspace of dimension $N+1$ and let $\Theta = \{\theta_0, \ldots, \theta_N\} \subset V'$ be a finite set of (linearly independent) linear functionals such that the matrix

$$V_N = [\theta_j (f_k)]_{j,k=0,\ldots,N}$$

is nonsingular, where $f_0, \ldots, f_N$ is a basis of $V_N$. Then there exists *triangular basis with respect to* $\Theta$ for $V$, i.e., elements $g_0, \ldots, g_N \in V$ such that

$$\theta_j(g_k) \begin{cases} = 0 & j < k, \\ \neq 0 & j = k, \end{cases}$$

at least after renumbering the functionals properly. Again, this basis may be obtained by a Gram–Schmidt like orthogonalization process, but, more important, there is a variety of interpolation problems where a triangular basis is known explicitly and easy to manipulate, as for example in the interpolation systems of Gasca and Maeztu [12]. In this case the finite difference $[\theta_0, \ldots, \theta_k, \theta]\, f$, $k = -1, \ldots, N$, which is defined for any $f \in V$ and any linear functional $\theta \in V'$ as

$$[\theta] f \;=\; \theta(f) \tag{7}$$

$$[\theta_0,\ldots,\theta_k,\theta]\,f \;=\; \frac{\theta_k(g_k)\,[\theta_0,\ldots,\theta_{k-1},\theta]\,f+\theta(g_k)\,[\theta_0,\ldots,\theta_k]\,f}{\theta_k(g_k)} \tag{8}$$

represents the coefficients with respect to the basis $g_0,\ldots,g_N$. In other words, the Newton formula for the interpolation operator $L_\Theta : V \to V_N$ is

$$L_\Theta f=\sum_{j=0}^{N}[\theta_0,\ldots,\theta_j]\,f\frac{g_j}{\theta_j(g_j)} \tag{9}$$

and, for any $\theta \in V'$ we have the error formula

$$\theta(f-L_\Theta f)=[\theta_0,\ldots,\theta_N,\theta]\,f. \tag{10}$$

Let us briefly prove these formulas by induction on $N$; the case $N=0$ can be verified easily. To advance with the induction hypothesis, let $N \geq 1$ and assume that (9) and (10) are valid for $N-1$. Since $\theta_j(g_N)=0$, $j=0,\ldots,N-1$, we have for any $k \leq N-1$ that

$$\begin{aligned}\theta_k\left(\sum_{j=0}^{N}[\theta_0,\ldots,\theta_j]\,f\frac{g_j}{\theta_j(g_j)}\right) &= \sum_{j=0}^{N}[\theta_0,\ldots,\theta_j]\,f\frac{\theta_k(g_j)}{\theta_j(g_j)}\\ &= \sum_{j=0}^{N-1}[\theta_0,\ldots,\theta_j]\,f\frac{\theta_k(g_j)}{\theta_j(g_j)}=\theta_k\left(L_{\Theta\setminus\{\theta_N\}}f\right)\\ &= \theta_k(f)\end{aligned}$$

and, by using the induction hypothesis for (9) and (10),

$$\begin{aligned}\theta_N\left(\sum_{j=0}^{N}[\theta_0,\ldots,\theta_j]\,f\frac{g_j}{\theta_j(g_j)}\right) &= [\theta_0,\ldots,\theta_N]\,f+\sum_{j=0}^{N-1}[\theta_0,\ldots,\theta_j]\,f\frac{\theta_N(g_j)}{\theta_j(g_j)}\\ &= \theta_N\left(f-L_{\Theta\setminus\{\theta_N\}}f\right)+\theta_N\left(L_{\Theta\setminus\{\theta_N\}}f\right)\\ &= \theta_N(f),\end{aligned}$$

which proves (9) for $N$. To extend (10) we consider for any $\theta \in V'$

$$\begin{aligned}\theta(f-L_\Theta f) &= \theta\left(f-L_{\Theta\setminus\{\theta_N\}}f\right)-\theta\left(L_\Theta f-L_{\Theta\setminus\{\theta_N\}}f\right)\\ &= [\theta_0,\ldots,\theta_{N-1},\theta]\,f-[\theta_0,\ldots,\theta_N,\theta]\,\frac{\theta(g_N)}{\theta_N(g_N)}\\ &= \frac{\theta_N(g_N)\,[\theta_0,\ldots,\theta_{N-1},\theta]\,f-\theta(g_N)\,[\theta_0,\ldots,\theta_N]}{\theta_N(g_N)}\\ &= [\theta_0,\ldots,\theta_N,\theta]\,f.\end{aligned}$$

This completes the inductional step. Finally, let me remark that the coefficients (or *"divided differences"*)

$$f[\theta_0, \ldots, \theta_k] = \frac{[\theta_0, \ldots, \theta_k] f}{\theta_k(g_k)}, \qquad k = 0, \ldots, N,$$

also satisfy a recurrence relation, namely,

$$f[\theta_0] = \frac{\theta_0(f)}{\theta_0(g_0)} \tag{11}$$

$$f[\theta_0, \ldots, \theta_k] = \frac{\theta_k(g_k) f[\theta_0, \ldots, \theta_{k-2}, \theta_k] - \theta_k(g_{k-1})[\theta_0, \ldots, \theta_{k-1}]}{\theta_k(g_k)}, \tag{12}$$

which follows easily from the recurrence for the finite difference.

This approach allows for the fast computation of the interpolation polynomials in many cases where the Newton fundamental polynomials are known explicitly and admit fast recursive evaluation schemes, cf. [7]. Therefore, the interpolation systems due to Gasca and Maeztu, combined with the recursive computation of the coefficients through a finite difference, give a very fast and efficient way to practically compute interpolation polynomials in a great variety of possible situations.

## 3 Minimal degree interpolation

Let us now focus on the question of what to do if there is a given set of knots which does not allow interpolation from some space $\Pi_n^d$, maybe because the number of points is not appropriate, or maybe because the points lie on an algebraic hypersurface of sufficiently low degree. However, independently of the position of the points the following result, which is easily proved, hints where to look for proper interpolation spaces.

**Proposition 3** *Let $\mathcal{X} = \{x_0, \ldots, x_N\} \subset \mathbb{R}^d$ consist of pairwise distinct points. Then for all $0 \leq k \leq N$ the non-square Vandermonde matrix*

$$V(\mathcal{X}, \Pi_k^d) = \left[x_j^\alpha\right]_{j=0,\ldots,N,\, |\alpha| \leq k}$$

*satisfies*

$$k+1 \leq \operatorname{rank} V(\mathcal{X}, \Pi_k^d) \leq N+1,$$

*and $\operatorname{rank} V(\mathcal{X}, \Pi_k^d) = N+1$ for any choice of $\mathcal{X} \subset \mathbb{R}^d$ of cardinality $N+1$ if and only if $k \geq N+1$.*

This proposition tells us that there always exist subspaces of $\Pi_N^d$ where the interpolation problem with respect to $\mathcal{X}$ is poised and that $\Pi_N^d$ is the smallest polynomial space to search in if there is nothing known about $\mathcal{X}$. Of course, there always exists a lot of subspaces which admit unique interpolation, so that one has to impose further restrictions to the interpolation space. The first to do so were de Boor and Ron who introduced the *least interpolant* in [3] as a particular choice of subspace with a lot of useful properties (see, in particular, [4]). Using some of these generic properties one can single out an interesting class of interpolation spaces which will be introduced as follows.

**Definition 4** *Let $\mathcal{X} \subset \mathbb{R}^d$. A polynomial space $\mathcal{P} \subset \Pi_N^d$ is called a* minimal degree interpolation space *with respect to $\mathcal{X}$ if*

1. *$\mathcal{P}$ is* interpolating, *i.e., for any function $f : \mathcal{X} \to \mathbb{R}$ there exists an unique polynomial $L(\mathcal{P}; f) \in \mathcal{P}$ such that $L(\mathcal{P}; f)(\mathcal{X}) = f(\mathcal{X})$.*

2. *$\mathcal{P}$ is* of minimal degree, *i.e., if $\mathcal{P} \subset \Pi_n^d$ and $n$ is minimal with this property, then there exists no subspace of $\Pi_{n-1}^d$ which is interpolating.*

3. *$\mathcal{P}$ is* degree reducing, *i.e., if $q \in \Pi^d$ is any polynomial, then the degree of $L(\mathcal{P}; f)$ is not larger than the degree of $q$.*

Again it is possible to construct (an only slightly more complicated) Newton approach by degree and point–by–point. Since the latter one allows for greater generalization, let us first briefly review the Newton approach by degree which was introduced and studied in [17]. The main idea is to split the nested family of multiindices $\mathbb{I}_k := \{\alpha : |\alpha| \leq k\}$, $k \in \mathbb{N}_0$, into two disjoint nested families $I_k, I_k' \subset \mathbb{I}_k$. We then say that a set of polynomials $\{p_\alpha : |\alpha| \leq n\}$ is *a Newton basis for* $\mathcal{X}$ if there exists two nested families of subsets $I_n, I_n' \subset \mathbb{I}_n$ and an indexation $\mathcal{X} = \{x_\alpha : \alpha \in I_n\}$ such that

1. $p_\beta(x_\alpha) = \delta_{\alpha,\beta}$, $\alpha, \beta \in I_n$, $|\alpha| \leq |\beta|$,

2. $p_\alpha(\mathcal{X}) = 0$, $\alpha \in I_n'$,

3. $\{p_\alpha : \alpha \in \mathbb{I}_n\}$ is a basis of $\Pi_n^d$.

In more algebraic terms this means that $p_\alpha$, $\alpha \in I_n$, is a Newton basis for some quotient space $\Pi^d/\mathcal{I}(\mathcal{X})$, where $\mathcal{I}(\mathcal{X}) = \{p \in \Pi^d : p(\mathcal{X}) = 0\}$ denotes the polynomial ideal induced by $\mathcal{X}$. Indeed, the connections between minimal degree interpolation spaces and Newton bases are very close as the following result indicates.

**Theorem 5** *A polynomial space $\mathcal{P} \subset \Pi^d$ is a minimal degree interpolation with respect to $\mathcal{X}$ if and only if it is spanned by a Newton basis for $\mathcal{X}$.*

Again, it is possible to define a finite difference for the minimal degree interpolation space and derive a Newton form like (4) and even a remainder formula similar to (5). Since most of the ideas are similar but the results are of more intricate nature and more difficult to state, and since all can be found in [17] anyway, I do not want to dwell on this any further.

Instead, let us consider a more specific construction of minimal degree interpolation spaces which will reveal a close relationship between certain minimal degree interpolation spaces and the Gröbner basis for the ideal $\mathcal{I}(\mathcal{X})$ induced by $\mathcal{X}$. For that purpose we turn to a "point-by-point" approach again and will use a more refined notion of "degree" of multivariate polynomials. Let $\prec$ denote any total ordering on $\mathbb{N}_0^d$ which has the property that it is compatible with addition, i.e., $\alpha \prec \beta$ implies that $\alpha+\gamma \prec \beta+\gamma$ for any $\alpha, \beta, \gamma \in \mathbb{N}_0^d$, and that 0 is minimal with respect to $\prec$. Such an ordering is frequently called a *well-ordering*. Examples are (inverse) lexicographical orderings as well as graded lexicographical orderings, where for the "degree" one might apply an arbitrary norm on $\mathbb{R}^d$ to the multiindices. In this respect, the total degree corresponds to the norm $|\cdot|_1$. Any well-ordering induces a term order on the monomials $x^\alpha$ by saying that $x^\alpha \prec x^\beta$ if and only if $\alpha \prec \beta$. This in turn implies, in a natural way, the *leading term* $\Lambda_p$ of any polynomial $p \in \Pi^d$ as the maximal term with respect to this term order which has a nonzero coefficient. Finally, we say for any pair of polynomials $p, q \in \Pi^d$ that $p \prec q$ if and only if $\Lambda_p \prec \Lambda_q$.

The minimal degree interpolation space which we want to consider now is the one which uses a minimal number of monomials (i.e., $N+1$) which are, in addition, minimal with respect to the term order $\prec$. This space, denoted by $\mathcal{P}_\prec(\mathcal{X})$ is *uniquely* defined by $\mathcal{X}$ through the following properties:

1. $\mathcal{P}_\prec(\mathcal{X})$ is an interpolation space for $\mathcal{X}$.

2. $\mathcal{P}_\prec(\mathcal{X})$ is $\prec$*-minimal*, i.e., if

$$\alpha = \max_\prec \{\Lambda_p : p \in \mathcal{P}_\prec(\mathcal{X})\},$$

then the interpolation problem with respect to $\mathcal{X}$ is not solvable in any subspace of $\{p \in \Pi^d : \Lambda_p \prec x^\alpha\}$.

3. $\mathcal{P}_\prec(\mathcal{X})$ is $\prec$*-reducing*, i.e., for any $q \in \Pi^d$ the interpolating polynomial $L(\mathcal{P}_\prec(\mathcal{X}); q)$ satisfies

$$L(\mathcal{P}_\prec(\mathcal{X}); q) \prec q.$$

However, we also have the following characterization of $\mathcal{P}_\prec(\mathcal{X})$, which has been observed in [17] for the graded lexicographical order, and was suggested by a similar property for the least interpolation from [3].

**Theorem 6** *Let $\mathcal{X} \subset \mathbb{R}^d$ be finite and let $\prec$ be any term order. Then*

$$\mathcal{P}_\prec(\mathcal{X}) = \bigcup_{q(\mathcal{X})=0} \ker \Lambda_q(D). \tag{13}$$

Here, $\Lambda_q(D) = \frac{\partial^{|\alpha|}}{x^\alpha}$ is the partial differential operator induced by the exponent in the leading term $\Lambda_q = c_\alpha x^\alpha$ of $q$. Let us remark that from equation (13) and the fact that $\{q : q(\mathcal{X}) = 0\}$ is a polynomial ideal it immediately follows that $\mathcal{P}_\prec(\mathcal{X})$ is *closed under differentiation* (or *D-invariant*) and that therefore the set

$$\left\{\alpha \in \mathbb{N}_0^d : x^\alpha \in \mathcal{P}_\prec(\mathcal{X})\right\}$$

is a *lower set*. Recall that $I \subset \mathbb{N}_0^d$ is called a *lower set* if for any choice $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ with $\alpha \in I$ the existence of a representation $\alpha = \beta + \gamma$ implies that $\beta, \gamma \in I$. Moreover, it can be easily verified that the choice of $\mathcal{P}_\prec(\mathcal{X})$ is independent of scaling and translating $\mathcal{X}$, i.e., for any real number $c \neq 0$ and any $y \in \mathbb{R}^d$ we have

$$\mathcal{P}_\prec(c\mathcal{X} + y) = \mathcal{P}_\prec(\mathcal{X}).$$

Despite of these common properties, the choice of $\mathcal{P}_\prec(\mathcal{X})$ depends strongly on the term order "$\prec$". For an example we consider just interpolation at 4 points in $\mathbb{R}^2$ where we have the following "generic" situations for some exemplary choices of "$\prec$":

1. if $\prec$ is the inverse lexicographical order, then the space $\mathcal{P}_\prec(\mathcal{X})$ is spanned by $1, x, x^2, x^3$.

2. if $\prec$ is the lexicographical order, then the space $\mathcal{P}_\prec(\mathcal{X})$ is spanned by $1, y, y^2, y^3$.

3. if $\prec$ is the graded lexicographical order, then the space $\mathcal{P}_\prec(\mathcal{X})$ is spanned by $1, x, y, x^2$.

4. if $\prec$ is a modified graded lex order in the sense that as "degree" of $x^\alpha$ one takes $|\alpha|_\infty$, then the space $\mathcal{P}_\prec(\mathcal{X})$ is spanned by $1, x, y, xy$.

Let us finally connect minimal degree interpolation spaces to Gröbner bases, an important tool in Computational Algebra. Gröbner bases have been introduced by Buchberger [5, 6] for an algorithmic treatment of the quotient space

$\Pi^d/\mathcal{I}$, where $\mathcal{I} \subset \Pi^d$ is a polynomial ideal. To review the definition of a Gröbner basis, let us make use of the following standard notation for the ideal generated by a finite set $p_1, \ldots, p_n$ of polynomials:

$$\langle p_1, \ldots, p_n \rangle = \left\{ \sum_{j=1}^{n} q_j p_j : q_1, \ldots, q_n \in \Pi^d \right\}.$$

**Definition 7** *Let $\mathcal{I} \subset \Pi^d$ be a polynomial ideal.*

1. *A finite set $\mathcal{B} = \{p_1, \ldots, p_n\} \subset \Pi^d$ is called a* basis *of $\mathcal{I}$ if $\mathcal{I} = \langle \mathcal{B} \rangle = \langle p_1, \ldots, p_n \rangle$.*

2. *A finite set $\mathcal{G} = \{p_1, \ldots, p_n\}$ is called a* Gröbner basis *of $\mathcal{I}$ if*

$$\langle \Lambda_{p_1}, \ldots, \Lambda_{p_n} \rangle = \langle \Lambda_p : p \in \mathcal{I} \rangle .$$

3. *A Gröbner basis $\mathcal{G} = \{p_1, \ldots, p_n\}$ is called a* minimal Gröbner basis *of $\mathcal{I}$ if no proper subset of $\mathcal{G}$ is basis for $\mathcal{I}$ any more.*

As indicated by the name, any Gröbner basis of an ideal $\mathcal{I}$ is indeed a basis of $\mathcal{I}$. Moreover, it is worthwhile to note that neither the Gröbner basis nor the minimal basis is unique for an ideal $\mathcal{I}$ in general; a unique choice would be the so–called *reduced Gröbner basis* which can be constructed by Buchberger's algorithm. For a detailed exposition see [10], but let me just remark that the heart of this algorithm is a process called *reduction* which can be regarded as a natural extension of the division with remainder to multivariate polynomials. Without insisting in details, reduction of a polynomial $p$ with respect to a finite set $\{p_1, \ldots, p_n\}$ means to find a polynomial $q \in \Pi^d$ such that $p - q \in \langle p_1, \ldots, p_n \rangle$ and such that no term in $q$ is belongs to $\langle \Lambda_{p_1}, \ldots, \Lambda_{p_n} \rangle$. It can be shown that $q$ is unique if $\{p_1, \ldots, p_n\}$ is a Gröbner basis for $\mathcal{I}$, in which case it makes sense to talk about *the* polynomial $q = p \rightarrow_{\mathcal{G}}$ as the *reduction of $p$* with respect to a Gröbner basis $\mathcal{G}$.

**Definition 8** *A Gröbner basis $\mathcal{G}$ for an ideal $\mathcal{I}$ is called* reduced *if for any $p \in \mathcal{G}$ one has*

$$p = p \rightarrow_{\mathcal{G}}$$

*and if, for any $p \in \mathcal{G}$ the leading term $\Lambda_p$ is has coefficient 1.*

The (unique) reduced Gröbner basis for the ideal $\mathcal{I}(\mathcal{X})$, denoted by $\mathcal{G}(\mathcal{X})$ now connects closely to the minimal degree interpolation space $\mathcal{P}_\prec(\mathcal{X})$. Indeed, from equation (13) and the properties of a Gröbner basis it follows that

$$\mathcal{P}_\prec(\mathcal{X}) = \bigcap_{q \in \mathcal{G}(\mathcal{X})} \Lambda_q(D), \tag{14}$$

i.e., the interpolation space can be defined directly from the Gröbner basis. Since, due to (14), interpolation and reduction are identities on $\mathcal{P}_\prec(\mathcal{X})$ and both map

$$p + \sum_{g \in \mathcal{G}(\mathcal{X})} q_g\, g, \qquad p \in \mathcal{P}_\prec(\mathcal{X}), \quad q_g \in \Pi^d,\ g \in \mathcal{G}(\mathcal{X})$$

to $p$, we can conclude the following result.

**Theorem 9** *Let $\mathcal{X} \subset \mathbb{R}^d$ be finite and let $\mathcal{G}(\mathcal{X})$ be the reduced Gröbner basis for $\mathcal{I}(\mathcal{X})$. Then, for any $p \in \Pi^d$ we have*

$$L\left(\mathcal{P}_\prec(\mathcal{X}); p\right) = p \to_{\mathcal{G}(\mathcal{X})} . \tag{15}$$

To "invert" (14) we observe that, knowing the interpolation space then we can write the ideal $\mathcal{I}(\mathcal{X})$ as

$$\mathcal{I}(\mathcal{X}) = \left\{p - L\left(\mathcal{P}_\prec(\mathcal{X}); p\right) : p \in \Pi^d\right\}. \tag{16}$$

However, the Gröbner basis can also be characterized in the same way: let $I \subset \mathbb{N}_0^d$ denote the corners of the *upper set*

$$\mathbb{I}' = \mathbb{N}_0^d \setminus \left\{\alpha \in \mathbb{N}_0^d : x^\alpha \notin \mathcal{P}_\prec(\mathcal{X})\right\},$$

i.e.,

$$I = \mathbb{I}' \setminus \left( \bigcup_{0 \neq \alpha \in \mathbb{N}_0^d} \mathbb{I}' + \alpha \right),$$

then

$$\mathcal{G}(\mathcal{X}) = \left\{x^\alpha - L\left(\mathcal{P}_\prec(\mathcal{X}); (\cdot)^\alpha\right) : \alpha \in I\right\}. \tag{17}$$

In other words, the equations (14) and (17) show that there is a strong duality between the minimal degree interpolation space associated to a term order and the reduced Gröbner basis of the ideal $\mathcal{I}(\mathcal{X})$ with respect to this term order. In particular, one can obtain the reduced Gröbner basis as a by-product of the Gram–Schmidt construction of the Newton basis for the minimal degree interpolation space.

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does as much and no less than the others. A simple example will illustrate the matter. To solve the Laplace equation with homogeneous Dirichlet data,

$$-\Delta u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

one must write the following program

```
solve(u) begin
  onbdy(1,2) u = 0;
  pde(u) - laplace(u) = f;
end;
```

Of course many things are missing, such as a description of $\Omega$ and $f$. If $f$ is an analytic function like $\sin(x_1) * \sin(x_2)$ one simply writes

```
f = sin(x) * sin(y) ;
```

To describe a domain one gives a piecewise description of its boundary in the direction that leaves $\Omega$ on its left. For instance a disk with a whole inside would be

```
border(1,0,2*pi,80)
{
  x:= cos(t);
  y:= sin(t);
};
border(2,0,2*pi,20)
{
  x:= 0.3+0.5*cos(-t);
  y:= 0.5*sin(-t);
};
buildmesh(820);
```

The first number after "border" is the identification number of the boundary and the last one is the number of discretization points on it. The 2 middle values are the min and max of parameter "t". The keyword "buildmesh" calls an automatic mesh generator.
The functionalities of Gfem are

- To operate on scalar functions of 2 variables $x, y$ defined on a domain $\Omega$.
- Domains are represented by triangulations and functions by piecewise linear continuous functions on the triangulations.

- The major operations on functions are: algebraic, derivatives, convection, inverse of elliptic operators (scalar and vectorial).

For example the program

$$plot(dx(sin(x+y)));$$

will display the level curves of the linear interpolate of $cos(x+y)$ on the domain which was defined earlier by a triangulation.

### 2.1 Triangulations

Automatic mesh generation using triangular element is performed by the Delaunay–Voronoi algorithm for a domain whose boundary is input piecewise analytically.

For example let us return to the previous program which triangulates the unit circle with a whole inside of radius 0.1 and a maximum of 820 vertices. The outer boundary has a reference number equal to 1; it is discretised with 80 points The inner boundary has a reference number equal to 2; it is discretised with 100 points. Notice that the boundaries are described by their analytical parameterization in the direction which leaves the interior of the domain on its left. The fineness of the mesh is controlled by the size of the nearest boundary edges. It can also be controlled by artificial boundaries with ID zero. Here an extra boundary very near the inner circle has been added so as to simulate a boundary layer.

```
border(1,0,2*pi,80)
{
  x:= cos(t);
  y:= sin(t);
};
border(2,0,2*pi,100)
{
  x:= 0.3+0.5*cos(-t);
  y:= 0.5*sin(-t);
};
border(0,0,2*pi,100)
{
  x:= 0.3+0.55*cos(-t);
  y:= 0.55*sin(-t);
};
buildmesh(1000);
```

Domains with corners can be treated by 3 methods:

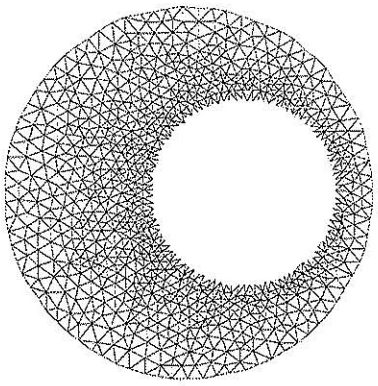

Figure 2: Triangulation of a disk with a whole

- In case of a polygon by reading from a file a list of vertices.
- Using the if...then...else to describe the border piecewise.
- See the border as many borders with same numbers.

Here is the famous backward step of CFD:

```
border(1,0,1,6)  begin x:=0;       y:=1-t    end;
border(2,0,1,15) begin x:=2*t;     y:=0      end;
border(2,0,1,10) begin x:=2;       y:=-t     end;
border(2,0,1,20) begin x:=2+3*t;   y:=-1     end;
border(2,0,1,35) begin x:=5+15*t;  y:=-1     end;
border(3,0,1,10) begin x:=20;      y:=-1+2*t end;
border(4,0,1,35) begin x:=5+15*(1-t); y:=1 end;
border(4,0,1,40) begin x:=5*(1-t);y:=1     end;

buildmesh(1500);
```

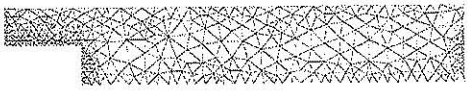

Figure 3: Course mesh for the backward step

More complex domains can be generated such as shown later. For domains representing multi-materials such as copper, iron and air in electromagnetic, the concept of "region" exists and coincided with simply connected components defined by outer and inner boundaries.

### 2.2 Simple operations

Functions are generated by assignments such as in

$$f = x * y;$$

This is different from the statement $pi := 4 * atan(1)$ because $f$ is a function while $pi$ is a scalar. When Gfem see a '=' it checks whether the function exists already. If it does not it allocates an array of values at the vertices. With ':=', no array is allocated.

Functions can be operated upon with '*,/,+,-', the usual functions sin, cos ... and new operators such as 'plot', 'dx,dy' (for partial derivatives)

### 2.3 Boundary conditions for PDEs

Analytical formulae can be used in Gfem to define boundary conditions. Consider the case of a potential flow over a cylinder solved with a stream function v (see the first figure). Refering to the first triangulation, we will write:

```
onbdy(1) v = u2 * y - u1 * x ;
onbdy(2) v = 0;
```

Neumann and Robin / Fourier boundary conditions are treated similarly with the reserved word id(), meaning identity operator. For example $\frac{v}{r} + \frac{\partial v}{\partial n} = 0$ would become

```
onbdy(1) id(v) / sqrt(x*x + y*y) + dnu(v) = 0
```

### 2.4 The Solver

At present Gfem has a general solver for a scalar and vector elliptic PDE, of the form

$$bv + a_1 \frac{\partial v}{\partial_x} + a_2 \frac{\partial v}{\partial_x} - \nabla \cdot (\nu \nabla v) = f$$

It uses a Gauss factorisation and it is called by writing

```
solve(v) begin
  pde(v) id(v)*b + dx(v)*a1 + dy(v)*a2 - laplace(v)*nu = f;
  ...
  end;
```

In the case of a 2-vector coupled Helmoltz equation we have for instance

```
solve(u,v) begin
   onbdy(1) id(u) + dnu(v) = 0;
   onbdy(1) v = 0;
   pde(u) v - laplace(u) = 0;
   pde(v) u - laplace(v) = f;
end;
```

As matrices are stored, it is possible to solve several PDEs at once and remember when necessary that the matrices have been factorized.

The power of Gfem is for time dependent problems because it is possible to loop on some part of the program with the instruction

```
iter{...}
```

Here are 10 time steps for the heat equation:

```
dt := 0.1; v = 1; k := 1;
iter{10} begin
  solve(v,k) begin
           onbdy(1) v = 0;
           de(v) id(v)/dt -laplace(v)*nu = v/dt;
  end;
  k:=-1;
end;
```

By the way, for convection problems there is a special operator "convect" which implements the modified method of characteristics. Here is the Navier-Stokes equations solved by a projection algorithm:

```
i:=1; j:=2; k:=3;
iter(80)
{ /* '{' is the same as begin */
 f=convect(un,u,v,dt);  g=convect(vn,u,v,dt);

 solve(u,i){ /*Horizontal velocity*/
  onbdy(1) u = y*(1-y);
  onbdy(2,4) u = 0;
  onbdy(3)dnu(u)=0;
   pde(u) id(u)/dt-laplace(u)*nu = f/dt -dx(p);
 };

 solve(v,j){ /* Vertical velocity */
```


Figure 4: Flow over a backward step

```
 onbdy(1,2,3,4) v = 0;
  pde(v) id(v)/dt-laplace(v)*nu = g/dt -dy(p);
};

solve(p,k) {  /*  Pressure */
 onbdy(1,2,4) dnu(p) = 0;
 onbdy(3) p=0;
  pde(p)  -laplace(p)= -(dx(f) + dy(g))/dt;
 };
un = u;  vn = v; i:=-1; j:=-2; k:=-3;
} ;
```

## 2.5 Variational Formulation

As it turns out a pointwise description of a PDE system is not very good. For instance in elasticity, unless the variational form is given, one is likely to make mistakes on the boundary conditions. So we added variational formulations to Gfem. Below are the Lamé equations of linear elasticity:

```
E= 21.5;  sigma := 0.29;
mu =E/(2*(1+sigma));
lam = E*sigma/((1+sigma)*(1-2*sigma));
nu = lam+sigma;

varsolve(u,v;w,s)  begin
     onbdy(1) u=0;
     onbdy(1) v=0;
e11 = dx(u);
e22 = dy(v);
e12 = 0.5*(dx(v)+dy(u));
e21 = e12;
dive = e11 + e22;
s11w=2*(lam*dive+2*mu*e11)*dx(w);
s22s=2*(lam*dive+2*mu+e22)*dy(s);
s12s = 2*mu*e12*(dy(w)+dx(s));
```

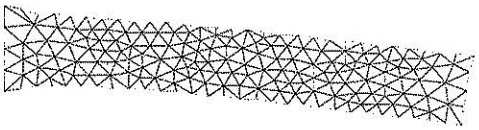

Figure 5: Bending of a beam by its own weigth

```
s21w = s12s;
a = s11w+s22s+s12s+s21w +0.1*s;
end : intt[a];  /* means volume intergral of a */

x = x + 0.1 * u; /* will display deformed mesh */
y = y + 0.1 * v;
```

The power of languages like Gfem can be impressive. For instance the author was given an electromagnetic problem coupled with turbulent flow and was able to give an approximate answer in less than an afternoon. It was the study of natural convection in a electrolytic metal which involved Maxwell-Helmholtz equations (with complex coefficients) driving the metal by a left hand side in Navier-Stokes equations. The problem had a free boundary and FreeFem could not handle that part (now it can). Also both equations should be solved in the same block and that it could not do because the vector systems are limited to 2 in dimensions; but the next version will not have such limits (see Hecht-Pironneau (1996)).

## 3 Graphics

The following features apply to Gfem but not to FreeFem:

Three resizeable windows display:

- the program
- the triangulation
- the level curves or color map or 1D cross section

Graphic windows are zoomable without limit; a one dimensional cross section along a user specified segment can be plot to display accurately a function.

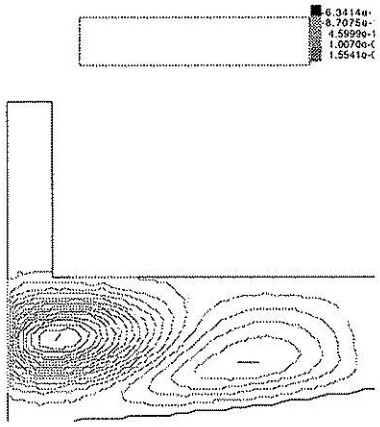

Figure 6: Stream lines of the convection current in the metal and in the scories

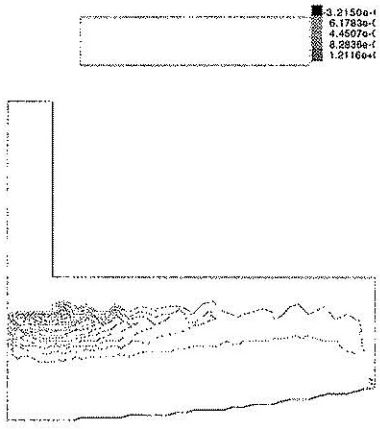

Figure 7: Electromagnetic forces that drive the convection in the metal

### 3.1 Export

Each window's content can be stored in text format. All graphics can be copied in the clipboard and pasted into another application as an EPFS pict file.

### 3.2 Fine graphics

All graphics can also be stored in Postscript files with headers to include them in plain TEX-files in a resizeable form.

For those who do not feel like paying for a software there is a simple way to enhance FreeFem graphics by writing an interface to gnuplot and add a small module to FreeFem to write gnuplot files. This exercise takes about 2 hours and is given to students every year; FreeFem must be recompiled but the source files are given.

## 4 Future versions

Now FreeFem is well debugged but it has grown a bit out of proportion and the main file in particular (syntaxic.cpp) is a bit like an Italian spaghetti plate: C++ experts mask their face and laugh.

On the other hand there is a big demand for a 3D FreeFem. D. Bernardi wrote a new interpreter, in clean C++, based on Hecht's formal functions. I wrote a new solver based on templates in C++ so as to be able to do an unlimited number of coupled vector systems, complex or real (see Hecht-Pironneau (1996)). We will need both a direct and an iterative solver like GMRES.

But the real challenge is for the description of 3D geometries.

We think that VRML has a good future. It is likely that there will be dozen of cheap VRML visualizers and input packages on all 3 major systems (Mac, PC and UNIX). There is also a 3D automatic mesh generator at INRIA and soon will be a mesh adaption package for it, based on variable metrics as in 2D. This software requires a triangulation of the boundary as input, therefore the missing link is a boundary mesh generator when the geometry is given in VRML format.

But VRML is made for image synthesis mostly; objects are described by CSG (constructive solid geometry) as unions, intersections and extrusions of simple objects such as planes, bricks, spheres, cones and torus and these objects are *displayed* but never are the intersection curves computed. Hence this description is not enough. Our plans are to construct the triangulation of scenes from the triangulation of the surfaces of the simple objects. It is theoretically possible

by intersecting each pair of triangles, but of course this would be too long so one has to be more clever.

Possibilities are infinite and to forsee the right options for an intelligent software is not easy. For instance several people have asked us for splines, higher degrees of elements, other solvers, handling of two mesh at once, links with Mathlab or Mathematica... (By the way the response to FreeFem is very good! it appears that on the average one web surfer transfers it every day and we receive about one question every week.) The next 2D version will have splines for the boundaries, perhaps other solvers, more variational formulations but not $P^2$ elements because that would force us to type the language. Another line of development is to provide the users with a library rather than a complete product. Such users are few because you need to be fairly fluent in C++ and willing to investigate other people's C-classes. But, refering again to Hecht-Pironneau (1996), it is a very powerful direction especially if it is combined with a package like FreeFem for input/output.

# References

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**Current Trends in Applied Mathematics**
*Miguel A. Herrero y Enrique Zuazua (Eds.)*
*Editorial Complutense, 1996*
*260 páginas, ISBN 84-89365-94-6*

Las seis contribuciones que recoge este libro tienen su origen en las Conferencias que sus autores dieron en Aguadulce (Almería) en 1993, dentro de los Cursos de Verano de El Escorial. En cada una de ellas los autores, reconocidos especialistas en su campo, reflejan el estado actual de alguna de las áreas de investigación de más reciente y rápido desarrollo en relación con problemas básicos de Matemáticas, Física e Ingeniería. El rango de éstos va desde problemas de transporte de carga en semiconductores y la teoría matemática de cristales líquidos en Física del Estado Sólido, hasta la dinámica de dos líquidos miscibles incompresibles en Mecánica de Fluidos y la teoría matemática del colapso gravitacional, pasando por los aspectos matemáticos de los problemas de frontera libre y la Teoría de Control en Sistemas Distribuidos. Si aisladamente cada contribución es valiosa por su presentación uniforme de resultados que de otra forma sólo serían accesibles a través de una literatura muy dispersa, el conjunto de la obra ilustra una gran diversidad de técnicas matemáticas asi como refleja la naturaleza interdisciplinar de la investigación en un campo como el de la Matemática Aplicada.

CONTENIDOS: 1. Some Problems of Charge Transport in Semiconductors (por Luis L. Bonilla, 71 pp.), 2. An Introduction to Free Boundary Problems (por José Carrillo, 47 pp.), 3. The Mathematical Theory of Gravitational Collapse (por Demetrios Christodoulou, 30 pp.), 4. Non-solenoidal Velocity Effects and Korteweg Stresses in Simple Mixtures of Incompressible Liquids (por Daniel D. Joseph, 30 pp.), 5. Control for Distribuited Systems: The Microlocal Approach (por Gilles Lebeau, 38 pp.), 6. Mathematical Theory of Liquid Crystals (por Fang Hua Lin, 41 pp.).

L. M. Abia

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En este mes de septiembre se celebra en Vigo el XV Congreso de Ecuaciones Diferenciales y Aplicaciones / V Congreso de Matemática Aplicada, reunión científica bianual que constituye el punto de encuentro de nuestra sociedad. Con esta ocasión se celebrará la reunión anual prevista en nuestros estatutos y se procederá a la elección de nuevos miembros. Deseo enviar un cordial saludo a todos los socios que acudan al evento y los mejores deseos del comité ejecutivo de la sociedad a los organizadores para que se consiga un nivel de exposición y discusión científica acorde con el nivel que las matemáticas y sus aplicaciones han alcanzado en nuestro país.

Como decíamos en el artículo de presentación del anuario de este año, hay aún un gran esfuerzo por hacer en el anuncio y apoyo a toda otra suerte de cursos, *workshops*, congresos, etc., que los más diversos grupos de la Sociedad organizan. La Matemática Aplicada es plural por naturaleza y por ello la descentralización de actividades debe ir acompañada de la agilidad y fluidez de información. La discusión sobre cómo articular la pluralidad de intereses de tantos investigadores trabajando en líneas diversas con la coordinación mínima que favorezca la interacción mutua y sirva de espejo a la calidad relativa se ve muy obstaculizada por una cierta tendencia al ensimismamiento, cuando no a la pura endogamia, en el actual panorama científico. Es preciso que pongamos manos a la obra para mejorar nuestra "interfase social-científica". En este sentido nuestra hoja WEB (**http://www.uam.es/sema**) ha tenido una estupenda acogida y os invitamos a suministrar ideas de como hacerla más efectiva, así como el boletín que teneis en las manos.

## Sociedades españolas

Nuestra sociedad sigue mantiendo buenas relaciones con otros modelos asociativos que se dan en las matemáticas españolas. En visita al CEMNI en Barcelona realizada el pasado mayo he tenido ocasión de debatir con el Prof. Oñate, presidente de SEMNI, sociedad de métodos numéricos en la ingeniería, las formas de establecer mayores lazos de colaboración entre ambas sociedades, lo que esperamos ver plasmado en breve en forma de un convenio para favorecer la afilaciación a las dos sociedades y la organización de actividades comunes. Por otra parte estamos a la espera de la evolución de la propuesta de reconstitu-

ción de la Real Sociedad Matemática Española, dirigida por el profesor Antonio Martínez Naveira, de la Univ. de Valencia, Presidente de la Comisión Gestora. De la comisión forman parte nuestros compañeros Enrique Fernández Cara (que representa oficialmente a nuestra sociedad), Ildefonso Díaz y Jesús María Sanz Serna.

## Relaciones internacionales

En el marco internacional he de reseñar la celebración el 29/30 de julio en Sydney, Australia, de la reunión de CICIAM, Committee for the International Conferences on Industrial and Applied Mathematics, del que somos miembros. El principal tema de trabajo era la organización de ICIAM99, el gran congreso internacional de Matematica Aplicada e Insustrial a celebrar en Edimburgo en 1999. Este congreso cuenta con un comité organizador independiente presidido por el profesor Julian Hunt y en breve aparecerán informaciones concretas sobre las actividades programadas en nuestra hoja WEB. En la reunión de Sydney nuestra sociedad fue representada por el presidente de CICIAM, Prof. G. Mennicken. Por otra parte la sociedad ECCOMAS tiene prevista su próxima reunión en octubre próximo en Atenas.

## Reflexión hacia el furturo

En la vida académica nacional existe un cierto desasosiego e incertidumbre por diversas razones, a saber, la repercusión de los nuevos planes de estudios, el incorrecto funcionamiento del sistema de concursos para la promoción del profesorado (que es, en no pocos casos, literalmente escandaloso) y otros temas de la organización académica, cuyo cambio se discute o proyecta. A lo que hay que añadir la histórica carencia de un plan nacional que oriente la inversión científica en matemática aplicada, o "matemáticas y sus aplicaciones" si se prefiere. He de reconocer que nuestra joven sociedad aún no ha encontrado la forma de encauzar el debate sobre tan importantes temas, pero sin duda ésta es una de nuestras tareas importantes. Un cordial saludo a todos.

JUAN LUIS VÁZQUEZ, PRESIDENTE DE SEMA

# PRESENTACIÓN DE SMAI

La *Société de Mathématiques Appliquées et Industrielles* (SMAI) se creó en 1983, cuando unos matemáticos aplicados franceses sintieron la especificidad de su diciplina y la necesidad de desarrollar su influencia. Hoy cuenta con cerca de

1.400 miembros, la mayoría de ellos practican la enseñanza superior o la investigación; a ellos cabe agregar colegas trabajando en la industria. Son especialistas en cálculo científico, análisis numérico, probabilidades aplicadas, estadística, control, automática, optimización, matemáticas discretas, etc... Además de miembros individuales, la sociedad cuenta con "miembros institucionales", así como laboratorios universitarios o empresas privadas.

La Sociedad se dedica a contribuir al desarrollo de las matemáticas aplicadas tanto en la investigación o las aplicaciones industriales, como en la formación de ingenieros, en particular a nivel de formación continua.

La SMAI tiene un Consejo de Administración elegido por el conjunto de sus miembros ; el Consejo se reúne dos o tres veces por año, para definir las líneas generales de la actividad. Un buró se encarga de aplicar esa política y de resolver los problemas corrientes.

En la misma SMAI existen grupos constituidos alrededor de intereses científicos particulares. Hoy día hay tres grupos:

- El GAMNI, *Groupe pour l'Avancement des Méthodes Numériques de l'Ingénieur*, que se propone desarrollar los métodos del análisis numérico en la industria.
- El grupo MAS, *Modélisation Aléatoire et Statistique*, promueve los métodos de estadística y probabilidades aplicadas.
- El grupo MODE, *Mathématiques de l'Optimisation et de la Décision*, considera temas como el análisis no lineal, la optimización, la investigación operacional, con applicaciones en la economía, las finanzas y las ciencias sociales.

## Las acciones de la SMAI

La SMAI desarrolla una actividad de edición y de organización de congresos. Publica cuatro veces al año un boletín llamado *Matapli*, en el cual, junto a artículos científicos –generalmente de síntesis–, se encuentran informaciones sobre los congresos, las tesis, los libros recién publicados y la vida de la comunidad en la Universidad y en el CNRS. La collección de la SMAI *Mathématiques & Applications* publica monografías, particularmente a nivel de tercer ciclo o de último año de escuela de ingenieros: 24 títulos están ya disponibles.

Y por último, la SMAI edita varias revistas: por una parte la revista sobre papel M2AN (*Méthodes Mathématiques de l'Analyse Numérique*), y por otra

tres revistas electrónicas, que tienen la misma exigencia científica, y el mismo tipo de selección que para las revistas sobre papel (comité de redacción, sistema estricto de referee), los artículos están disponibles - para ser leídos e impresos por el mismo lector - a través de la red Internet (`http://www.emath.fr`). Se trata de *Esaim : Contrôle Optimal et Calcul de Variations, Esaim : Probabilités et Statistique* y *Esaim : Actes de Congrès*.

La SMAI es miembro del CICIAM, que organiza cada 4 años el ICIAM (International Congress for Industrial and Applied Mathematics).

La SMAI organiza cada año el *Congrès d'Analyse Numérique*, que reúne más de 300 congresistas, y sus grupos MAS y MODE tienen cada uno sus "Journées". En todas esas ocasiones, se realizan conferencias de alto nivel, y se da la oportunidad a numeros jóvenes –con tesis recién acabadas o para acabar– de presentar sus trabajos y hacerse conocer por los colegas de todas las Universidades francesas. El GAMNI es cofundador del congreso Eccomas y organiza varios encuentros científicos cada año.

La SMAI organiza también cursos, en particular en colaboración con la Formation Continue del CNRS. En 1996, por ejemplo, hubo un curso de 3 semanas sobre *Recherches Avancées en Calcul Scientifique*.

La SMAI y el GAMNI financian un premio científico, el Prix Blaise Pascal, otorgado cada año por la Academia de Ciencias francesa para recompensar a un investigador en el ámbito del cálculo numérico y de las ciencias para ingenieros. La SMAI financia también el Premio Lagrange, otorgado cada cuatro años por el CICIAM. Este premio recompensa a un colega de alto nivel por el conjunto de su carrera.

En Francia la SMAI coordina sus esfuerzos con la *Société Mathématique de France* (SMF), la *Association pour la Statistique et ses Utilisations* (ASU), la *Association Française pour la Cybernétique Économique et Technique* (AFCET), etc ...

La investigación –tanto teórica como aplicada– tiene hoy una dimensión internacional. Por eso, la SMAI es miembro de la *European Mathematical Society* (EMS) y de *Eccomas*, al mismo tiempo que desarrolla intercambios con sociedades, que tienen el mismo objetivo en otros países : AMS y SIAM en los EE.UU., SEMA en España, IMA en Gran Bretaña, DMV y GAMM en Alemania, SIMAI en Italia.

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## COMPORTAMIENTO ASINTÓTICO DE UNA ECUACIÓN DE CONVENCCIÓN-DIFUSIÓN CON DIFUSIÓN VARIABLE

**Doctorando:** Gema Fabiola Duro Carralero.

**Director/es:** Enrique Zuazua Iriondo.

**Defensa:** 6 de marzo de 1997, Universidad Complutense de Madrid.

**Calificación:** Apto cum laude.

*Resumen:* Esta Tesis doctoral está dedicada al estudio del comportamiento asintótico cuando $t \to \infty$ de las soluciones de ecuaciones de convección-difusión del tipo:

$$(\mathcal{P}) \begin{cases} u_t - \operatorname{div}(a(x)\nabla u) = d \cdot \nabla(|u|^{q-1}u) & \text{en } (0,\infty) \times \mathbf{R} \\ u(0,x) = u_0(x) \end{cases}$$

con $d \in \mathbf{R}$, $q \geq 1$.
Nos centramos esencialmente en coeficientes de la forma $a(x) = 1 + b(x)$, donde $b(x)$ es una función *integrable*, o bien, *periódica*.
En el caso de coeficientes integrables hemos tratado los siguientes casos:

- Cuando $q > 1 + 1/N$, hemos probado que el comportamiento asintótico de las soluciones de $(\mathcal{P})$ con dato inicial $u_0(x) \in L^1(\mathbf{R}) \cap L^p(\mathbf{R})$ con $p > Nq/(N+2)$, viene dado por la solución fundamental de la ecuación del calor con masa

$$M = \int_{\mathbf{R}} u_0(x)dx.$$

- Cuando $q = 1 + 1/N$, hemos demostrado que el comportamiento asintótico de las soluciones de $(\mathcal{P})$ con dato inicial $u_0(x) \in L^1(\mathbf{R})$ viene dado por la solución autosemejante con masa $M$ de la ecuación completa con $a \equiv 1$. Además en este caso hemos calculado el segundo término del desarrollo asintótico.

En el caso de coeficientes periódicos también hemos obtenido el primer término del desarrollo asintótico. Para abordar este problema hemos usado técnicas clásicas en la Teoría de Homogenización.
Por último, en el caso particular de una dimensión espacial hemos considerado además otro tipo de coeficientes en los que $b(x)$ es una función *decreciente*. En este caso además hemos estudiado el comportamiento asintótico cuando $1 < q < 2$.

## REDES NEURONALES EN CLASIFICACIÓN

**Doctorando:** María Sagrario Sánchez Pastor.

**Director/es:** Luis A. Sarabia Peinador, Miguel A. Revilla Ramos.

**Defensa:** 23 de mayo de 1997, Universidad de Valladolid.

**Calificación:** Apto cum laude.

*Resumen:* El cálculo neuronal es una alternativa para problemas de clasificación multivariante (conocida la pertenencia de $n$ vectores $p$-dimensionales a $k$ categorías, construir la regla de decisión para asignar un nuevo vector a una de ellas) si no se desea recurrir a modelos de tipo probabilístico y/o funcional basados en información apriorística sobre los datos.
En este tipo de problemas, una red ha de ser entrenada para minimizar los errores de clasificación. Esta tarea supone manejar frecuencias (función respuesta discreta) y no es adecuada la regla de propagación de errores hacia atrás para vincular las modificaciones en los pesos de la red a las modificaciones de las frecuencias de error en clasificación obtenidas al progresar el aprendizaje.
Dada una red multinivel siempre hacia delante (MLF), se propone un mecanismo de aprendizaje basado en la evolución estocástica de sus pesos, como alternativa al método más usado, basado en el gradiente descendente para propagar los errores hacia atrás (backpropagation).
La red con aprendizaje estocástico se probará con problemas-tipo complejos de clasificación, tanto reales como simulados, incidiendo especialmente en su comportamiento como test de hipótesis no paramétrico.

## Polinomios de Chebyshev; algoritmos y aplicación en la determinación y compresión de órbitas

**Doctorando:** Roberto Barrio Gil.
**Director/es:** Antonio Elipe Sánchez.
**Defensa:** 28 de mayo de 1997, Universidad de Zaragoza.
**Calificación:** Apto cum laude.

*Resumen:* En esta memoria se analizan distintas aplicaciones de los polinomios de Chebyshev en el análisis numérico y en la mecánica celeste. En el primer capítulo se repasan los algoritmos para la manipulación de combinaciones lineales de polinomios de Chebyshev y se aportan nuevos algoritmos para la determinación directa de la integral y derivada m-ésima de dichas combinaciones lineales. Así mismo, se introduce una fórmula para la evaluación de la derivada m-ésima de combinaciones lineales de polinomios de Jacobi sin necesidad de calcular explícitamente el polinomio derivada, lo cual nos permite ahorrar operaciones. En el segundo capítulo se analizan las estabilidades *backward* y *forward* de los algoritmos de evaluación comentados en el capítulo anterior.
Los dos siguientes capítulos están dedicados a la resolución numérica de ecuaciones diferenciales ordinarias utilizando aproximaciones polinómicas de las solución, expresadas como combinaciones lineales de polinomios de Chebyshev de primera especie. Utilizando las relaciones existentes con los métodos de colocación y los métodos Runge-Kutta, se obtienen resultados sobre el orden y la A-estabilidad de los métodos de colocación basados en los ceros de los polinomios ultraesféricos o de Gegenbauer.
Se aplican los métodos de integración de EDO al problema del satélite artificial, utilizando para ello formulaciones específicas, como la de Dziobek-Brouwer, que permiten usar grandes pasos de integración y métodos de grado alto. Esta formulación reduce sensiblemente el tiempo de computación para la determinación de efemérides de satélites altos y de baja excentricidad, como es el caso de los satélites geoestacionarios, uno de los más comunes. Por último, se analiza el problema de la compresión de datos

por medio de combinaciones lineales de polinomios de Chebyshev. Se presentan los algoritmos de la compresión simple y doble con estimadores del error, lo cual permite la automatización de todo el proceso.

ESTABILIDAD ORBITAL DE SATÉLITES ESTACIONARIOS

**Doctorando:** Teodoro López Moratalla.
**Director/es:** André Deprit.
**Defensa:** 19 de junio de 1997, Universidad de Zaragoza.
**Calificación:** Apto cum laude.

*Resumen:* Se realiza un estudio de la estabilidad de las órbitas estacionarias de un satélite artificial alrededor de un planeta que gira con velocidad angular constante en la dirección de su eje principal de inercia de momento mayor. Dichas órbitas son los equilibrios del sistema dinámico que define el movimiento del satélite en un sistema de referencia móvil solidario al planeta. Se desarrolla un tratamiento totalmente analítico del problema con la ayuda de un procesador algebraico de carácter general.
En primer lugar se considera el caso de las órbitas ecuatoriales. Se establece bajo qué condiciones pueden existir los equilibrios y se realiza un estudio de su estabilidad lineal, concluyéndose que solo dos de ellos pueden ser linealmente estables, dependiendo de los valores que adopten los parámetros del problema. En los casos en que se verifica la estabilidad lineal, tras la apropiada normalización efectuada mediante transformaciones de Lie, se aplica el teorema de Arnold sobre formas cuadráticas no definidas. Los criterios de establidad obtenidos se aplican a la Tierra y a Marte, concluyendo que, en ambos casos, los equilibrios son estables en el sentido de Liapunov.
A continuación se trata el problema tridimensional. Aquí no es de aplicación el teorema de Arnold, por tratarse de un sistema con tres grados de libertad. El estudio de la estabilidad se realiza transformando el orden cero del hamiltoniano en la suma de un oscilador elíptico y un oscilador armónico. Esta transformación es posible en planetas de características similares a la Tierra. Una tranformación de Lie efectúa una doble normalización del hamiltoniano, reduciendo a uno los grados de libertad del sistema. Del estudio del flujo en el espacio fásico normalizado se concluye la estabilidad de los equilibrios.
Por último, y motivada por la obtención de los equilibrios del problema, se presenta una técnica de resolución simbólica de funciones implícitas mediante series de Lie.

APROXIMACIÓN POR SPLINES CÚBICOS Y ANALITICIDAD PARA PAQUETES DE VORTICIDAD CONSTANTE

**Doctorando:** Eliseo Chacón Vera.
**Director/es:** Tomás Chacón Rebollo, Russel E. Caflisch.
**Defensa:** 3 de julio de 1997, Universidad de Murcia.
**Calificación:** Apto cum laude.

*Resumen:* Se considera en esta tesis la solución débil de las Ecuaciones de Euler en dos dimensiones conocida por paquetes de vorticidad constante. La vorticidad inicial es una función de soporte compacto, localmente constante y toma un número finito de valores; la vorticidad evoluciona en tiempo permaneciendo con soporte compacto y tomando el mismo número finito de valores debido a su conservación a lo largo de

las trayectorias de las partículas. Para un paquete de vorticidad constante ocupando una región simplemente conexa el problema se reduce a determinar la posición de su frontera. La Ecuación de Dinámica de Contorno (CDE) describe la evolución de esta frontera.
Debido a la conservación de la vorticidad, el área del paquete de vorticidad permanece constante en tiempo, pero otras magnitudes geométricas cambian con el tiempo. La longitud y la curvatura de la frontera pueden llegar crecer muy rápido debido a la formación de pequeños filamentos de vorticidad que son expelidos del paquete de vorticidad hacia el fluido irrotacional que lo rodea. Este proceso da lugar a la creación de puntos de gran curvatura en la frontera del paquete y, en general, a una gran dificultad a la hora de la simulación numérica de la evolución de estos fluidos. Persistencia de la regularidad de la frontera para todo instante de tiempo ha sido un tema de debate que solo recientemente ha recibido una respuesta positiva. Se ha probando que un contorno inicial $C^\infty$ permanece siempre $C^\infty$ y que un contorno inicial $C^{1,\gamma}$, para $0 < \gamma < 1$, garantiza un contorno $C^{1,\gamma}$ para todo tiempo.
Dos resultados originales son presentados en esta tesis acerca de la evolución de paquetes de vorticidad constante:
Basandonos en el estudio teórico sobre la CDE, se introduce en este trabajo un nuevo método para resolver esta ecuación por medio de una interpolacion por splines cúbicos globales entre nodos y probamos convergencia del método para todo intervalo de tiempo. La idea del método es que la aproximación numérica a la frontera que se usa debe de tener alguna regularidad global mínima para así poder utilizar el análisis llevado a cabo. Este método es contrastado con simulaciones numéricas sobre la solución exacta conocida por elipse de Kirchhoff.
El segundo resultado original introducido en esta tesis es el hecho de que una frontera inicialmente analítica, permanece analítica por un intervalo de tiempo positivo. Este resultado es el primer paso en un proyecto más general que pasamos a exponer brevemente: Cualquier curva de Jordan analítica y no singular $C$ en el plano puede ser representada de manera única por una función analítica e inyectiva en un entorno de la curva. Esta función es conocida como la función de Schwarz de la curva y da reflexión con respecto a la curva. Si se denota por $S$, es la única función analítica tal que $S(z) = \overline{z}, \quad z \in C$. En esta tesis y en el futuro trabajo ya en progreso se intenta probar que analiticidad es una propiedad que se conserva en todo intervalo de tiempo y que el proceso de filamentación en un paquete de vorticidad constante está relacionado con el tipo y comportamiento de las singularidades complejas de su función de Schwarz. Básicamente se intenta probar que los filamentos son debidos a singularidades complejas de la función de Schwarz que se aproximan a los puntos de creciente curvatura en el contorno.

## Métodos de contorno para un problema de flujo estacionario alrededor de un túnel

**Doctorando:** Ricardo Celorrio de Pablo.
**Director/es:** Francisco Lisbona Cortés, Francisco Javier Sayas González.
**Defensa:** 5 de septiembre de 1997, Universidad de Zaragoza.
**Calificación:** Apto cum laude.

*Resumen:* En esta memoria se plantea un problema de infiltración en medio poroso

alrededor de un túnel, en situación estacionaria. Típicamente este problema se formula como un problema de contorno no lineal en derivadas parciales de segundo orden, en un dominio no acotado. De entre los diversos modelos no lineales para flujos no saturados nos centramos en el llamado modelo casilineal. La formulación de dicho problema en presión y la adopción del modelo casilineal derivan en el planteamiento de dos problemas exteriores para la ecuación de Helmholtz: uno con condición de contorno de Dirichlet y otro con condición de tipo Robin. Al proceder a la formulación en la frontera de dichos problemas, mediante el potencial de capa simple, se obtienen dos ecuaciones integrales: una de primer tipo con núcleo logarítmico y otra de segundo tipo.

En los capítulos 2–5 se estudian métodos numéricos para ecuaciones integrales que incluyen a las anteriores. Nos hemos centrado en métodos de colocación y algunas de sus variantes completamente discretizadas. De ellos se amplia el análisis del error conocido, demostrándose en todos los casos la existencia de desarrollos asintóticos del error.

Una vez resueltas numéricamente las ecuaciones integrales de frontera, se inserta la solución en la fórmula del potencial de capa simple, obteniéndose una reconstrucción de la solución del problema exterior para la ecuación de Helmholtz, para la que se tienen de nuevo expresiones asintóticas del error. Con ello se justifica el empleo de extrapolación de Richardson para la aceleración de la convergencia y la estimación a posteriori del error. Por último, en los capítulos 6 y 7, se aplican los resultados obtenidos al problema de infiltración alrededor del túnel y se exponen ensayos numéricos con los métodos analizados para distintas situaciones.